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APPLICATION OF CHAPLIGIN'S EQUATIONS FOR A NON-HOLONOMIC RIGID BODY

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Chapligin's equations are applied for a non-holonomic rigid body rolling without sliding on a fixed horizontal plane, not in their classic form but in an equivalent form more convenient for the application. The parameters of the system are the Eulerian angles φ, ψ, θ the coordinates x_a, y_a, z_a of the centre of gravity, the latter being the dependent parameters.

The aim of the present paper is to derive, by the aid of Chapligin's equations for non-holonomic systems, the equations of motion of a rigid body, which rolls without sliding on a fixed horizontal plane. At that we shall apply Chapligin's equations not in their classic form, but in an equivalent form more convenient for the applications.

Let q_ϱ $(\varrho=1, 2, \dots, z)$ be dependent and $q_\mu(\mu=r+1, \dots k)$ independent parameters for the non-holonomic system given, for which

$$\dot{q}_\varrho = \sum_{\mu=rn}^{k} B_\mu^{(\varrho)} \dot{q}_\mu \quad (\varrho=1, \dots, r) \tag{1}$$

holds, where the coefficients $B_\mu^{(\varrho)}$ depend only from $q_1, q_2, \dots, q_r$ but not from $q_{r+1}, q_{r+2}, \dots, q_k$. The kinetic energy T and the field function U, i. e. $U=-V$, where V denotes the potential energy function, also do not depend on $q_1, q_2, \dots, q_r$, i. e. $T=T(q_\mu, \dot{q}_\mu, \dot{q}_\varrho)$, $U=U(q_\mu)$ $(\mu=r+1, \dots, k)$. Moreover, these quantities do not depend on the time explicitly. Under these conditions Chapligin's equations may be written in the form [1, p. 892]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\mu}\right) - \frac{\partial T}{\partial q_\mu} - \frac{\partial U}{\partial q_\mu} + \sum_{\varrho=1}^{r} B_\mu^{(\varrho)} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\varrho}\right) = 0. \tag{2}$$

1. Constraints of a rigid body rolling without sliding on a horizontal plane. We shall fix inertial system of reference $Oxyz$, the plane Oxy being horizontal. Let $G\xi\eta\zeta$ be a moving system of reference, invariably connected with the body, where G denotes the centre of gravity and $G\xi$, $G\eta$, $G\zeta$ are the principal central axes of inertia for the body. Then the parameters of the rolling body will be the coordinates x_G, y_G, z_G of the centre of gravity of the body and Eulerian angles of the moving system of reference.

Let the equation of the surface of the body with respect to the system $G\xi\eta\zeta$ be

$$F(\xi, \eta, \zeta) = 0. \tag{3}$$

The transformation formulas for the systems of reference imply $z=z_G+\sin\theta\sin\varphi\xi+\sin\theta\cos\varphi\eta+\cos\theta\zeta$. Now $z=0$ provides the equation of the tangential plane of the body with respect to the moving system of reference

(4) $$\sin\theta\sin\varphi\xi+\sin\theta\cos\varphi\eta+\cos\theta\zeta+z_G=0.$$

Let ξ_0, η_0, ζ_0 be the coordinates of the point of contact P of the body with respect to $G\xi\eta\zeta$. Since the surface (3) and the plane (4) possess a common normal vector, we get

(5) $$F'_\xi(\xi_0,\eta_0,\zeta_0)/\sin\varphi\sin\theta=F'_\eta(\xi_0,\eta_0,\zeta_0)/\cos\varphi\sin\theta=F'_\zeta(\xi_0,\eta_0,\zeta_0)/\cos\theta.$$

Solving together equations (3), (5) with respect to ξ_0, η_0, ζ_0 we find

(6) $$\xi_0=\xi_0(\varphi,\theta),\quad \eta_0=\eta_0(\varphi,\theta),\quad \zeta_0=\zeta_0(\varphi,\theta).$$

Now (4) and (6) imply

(7) $$z_G=f(\varphi,\theta).$$

For the given surface (3) the function $f(\varphi,\theta)$ is completely determined. We shall express the coordinates of the point of contact by the aid of this function. To this end we represent the surface (3) as the envelope of the system of the tangential planes (4), where z_G is given by (7). Hence ξ_0, η_0, ζ_0 satisfy the following system of equations:

