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ALMOST HERMITIAN MANIFOLDS OF CONSTANT TYPE AND VANISHING GENERALIZED TENSOR OF BOCHNER

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Almost Hermitian manifolds of class AH_3 with vanishing generalized tensor of Bochner are considered. The form of the curvature tensor is found. Some properties of the generalized tensor of Bochner similar to the Weyl conformal curvature tensor and the Bochner curvature tensor are established. A new characterization for a Kähler manifold with vanishing Bochner curvature tensor is given.

In [10] L. Vanhecke and K. Yano established some properties of the almost Hermitian manifolds of the class AH_1 with vanishing tensor of Bochner. We consider AH_3-manifolds of constant type and vanishing generalized tensor of Bochner. We prove that some of the properties established in [10] are also valid for these manifolds.

Let M be an almost Hermitian manifold with dim $M=2n$, a metric tensor g, an almost complex structure J and a Levi—Civita connection ∇. The curvature tensor R of the connection ∇ is given by $R(X, Y)=[\nabla_X, \nabla_Y]-\nabla_{[X,Y]}$ for arbitrary vector fields X, Y. We denote $R(X, Y, Z, U)=g(R(X, Y)Z, U)$, where X, Y, Z, U are arbitrary vector fields. The classes AH_i $(i=1, 2, 3)$ are determined respectively by the following identities for the curvature tensor:

1) $R(X, Y, Z, U)=R(X, Y, JZ, JU)$;

2) $R(X, Y, Z, U)=R(X, Y JZ, JU)+R(X, JY, Z, JU)+R(JX, Y, Z, JU)$;

3) $R(X, Y, Z, U)=R(JX, JY, JZ, JU)$.

It is known that $AH_1\subset AH_2\subset AH_3$.

If α is an arbitrary section in the tangential space T_pM, its curvature is given by the equality $K(\alpha, p)=K(x, y)=R(x, y, y, x)$, where $\{x, y\}$ is an orthonormal basis for α. With respect to this basis the angle θ between α and $J\alpha$ is given by the equality $\cos\theta=|g(x, Jy)|$. The angle $\theta\in[0, \pi/2]$, and if $\theta=0$ $(\theta=\pi/2)$ the section is called holomorphic (antiholomorphic). If α is a holomorphic section with an orthonormal basis $\{x, Jx\}$, its curvature is denoted by $K(x, Jx)=H(x)$. Let $E^m(m\geq 2)$ be an m-dimensional linear subspace (m-plane) of T_pM and $\{u_1, \ldots, u_m\}$ be an orthonormal basis of E^m. Then the curvature of E^m is given by the formula [2]

$$(1)\qquad K(E^m, p)=\sum_{1\leq i\neq j\leq m} K(u_i, u_j).$$

A linear subspace E of T_pM is called holomorphic (antiholomorphic) if $JE=E(JE\perp E)$.

Let M be a Kählerian manifold. The tensor of Bochner in $p \in M$ has the form

$$\begin{aligned} B(x, y, z, u) = R(x, y, z, u) - \frac{1}{2(n+2)} \{ & g(y, z)S(x, u) - g(x, z)S(y, u) \\ & + g(Jy, z)S(Jx, u) - g(Jx, z)S(Jy, u) - 2g(Jx, y)S(Jz, u) \\ & + g(x, u)S(y, z) - g(y, u)S(x, z) \\ & + g(Jx, u)S(Jy, z) - g(Jy, u)S(Jx, z) - 2g(Jz, u)S(Jx, y) \} \\ & + \frac{S(p)}{4(n+1)(n+2)} R_0(x, y, z, u), \end{aligned} \tag{2}$$

where $S(y, z)$ and $S(p)$ are the Ricci tensor and the scalar curvature respectively, and

$$\begin{aligned} R_0(x, y, z, u) = & g(y, z)g(x, u) - g(x, z)g(y, u) \\ & + g(Jy, z)g(Jx, u) - g(Jx, z)g(Jy, u) - 2g(Jx, y)g(Jz, u). \end{aligned}$$

We shall establish a characteristic for a Kählerian manifold with vanishing tensor of Bochner.