(8) $$\begin{aligned}&\sin\theta\sin\varphi\xi_0+\sin\theta\cos\varphi\eta_0+\cos\theta\zeta_0+f(\varphi,\theta)=0,\\&\sin\varphi\cos\theta\xi_0+\cos\varphi\cos\theta\eta_0-\sin\theta\zeta_0+f'_\theta=0,\\&\sin\theta\cos\varphi\xi_0-\sin\theta\sin\varphi\eta_0+f'_\varphi=0.\end{aligned}$$

Now (8) imply

(9) $$\begin{aligned}&\xi_0=-f\sin\theta\sin\varphi-f'_\theta\sin\varphi\cos\theta-f'_\varphi\cos\varphi/\sin\theta,\\&\eta_0=-f\sin\theta\cos\varphi-f'_\theta\cos\varphi\cos\theta+f'_\varphi\sin\varphi/\sin\theta,\\&\zeta_0=-f\cos\theta+f'_\theta\sin\theta.\end{aligned}$$

Since the body is rolling without sliding, the velocity of the point of contact P of the body equals zero, i. e.

(10) $$\bar{v}_p=\bar{v}_G+\bar{\omega}\times\overline{GP}=0,$$

where ω denotes the instantaneous angular velocity of the body. The components of ω with respect to $G\xi\eta\zeta$ are

(11) $$\begin{aligned}&p=\sin\varphi\sin\theta\dot{\psi}+\cos\varphi\dot{\theta},\\&q=\cos\varphi\sin\theta\dot{\psi}-\sin\varphi\dot{\theta},\\&r=\cos\theta\dot{\psi}+\dot{\varphi}.\end{aligned}$$

Let u, v, w be the components of the velocity $\bar{v}_G$ with respect to the system $G\xi\eta\zeta$. Then (10) implies

(12) $$u=\eta_0 r-\zeta_0 q,\quad v=\zeta_0 p-\xi_0 r,\quad w=\xi_0 q-\eta_0 p.$$

Obviously

$$\bar{v}_G = \dot{x}_G \bar{i} + \dot{y}_G \bar{j} + \dot{z}_G \bar{k} = u \bar{\xi}_0 + v \bar{\eta}_0 + w \bar{\zeta}_0, \tag{13}$$

where $\bar{i}$, $\bar{j}$, $\bar{k}$ and $\bar{\xi}_0$, $\bar{\eta}_0$, $\bar{\zeta}_0$ denote the unite vectors of the axes of $Qxyz$ and $G\xi\eta\zeta$ respectively. Now (13) implies

$$\begin{aligned} \dot{x}_G &= a_{11} u + a_{12} v + a_{13} w, \\ \dot{y}_G &= a_{21} u + a_{22} v + a_{23} w, \\ \dot{z}_G &= a_{31} u + a_{32} v + a_{33} w. \end{aligned} \tag{14}$$

Substituting (11) and (12) in (14), we get

$$\begin{aligned} \dot{x}_G = {} & \dot{\varphi}(a_{11}\eta_0 - a_{12}\xi_0) + \dot{\theta}\,[a_{11}\zeta_0 \sin\varphi + a_{12}\zeta_0 \cos\varphi - a_{13}(\xi_0 \sin\varphi + \eta_0 \cos\varphi)] \\ & + \dot{\psi}[a_{11}(\eta_0 \cos\theta - \zeta_0 \sin\theta \cos\varphi) + a_{12}(\zeta_0 \sin\theta \sin\varphi - \xi_0 \cos\theta) \\ & + a_{13} \sin\theta(\xi_0 \cos\varphi - \eta_0 \sin\varphi)], \\ \dot{y}_G = {} & \dot{\varphi}(a_{21}\eta_0 - a_{22}\xi_0) + \dot{\theta}\,[a_{21}\zeta_0 \sin\varphi + a_{22}\zeta_0 \cos\varphi - a_{23}(\xi_0 \sin\varphi + \eta_0 \cos\varphi)] \\ & + \dot{\psi}\,[a_{21}(\eta_0 \cos\theta - \zeta_0 \sin\theta \cos\varphi) + a_{22}(\zeta_0 \sin\theta \sin\varphi - \xi_0 \cos\theta) \\ & + a_{23} \sin\theta(\xi_0 \cos\varphi - \eta_0 \sin\varphi)]. \end{aligned} \tag{15}$$

The third equation (14) implies

$$\dot{z}_G = \dot{\theta} f'_\theta + \dot{\varphi} f'_\varphi, \tag{16}$$

i. e. the holonomic constraint (7). Hence the non-holonomic constraints for the parameters are the equations (15). Therefore, according to (7) and (15) the independent parameters for the body are Eulerian angles φ, ψ, θ and the dependent parameters are x_G, y_G, z_G. According to (1) we have

$$\begin{aligned} \dot{x}_G &= B_\varphi^{(x)}\dot{\varphi} + B_\psi^{(x)}\dot{\psi} + B_\theta^{(x)}\dot{\theta}, \\ \dot{y}_G &= B_\varphi^{(y)}\dot{\varphi} + B_\psi^{(y)}\dot{\psi} + B_\theta^{(y)}\dot{\theta}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} B_\varphi^{(x)} &= a_{11}\eta_0 - a_{12}\xi_0, \\ B_\psi^{(x)} &= a_{11}(\eta_0 \cos\theta - \zeta_0 \sin\theta \cos\varphi) + a_{12}(\zeta_0 \sin\theta \sin\varphi - \xi_0 \cos\theta) \\ & \quad + a_{13}(\xi_0 \cos\varphi - \eta_0 \sin\varphi) \sin\theta, \\ B_\theta^{(x)} &= a_{11}\zeta_0 \sin\varphi + a_{12}\zeta_0 \cos\varphi - a_{13}(\xi_0 \sin\varphi + \eta_0 \cos\varphi), \\ B_\varphi^{(y)} &= a_{21}\eta_0 - a_{22}\xi_0, \\ B_\psi^{(y)} &= a_{21}(\eta_0 \cos\theta - \zeta_0 \sin\theta \cos\varphi) + a_{22}(\zeta_0 \sin\theta \sin\varphi - \xi_0 \cos\theta) \\ & \quad + a_{23}(\xi_0 \cos\varphi - \eta_0 \sin\varphi) \sin\theta \\ B_\theta^{(y)} &= a_{21}\,\zeta_0 \sin\varphi + a_{22}\,\zeta_0 \cos\varphi - a_{23}(\xi_0 \sin\varphi + \eta_0 \cos\varphi), \end{aligned} \tag{18}$$

according to (15). Now (17), (18) display that Chapligin's conditions are satisfied by the coefficients $B_\mu^{(\varrho)}$.

2. Equations of motion of the body with respect to the parameters φ, ψ, θ. The kinetic energy of the moving body is given by

$$T=\frac{1}{2}(Ap^2+Bq^2+Cr^2)+\frac{1}{2}M(\dot{x}_G^2+\dot{y}_G^2+\dot{z}_G^2), \tag{19}$$

where A, B, C denote the principal central moments of inertia of the body and M its mass. The gravity function is obviously $U=-Mgz_G$ or, according to (7)

$$U=-Mgf(\varphi,\theta). \tag{20}$$

It follows from (19) and (20) that the functions T and U satisfy Chapligin's conditions, i. e. they do not depend on the parameters x_G and y_G.

Equations (2) imply

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\varphi}}\right)-\frac{\partial T}{\partial\varphi}-\frac{\partial U}{\partial\varphi}+\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}_G}\right)B_\varphi^{(x)}+\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{y}_G}\right)B_\varphi^{(y)}=0,\\
&\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\psi}}\right)-\frac{\partial T}{\partial\psi}-\frac{\partial U}{\partial\psi}+\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}_G}\right)B_\psi^{(x)}+\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{y}_G}\right)B_\psi^{(y)}=0,\\
&\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right)-\frac{\partial T}{\partial\theta}-\frac{\partial U}{\partial\theta}+\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}_G}\right)B_\theta^{(x)}+\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{y}_G}\right)B_\theta^{(y)}=0.
\end{aligned} \tag{21}$$