Theorem 1. *Let M be a Kählerian manifold with* $\dim M = 2n \geq 4$. *The manifold M has a vanishing tensor of Bochner if and only if the curvature of any antiholomorphic n-plane in T_pM does not depend on the n-plane for an arbitrary point $p \in M$.*

Proof. Let $B=0$ and E^n be an arbitrary antiholomorphic n-plane in T_pM. If $\{u_1, \ldots, u_n\}$ is an orthonormal basis for E^n, then $\{u_i, Ju_i\}$ $(i=1, \ldots, n)$ is an adapted basis for T_pM. The equality (1) gives

$$K(E^n, p) = 2 \sum_{1 \leq i < j \leq n} K(u_i, u_j). \tag{3}$$

From the given condition, using (2), we obtain

$$K(u_i, u_j) = R(u_i, u_j, u_j, u_i) = \frac{S(u_i)+S(u_j)}{2(n+2)} - \frac{S(p)}{4(n+1)(n+2)}, \quad i \neq j,$$

where $S(u_i)$ is the Ricci curvature of the direction, determined by the vector u_i. Substituting in (3) we find

$$K(E^n, p) = (n-1)\, S(p)/4(n+1),$$

which proves that $K(E^n, p)$ does not depend on the antiholomorphic n-plane E^n in T_pM.

For the inverse let E^n be an arbitrary antiholomorphic n-plane in T_pM with an orthonormal basis $\{u_1, \ldots, u_n\}$. Then $\{Ju_1, u_2, \ldots, u_n\}$ is also an orthonormal basis for an antiholomorphic n-plane $\bar{E}^n$ in T_pM. From $K(E^n, p) = K(\bar{E}^n, p)$ it follows that

$$\sum_{j=2}^{n} K(u_1, u_j) = \sum_{j=2}^{n} K(Ju_1, u_j) = \sum_{j=2}^{n} K(u_1, Ju_j). \tag{4}$$

In the last equality we used that every Kählerian manifold is an AH_3-manifold. Taking into account that

$$S(u_1) = \sum_{j=2}^{n} \{ K(u_1, u_j) + K(u_1, Ju_j)\} + H(u_1),$$

from (4) we obtain $2\Sigma_{j=2}^{n} K(u_1, u_j) = S(u_1) - H(u_1)$. Substituting the vector u_1 by any of the vectors $u_j (j=2, \ldots, n)$, and summing the corresponding equalities, we get

$$2K(E^n, p) = \frac{1}{2} S(p) - \sum_{i=1}^{n} H(u_i).$$

This equality implies that $\Sigma_{i=1}^{n} H(u_i)$ does not depend on the adapted basis. Hence $B=0$ [9].

Let M be an almost Hermitian manifold. In [3], [1] the following generalized curvature tensor is considered:

$$\begin{aligned} 16R^*(x, y, z, u) &= 3R(x, y, z, u) + 3R(Jx, Jy, z, u) + 3R(Jx, Jy, Jz, Ju) \\ (5) \qquad &+ 3R(x, y, Jz, Ju) - R(Jy, Jz, x, u) - R(Jz, Jx, y, u) + R(y, Jz, Jx, u) \\ &+ R(Jz, x, Jy, u) - R(y, z, Jx, Ju) - R(z, x, Jy, Ju) + R(Jy, z, x, Ju) + R(z, Jx, y, Ju) \end{aligned}$$

This tensor is uniquely determined by the following conditions:

1) $R^*(x, y, z, u) = -R^*(y, x, z, u)$;

2) $R^*(x, y, z, u) + R^*(y, z, x, u) + R^*(z, x, y, u) = 0$;

3) $R^*(x, y, z, u) = -R^*(x, y, u, z)$;

4) $R^*(x, y, z, u) = R^*(x, y, Jz, Ju)$;

5) $R^*(x, Jx, Jx, x) = R(x, Jx, Jx, x)$.