Now (11), (14), (19) and (20) imply

$$\begin{aligned}
&\frac{\partial T}{\partial\dot{x}_G}=M\dot{x}_G=M(a_{11}u+a_{12}v+a_{13}w),\\
&\frac{\partial T}{\partial\dot{y}_G}=M\dot{y}_G=M(a_{21}u+a_{22}v+a_{23}w);
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\frac{\partial T}{\partial\dot{\varphi}}=Cr+M\dot{z}_G\frac{\partial\dot{z}_G}{\partial\dot{\varphi}},\quad \frac{\partial T}{\partial\dot{\theta}}=Ap\cos\varphi-Bq\sin\varphi+M\dot{z}_G\frac{\partial\dot{z}_G}{\partial\dot{\theta}},\\
&\frac{\partial T}{\partial\dot{\psi}}=Ap\sin\theta\sin\varphi+Bq\sin\theta\cos\varphi+Cr\cos\theta;
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\frac{\partial T}{\partial\varphi}=Apq-Bqp+M\dot{z}_G\frac{\partial\dot{z}_G}{\partial\varphi},\quad \frac{\partial T}{\partial\psi}=0,\\
&\frac{\partial T}{\partial\theta}=Ap\cos\theta\sin\varphi\dot{\psi}+Bq\cos\theta\sin\varphi\dot{\psi}-Cr\sin\theta\dot{\psi}+M\dot{z}_G\frac{\partial\dot{z}_G}{\partial\theta};
\end{aligned} \tag{24}$$

$$\frac{\partial U}{\partial\varphi}=-Mgf'_\varphi,\quad \frac{\partial U}{\partial\psi}=0,\quad \frac{\partial U}{\partial\theta}=-Mgf'_\theta. \tag{25}$$

It is easily seen from (7) that

$$\begin{aligned}
&\frac{d}{dt}\left(M\dot{z}_G\frac{\partial\dot{z}_G}{\partial\dot{\varphi}}\right)-M\dot{z}_G\frac{\partial\dot{z}_G}{\partial\varphi}=M\frac{d}{dt}(a_{31}u+a_{32}v+a_{33}w)f'_\varphi,\\
&\frac{d}{dt}\left(M\dot{z}_G\frac{\partial\dot{z}_G}{\partial\dot{\theta}}\right)-M\dot{z}_G\frac{\partial\dot{z}_G}{\partial\theta}=M\frac{d}{dt}(a_{31}u+a_{32}v+a_{33}w)f'_\theta.
\end{aligned} \tag{26}$$

Introducing (22)—(26) in (21), we get the system

$$C\dot{r}-(A-B)pq+Mgf'_{\varphi}+MB^{(x)}_{\varphi}\frac{d}{dt}(a_{11}u+a_{12}v+a_{13}w)$$

$$+MB^{(x)}_{\varphi}\frac{d}{dt}(a_{21}u+a_{22}v+a_{23}w)+Mf'_{\varphi}\frac{d}{dt}(a_{31}u+a_{32}v+a_{33}w)=0,$$

$$A\dot{p}\sin\theta\sin\varphi+B\dot{q}\sin\theta\cos\varphi+C\dot{r}\cos\theta+Ap\frac{d}{dt}(\sin\theta\sin\varphi)$$

$$+Bq\frac{d}{dt}(\sin\theta\cos\varphi)+Cr\frac{d}{dt}(\cos\theta)+MB^{(x)}_{\psi}\frac{d}{dt}(a_{11}u+a_{12}v+a_{13}w)$$

(27)

$$+MB^{(y)}_{\psi}\frac{d}{dt}(a_{21}u+a_{22}v+a_{23}w)=0,$$

$$A\dot{p}\cos\varphi-B\dot{q}\sin\varphi-Ap(\sin\varphi\dot{\varphi}+\cos\theta\sin\varphi\dot{\psi})$$

$$-Bq(\dot{\varphi}\cos\varphi=\dot{\psi}\cos\theta\cos\varphi)+Cr\sin\theta\dot{\psi}+Mgf'_{\theta}$$

$$+MB^{(x)}_{\theta}\frac{d}{dt}(a_{11}u+a_{12}v+a_{13}w)+MB^{(y)}_{\theta}\frac{d}{dt}(a_{21}u+a_{22}v+a_{23}w)$$

$$+Mf'_{\theta}\frac{d}{dt}(a_{31}u+a_{32}v+a_{33}w)=0.$$

3. Transformation of the equations of motion into a symmetric form. The equations of motion (27) remind Eulerian equations for a rigid body with a fixed point. The non-holonomic constraints provide naturally additional terms. In order to receive the equations in a symmetric form we multiply equations (27) by $-\sin\varphi\operatorname{cotg}\theta$, $\sin\varphi/\sin\theta$, and $\cos\varphi$ respectively and add

$$A\dot{p}-(B-C)qr+Mg(f'_{\theta}\cos\varphi-f'_{\varphi}\sin\varphi\operatorname{tg}\theta)$$

$$+M(-B^{(x)}_{\varphi}\sin\varphi\operatorname{cotg}\theta+\sin\varphi/\sin\theta B^{(x)}_{\psi}+\cos\varphi B^{(x)}_{\theta})\frac{d}{dt}(a_{11}u+a_{12}v+a_{13}w)$$

(28)

$$+M(-B^{(y)}_{\psi}\sin\varphi\operatorname{cotg}\theta+\sin\varphi/\sin\theta B^{(y)}_{\psi}+\cos\varphi B^{(y)}_{\theta})\frac{d}{dt}(a_{21}u+a_{22}v+a_{23}w)$$

$$+M(-B^{(z)}_{\varphi}\sin\varphi\operatorname{cotg}\theta+\sin\varphi/\sin\theta B^{(z)}_{\psi}+\cos\varphi B^{(z)}_{\theta})\frac{d}{dt}(a_{31}u+a_{32}v+a_{33}w).$$

Then we multiply equations (27) by $-\cos\varphi\operatorname{cotg}\theta$, $\cos\varphi/\sin\theta$ and $-\sin\varphi$ respectively and add

$$B\dot{q}-(C-A)rp-Mg(f'_{\theta}\sin\varphi+f'_{\varphi}\cos\varphi\operatorname{cotg}\theta)$$

$$+M[-B^{(x)}_{\varphi}\cos\varphi\operatorname{cotg}\theta+\cos\varphi/\sin\theta B^{(x)}_{\psi}-\sin\varphi B^{(x)}_{\theta}]\frac{d}{dt}(a_{11}u+a_{12}v+a_{13}w)$$

(29)

$$+M[-B^{(y)}_{\varphi}\cos\varphi\operatorname{cotg}\theta+\cos\varphi/\sin\theta B^{(y)}_{\psi}-\sin\varphi B^{(y)}_{\theta}]\frac{d}{dt}(a_{21}u+a_{22}v+a_{23}w)$$

$$+M[-f'_{\varphi}\operatorname{cotg}\theta\cos\varphi-\sin\varphi f'_{\theta}]\frac{d}{dt}(a_{31}u+a_{32}v+a_{33}w)=0.$$

Equations (8) and (18) imply

$$
\begin{aligned}
&-B^{(x)}_{\varphi}\sin\varphi\operatorname{cotg}\theta+\sin\varphi/\sin\theta B^{(x)}_{\psi}+\cos\varphi B^{(x)}_{\theta}=a_{12}\zeta_0-a_{13}\eta_0,\\
(30)\qquad &-B^{(y)}_{\varphi}\sin\varphi\operatorname{cotg}\theta+\sin\varphi/\sin\theta B^{(y)}_{\psi}+\cos\varphi B^{(y)}_{\theta}=a_{22}\zeta_0-a_{23}\eta_0,\\
&-f'_{\varphi}\sin\varphi\operatorname{cotg}\theta+f'_{\theta}\cos\varphi=a_{32}\zeta_0-a_{33}\eta_0
\end{aligned}
$$

and

$$
\begin{aligned}
&-B^{(x)}_{\varphi}\cos\varphi\operatorname{cotg}\theta+B^{(x)}_{\psi}\cos\varphi/\sin\theta-B^{(x)}_{\theta}\sin\varphi=a_{13}\xi_0-a_{11}\zeta_0,\\
(31)\qquad &-B^{(y)}_{\varphi}\cos\varphi\operatorname{cotg}\theta+B^{(y)}_{\psi}\cos\varphi/\sin\theta-B^{(y)}_{\theta}\sin\varphi=a_{23}\xi_0-a_{21}\zeta_0,\\
&=f'_{\varphi}\cos\varphi\operatorname{cotg}\theta-f'_{\theta}\sin\varphi=a_{33}\xi_0-a_{31}\zeta_0.
\end{aligned}
$$