We shall call an LC-tensor any tensor satisfying the conditions 1), 2), 3). With respect to the tensor R^* a curvature $K^*(\alpha, p)$ of a section α in T_pM, a Ricci tensor $S^*(x, y)$, a Ricci curvature $S^*(x)$, a scalar curvature $S^*(p)$ can be defined in the same way as with respect to the curvature tensor R. The generalized tensor of Bochner associated with R^* is given by the equality

$$\begin{aligned} B^*(x, y, z, u) = R^*(x, y, z, u) - \frac{1}{2(n+2)}\{ &g(y, z)S^*(x, u) - g(x, z)S^*(y, u) \\ &+ g(Jy, z)S^*(Jx, u) - g(Jx, z)S^*(Jy, u) - 2g(Jx, y)S^*(Jz, u) \\ (6) \qquad &+ g(x, u)S^*(y, z) - g(y, u)S^*(x, z) \\ &+ g(Jx, u)S^*(Jy, z) - g(Jy, u)S^*(Jx, z) - 2g(Jz, u)S^*(Jx, y)\} \\ &+ \frac{S^*(p)}{4(n+1)(n+2)} R_0(x, y, z, u). \end{aligned}$$

If $E^m (m \geq 2)$ is an m-plane in T_pM with an orthonormal basis $\{u_1, \ldots, u_m\}$, we define a generalized curvature $K^*(E^m, p)$ of E^m by the equality

$$(7) \qquad K^*(E^m, p) = \sum_{1 \leq i \neq j \leq m} K^*(u_i, u_j).$$

Similarly to Theorem 1 we obtain the following theorem.

Theorem 2. *Let M be an almost Hermitian manifold with* $\dim M = 2n \geq 4$. *M has a vanishing generalized tensor of Bochner if and only if the generalized curvature of any antiholomorphic n-plane in T_pM does not depend on the n-plane for an arbitrary point $p \in M$.*

In the theory of the almost Hermitian manifolds the tensor $R'(x, y, z, u) = R(x, y, Jz, Ju)$ is useful. With respect to this tensor we define a Ricci tensor by the equality

$$(8) \qquad S'(x, y) = \sum_{i=1}^{n} \{R'(u_i, x, y, u_i) + R'(Ju_i, x, y, Ju_i)\},$$

where $\{u_i, Ju_i\}$ $(i=1, \ldots, n)$ is an arbitrary adapted basis for T_pM. The Ricci curvature of the direction, determined by the vector x, is given by $S'(x) = S'(x, x)/g(x, x)$. If α is a section in T_pM, we define a curvature $K'(\alpha, p)$ of this section

$$K'(\alpha, p) = K'(x, y) = \frac{R(x, y, Jy, Jx)}{g(x, x)g(y, y) - g^2(x, y)},$$

where $\{x, y\}$ is a basis for α. With respect to an adapted basis we have

$$S'(u_i) = \sum_{j=1,\ j\neq i}^{n} \{K'(u_i, u_j) + K'(u_i, Ju_j)\} + H(u_i),\ i = 1, \ldots, n.$$

The bisectional curvature $H(u_i, u_j) = R(u_i, Ju_i, Ju_j, u_j)$ [5] and the curvature K' are connected by the following equality $H(u_i, u_j) = K'(u_i, u_j) + K'(u_i, Ju_j)$, $i \neq j$. Using the first Bianchi identity for the tensor R, from (8) we obtain $S'(x, y) = 2^{-1} \sum_{i=1}^{n} R(x, Jy, Ju_i, u_i)$. Finally, the scalar invariant associated with the tensor R' is given by $S'(p) = \sum_{i=1}^{n} \{S'(u_i) + S'(Ju_i)\}$.

Let $E^m (m \geq 2)$ be an m-plane in T_pM with an orthonormal basis $\{u_1, \ldots, u_m\}$. We define a curvature $K'(E^m, p)$ of E^m with respect to the tensor R' by the formula

$$(9) \qquad K'(E^m, p) = \sum_{1 \leq i \neq j \leq m} K'(u_i, u_j).$$

Corollary 3. *Let M be an AH_3-manifold with* $\dim M = 2n \geq 4$. *The manifold M has a vanishing generalized tensor of Bochner if and only if*

$$(10) \qquad K(E^n, p) + K'(E^n, p) = c(p)$$

for an arbitrary antiholomorphic n-plane E^n in T_pM, $p \in M$.