According to (30) and (31), equations (28), (29) and the first equation (27) may be given the form

$$
\begin{aligned}
&A\dot{p}-(B-C)qr+Mg(f'_{\theta}\cos\varphi-f'_{\varphi}\sin\varphi\operatorname{cotg}\theta)\\
&\quad+M(a_{12}\zeta_0-a_{13}\eta_0)\frac{d}{dt}(a_{11}u+a_{12}v+a_{13}w)\\
&\quad+M(a_{22}\zeta_0-a_{23}\eta_0)\frac{d}{dt}(a_{21}u+a_{22}v+a_{23}w)\\
&\quad+M(a_{32}\zeta_0-a_{33}\eta_0)\frac{d}{dt}(a_{31}u+a_{32}v+a_{33}w),\\
&B'_q-(C-A)rp-Mg(f'_{\theta}\sin\varphi+f'_{\varphi}\cos\varphi\operatorname{cotg}\theta)\\
(32)\qquad &\quad+M(a_{13}\xi_0-a_{11}\zeta_0)\frac{d}{dt}(a_{11}u+a_{12}v+a_{13}w)\\
&\quad+M(a_{23}\xi_0-a_{21}\zeta_0)\frac{d}{dt}(a_{21}u+a_{22}v+a_{23}w)\\
&\quad+M(a_{33}\xi_0-a_{31}\zeta_0)\frac{d}{dt}(a_{31}u+a_{32}v+a_{33}w),\\
&C\dot{r}-(A-B)pq+Mgf'_{\varphi}+M(a_{11}\eta_0-a_{12}\xi_0)\frac{d}{dt}(a_{11}u+a_{12}v+a_{13}w)\\
&\quad+M(a_{12}\eta_0-a_{22}\xi_0)\frac{d}{dt}(a_{21}u+a_{22}v+a_{23}w)\\
&\quad+M(a_{31}\eta_0-a_{32}\xi_0)\frac{d}{dt}(a_{31}u+a_{32}v+a_{33}w),
\end{aligned}
$$

respectively. They are completely symmetric with respect to p, q, r, to ξ_0, η_0, ζ_0 and to a_{ij}. Hence it suffices to develop the first of them only. We have

$$
\begin{aligned}
&A\dot{p}-(B-C)qr+Mg(f'_{\theta}\cos\varphi-f'_{\varphi}\sin\varphi\operatorname{cotg}\theta)\\
(33)\qquad &+M(a_{12}\zeta_0-a_{13}\eta_0)(a_{11}\dot{u}+a_{12}\dot{v}+a_{13}\dot{w})+M(a_{22}\zeta_0-a_{23}\eta_0)(a_{21}\dot{u}+a_{22}\dot{v}+a_{23}\dot{w})\\
&+M(a_{32}\zeta_0-a_{33}\eta_0)(a_{31}\dot{u}+a_{32}\dot{v}+a_{33}\dot{w})+M(a_{12}\zeta_0-a_{13}\eta_0)(\dot{a}_{11}u+\dot{a}_{12}v+\dot{a}_{13}w)\\
&+M(a_{22}\zeta_0-a_{23}\eta_0)(\dot{a}_{21}u+\dot{a}_{22}v+\dot{a}_{23}w)+M(a_{32}\zeta_0-a_{33}\eta_0)(\dot{a}_{31}u+\dot{a}_{32}v+\dot{a}_{33}w).
\end{aligned}
$$

Now (12) imply

$$
(34)\qquad
\begin{aligned}
\dot{u} &= \eta_0\dot{r} - \zeta_0\dot{q} + r\dot{\eta}_0 - q\dot{\zeta}_0,\\
\dot{v} &= \zeta_0\dot{p} - \xi_0\dot{r} + p\dot{\zeta}_0 - r\dot{\xi}_0,\\
\dot{w} &= \xi_0\dot{q} - \eta_0\dot{p} + q\dot{\xi}_0 - p\dot{\eta}_0.
\end{aligned}
$$

Let L be defined by

$$
(35)\qquad
\begin{aligned}
L = {}& M(a_{12}\zeta_0 - a_{13}\eta_0)(a_{11}\dot{u} + a_{12}\dot{v} + a_{13}\dot{w})\\
&+ M(a_{22}\zeta_0 - a_{23}\eta_0)(a_{21}\dot{u} + a_{22}\dot{v} + a_{23}\dot{w}) + M(a_{32}\zeta_0 - a_{33}\eta_0)(a_{31}\dot{u} + a_{32}\dot{v} + a_{33}\dot{w}).
\end{aligned}
$$

Introducing (34) in (35), we get

$$
\begin{aligned}
L = {}& M\dot{p}\,[(a_{12}\zeta_0 - a_{13}\eta_0)^2 + (a_{22}\zeta_0 - a_{23}\eta_0)^2 + (a_{32}\zeta_0 - a_{33}\eta_0)^2]\\
&+ M\dot{q}\,[(a_{12}\zeta_0 - a_{13}\eta_0)(a_{13}\xi_0 - a_{11}\zeta_0) = (a_{22}\zeta_0 - a_{23}\eta_0)(a_{23}\xi_0 - a_{21}\zeta_0)\\
&+ (a_{32}\zeta_0 - a_{33}\eta_0)(a_{33}\xi_0 - a_{31}\zeta_0)] + M\dot{r}\,[(a_{12}\zeta_0 - a_{13}\eta_0)(a_{11}\eta_0 - a_{12}\xi_0)\\
&+ (a_{22}\zeta_0 - a_{23}\eta_0)(a_{21}\eta_0 - a_{22}\xi_0) + (a_{32}\zeta_0 - a_{33}\eta_0)(a_{31}\eta_0 - a_{32}\xi_0)]\\
&+ M\dot{\xi}_0[(a_{12}\zeta_0 - a_{13}\eta_0)(a_{13}q - a_{12}r) + (a_{22}\zeta_0 - a_{23}\eta_0)(a_{23}q - a_{22}r)\\
&+ (a_{32}\zeta_0 - a_{33}\eta_0)(a_{33}q - a_{32}r)] + M\dot{\eta}_0[(a_{12}\zeta_0 - a_{13}\eta_0)(a_{11}r - a_{13}\,p)\\
&+ (a_{22}\zeta_0 - a_{23}\eta_0)(a_{21}r - a_{23}\,p) + (a_{32}\zeta_0 - a_{33}\eta_0)(a_{31}r - a_{31}\,p)]\\
&+ M\dot{\zeta}_0[(a_{12}\zeta_0 - a_{13}\eta_0)(a_{12}\,p - a_{11}q) + (a_{22}\zeta_0 - a_{23}\eta_0)(a_{22}\,p - a_{21}q)\\
&+ (a_{32}\zeta_0 - a_{33}\eta_0)(a_{32}\,p - a_{31}q)]
\end{aligned}
$$

or, by virtue of the relations between

$$
L = M\dot{p}(\eta_0^2 + \zeta_0^2) - M\dot{q}\xi_0\eta_0 - M\dot{r}\xi_0\zeta_0 - M\dot{\xi}_0(\eta_0 q + \zeta_0 r) + M\dot{\eta}_0\eta_0 p + M\dot{\zeta}_0\zeta_0\,p.
$$

This equality may be written in the form

$$
(36)\qquad L = M\dot{p}(\eta_0^2 + \zeta_0^2) - M\dot{q}\xi_0\eta_0 - M\dot{r}\xi_0\zeta_0 + \frac{M}{2}\,p\,\frac{d}{dt}\,(\xi_0^2 + \eta_0^2 + \zeta_0^2) - M\dot{\xi}_0(\,p\xi_0 + q\eta_0 + r\zeta_0).
$$

Let N be defined by

$$
(37)\qquad
\begin{aligned}
N = {}& M(a_{12}\zeta_0 - a_{13}\eta_0)(\dot{a}_{11}u + \dot{a}_{12}v + \dot{a}_{13}w) + M(a_{22}\zeta_0 - a_{23}\eta_0)(\dot{a}_{21}u + \dot{a}_{22}v + \dot{a}_{23}w)\\
&+ M(a_{32}\zeta_0 - a_{33}\eta_0)(\dot{a}_{31}u + \dot{a}_{32}v + \dot{a}_{33}w).
\end{aligned}
$$