Proof. Since M is an AH_3-manifold, from (5) it follows that

$$K^*(u_i, u_j) = \frac{1}{8}\{3K(u_i, u_j) - K(u_i, Ju_j) - 5K'(u_i, u_j) + K'(u_i, Ju_j)\}.$$

Then the proposition follows from Theorem 2 taking into account the equalities (7), (9) and the last equality. The function in (10) has the form

$$c(p) = \frac{n-1}{2(n+1)} S(p) + \frac{2n-1}{4(n+1)} \{S(p) - S'(p)\}.$$

In [6] A. Gray has considred nearly Kähler manifolds of constant type. This notion was extended for the class of the almost Hermitian manifolds [11]. We shall call that an almost Hermitian manifold is of constant type $\lambda(p)$ if for any antiholomorphic section α in T_pM holds $K(\alpha, p) - K'(\alpha, p) = \lambda(p)$, where $\lambda(p)$ does not depend on the section α.

L. Vanhecke proved in [11] that if M is an AH_3-manifold of constant type, then for arbitrary x, y in T_pM the following equality $R(x, y, y, x) - R'(x, y, y, x) = \lambda\{g(x, x)g(y, y) - g^2(x, y) - g^2(x, Jy)\}$ holds. This equality can be written in the form $K(\alpha, p) - K'(\alpha, p) = \lambda \sin^2\theta$, where α is an arbitrary section in T_pM and $\theta = \sphericalangle(\alpha, J\alpha)$.

Let M be a Kählerian manifold with vanishing tensor of Bochner. If $\{x, y\}$ is an orthonormal basis for any antiholomorphic section in T_pM, the quantities

$$\frac{4}{n+2}S(x)-H(x)\,;\ \frac{1}{n+2}\ \frac{S(x)+S(y)}{2}-K(x,y)$$

depend only on the point $p\in M$. The inverse is also true, i. e. if these quantities depend only on the point, then M is a manifold with vanishing tensor of Bochner [9; 3].

Now let M be an AH_3-manifold satisfying in every point $p\in M$ the conditions

$$\frac{4}{n+2}S(x)-H(x)=\mu(p)\,;\ \frac{1}{n+2}\ \frac{S(x)+S(y)}{2}-K(x,y)=\nu(p), \tag{11}$$

where $\{x, y\}$ is an orthonormal basis for any antiholomorphic section in T_pM. We shall find the form of the curvature tensor for such manifolds. We shall use the next theorem.

Theorem 4 [4]. *Let M be an almost Hermitian manifold with* $\dim M\geq 4$ *and $T:(T_pM)^4\to\mathbb{R}$ be a quadrilinear mapping satisfying the following conditions:*

1) *T is an LC-tensor;*

2) $T(x, y, z, u)=T(Jx, Jy, Jz, Ju)$;

3) *$T(x, y, y, x)=0$, where $\{x, y\}$ is a basis for an arbitrary holomorphic or antiholomorphic section in T_pM. Then $T=0$.*

Theorem 5. *If M is an AH_3-manifold satisfying in every point $p\in M$ the conditions* (11), *then the curvature tensor R has the form*

$$R(x,y,z,u)=\frac{1}{2(n+2)}R_{S,I}(x,y,z,u)-\nu R_1(x,y,z,u)-\frac{\mu-\nu}{3}R_2(x,y,z,u), \tag{12}$$

where

$$R_{S,I}(x,y,z,u)=g(y,z)S(x,u)-g(x,z)S(y,u)$$
$$+g(Jy,z)S(Jx,u)-g(Jx,z)S(Jy,u)-2g(Jx,y)S(Jz,u)+g(x,u)S(y,z)$$
$$-g(y,u)S(x,z)+g(Jx,u)S(Jy,z)-g(Jy,u)S(Jx,z)-2g(Jz,u)S(Jx,y);$$
$$R_1(x,y,z,u)=g(y,z)g(x,u)-g(x,z)g(y,u);$$
$$R_2(x,y,z,u)=g(Jy,z)g(Jx,u)-g(Jx,z)g(Jy,u)-2g(Jx,y)g(Jz,u)$$

and vice versa.

Proof. We consider the tensor $T=R-(1/2(n+2))R_{S,I}+\nu R_1+((\mu-\nu)/3)R_2$. By the given conditions this tensor satisfies the properties 1), 2), 3) in Theorem 4 and hence $T=0$. The inverse is an easy verification.