This equality may obviously be written in the form

$$
(38)\qquad
\begin{aligned}
N = {}& Mu[\zeta_0(\dot{a}_{11}a_{12} + \dot{a}_{21}a_{22} + \dot{a}_{31}a_{32}) - \eta_0(\dot{a}_{11}a_{13} + \dot{a}_{21}a_{23} + \dot{a}_{31}a_{33})]\\
&+ Mv[\zeta_0(\dot{a}_{12}a_{12} + \dot{a}_{22}a_{22} + \dot{a}_{32}a_{32}) - \eta_0(\dot{a}_{12}a_{13} + \dot{a}_{22}a_{23} + \dot{a}_{32}a_{33})]\\
&+ Mw[\zeta_0(\dot{a}_{13}a_{12} + \dot{a}_{23}a_{22} + \dot{a}_{33}a_{32}) - \eta_0(\dot{a}_{13}a_{13} + \dot{a}_{23}a_{23} + \dot{a}_{33}a_{33})].
\end{aligned}
$$

It is easily seen that the following relations hold

$$p=\dot a_{12}a_{13}+\dot a_{22}a_{23}+\dot a_{32}a_{33},$$
(39)
$$q=\dot a_{13}a_{11}+\dot a_{23}a_{21}+\dot a_{33}a_{31},$$
$$r=\dot a_{11}a_{12}+\dot a_{21}a_{22}+\dot a_{31}a_{32}.$$

Moreover, we obviously have

(40)
$$\dot a_{12}a_{12}+\dot a_{22}a_{22}+\dot a_{32}a_{32}=0,$$
$$\dot a_{13}a_{13}+\dot a_{23}a_{23}+\dot a_{33}a_{33}=0.$$

Introducing (39) and (40) in (37), we get

(41)
$$N=Mu(\zeta_0 r-\eta_0 q)-Mv\eta_0 p-Mw\zeta_0 p.$$

According to (12) we get from (41)

(42)
$$N=-M(\zeta_0 q-\eta_0 r)(\xi_0 p+\eta_0 q+\zeta_0 r).$$

Because of (36) and (42), the equation (33) takes the following final form

(43)
$$A\dot p-(B-C)qr+Mg(f_\theta'\cos\varphi-f_\varphi'\sin\varphi\cot g\,\theta)$$
$$+M\dot p(\eta_0^2+\zeta_0^2)-M\dot q\xi_0\eta_0-M\dot r\xi_0\zeta_0+\frac{M}{2}p\frac{d}{dt}(\overline{GP})^2$$
$$-M\dot\xi_0(\overline\omega\,.\,\overline{Gp})-M(\zeta_0 q-\eta_0 r)(\overline\omega\,.\,\overline{GP})=0.$$

By virtue of the symmetry of the equations (32), the equations remaining may be received by a cyclic change

(44)
$$B\dot q-(C-A)rp-Mg(f_\theta'\sin\varphi-f_\varphi'\cos\varphi\cot g\,\theta)+M\dot q(\zeta_0^2+\xi_0^2)-M\dot p\zeta_0\eta_0$$
$$-M\dot r\zeta_0\eta_0+\frac{M}{2}q\frac{d}{dt}(\overline{GP}^2)-M\dot\eta_0(\overline\omega\,.\,\overline{GP})-M(\xi_0 r-\zeta_0 p)(\overline\omega\,.\,\overline{GP})=0,$$

(45)
$$C\dot r-(A-B)pq+Mgf_\varphi'+M\dot r(\xi_0^2+\eta_0^2)-M\dot p\xi_0\zeta_0$$
$$-M\dot q\eta_0\zeta_0+\frac{M}{2}r\frac{d}{dt}(\overline{GP}^2)-M\dot\zeta_0(\overline\omega\,.\,\overline{GP})-M(\eta_0 p-\xi_0 q)(\overline\omega\,.\,\overline{GP})=0.$$

Now equations (43)—(45) represent a complete system of equations of movement of a rigid body rolling without sliding on a fixed horizontal plane. Furthermore, the equations (7) and (15) for the dependent parameters z_G, x_G, y_G must be added.

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