Corollary 6. *If M is an AH_3-manifold with* $\dim M\geq 4$ *satisfying in every point $p\in M$ the conditions* (11), *then*

a) *M is an AH_2-manifold;*

b) *M is of constant type.*

Proof. From Theorem 5 it follows that the curvature tensor has the form (12). By a straightforward calculation we obtain

$$R(x,y,z,u)-R(x,y,Jz,Ju)-R(x,Jy,z,Ju)-R(Jx,y,z,Ju)=0,$$

which proves a). From (12) we have $R(x,y,y,x)-R(x,y,Jy,Jx)=(\mu-4\nu)/3$ for an orthonormal basis of any antiholomorphic section in T_pM, i. e. M is of constant type $\lambda=(\mu-4\nu)/3$.

Corollary 7. *An AH_3-manifold M with* $\dim M \geqq 4$ *satisfying the conditions* (11) *is an AH_1-manifold if and only if $\mu = 4\nu$.*

Proof. Theorem 5 implies that the curvature tensor has the form (12). Using this formula by direct computation we find

$$R(x, y, y, x) - R(x, y, Jy, Jx)$$
$$= \frac{\mu - 4\nu}{3}\{g(y, z)g(x, u) - g(x, z)g(y, u) - g(Jy, z)g(Jx, u) + g(Jx, z)g(Jy, u)\}.$$

The proposition follows directly from this equality.

Corollary 8. *If the AH_3-manifold M with* $\dim M \geqq 4$ *satisfies the conditions* (11) *in every point $p \in M$ and $\mu = 4\nu$, then M is an AH_1-manifold with vanishing tensor of Bochner.*

Proof. In terms of an adapted basis the conditions (11) by $\mu = 4\nu$ are

$$\frac{4}{n+2} S(u_i) - H(u_i) = \mu; \quad \frac{1}{n+2} \frac{S(u_i)+S(u_j)}{2} - K(u_i, u_j) = \frac{\mu}{4}, \ i \neq j.$$

From these equalities by summing we obtain $\mu = S/(n+1)(n+2)$ and replacing in (12) we find $R = (1/2(n+2))\, R_{S,l} - (S/4(n+1)(n+2))R_0$, i. e. $B = 0$.

Let M be an AH_3-manifold satisfying in every point $p \in M$ the conditions (11). The functions $\mu(p)$ and $\nu(p)$ can be expressed by the scalar invariants $S(p)$ and $S'(p)$. In fact, using an adapted basis we have

$$(13) \qquad \frac{4}{n+2} S(u_i) - H(u_i) = \mu; \quad \frac{1}{n+2} \frac{S(u_i)+S(u_j)}{2} - K(u_i, u_j) = \nu, \ i \neq j.$$

Summing the second equalities on j we obtain

$$\frac{4}{n+2} S(u_i) - H(u_i) = \frac{S}{2(n+2)} - 2(n-1)\nu.$$

Comparing these equalities with the first equalities of (13) we find

$$(14) \qquad \mu + 2(n-1)\nu = S/2(n+2).$$

On the other hand $K(u_i, u_j) - K'(u_i, u_j) = (\mu - 4\nu)/3$, $i \neq j$. Summing these equalities we obtain

$$(15) \qquad \mu - 4\nu = 3(S - S')/4n(n-1).$$

From (14) and (15) we find

$$\mu = \frac{S}{(n+1)(n+2)} + \frac{3(S-S')}{4n(n-1)}; \quad \nu = \frac{S}{4(n+1)(n+2)} - \frac{3(S-S')}{8n(n^2-1)}.$$

The manifold is of constant type $\lambda = (\mu - 4\nu)/3 = (S - S')/4n(n-1)$.

Theorem 9. *Let M be an AH_3-manifold of constant type λ and* $\dim M \geqq 4$. *The following conditions are equivalent:*

a) *M is with vanishing generalized tensor of Bochner;*

b) *The curvature tensor has the form* (12).

Proof. Let $B^* = 0$. With respect to an adapted basis from (6) it follows that

$$(16) \qquad \begin{aligned} &4S^*(u_i)/(n+2) - H(u_i) = S^*/(n+1)(n+2); \\ &\frac{1}{n+2} \frac{S^*(u_i)+S^*(u_j)}{2} - K^*(u_i, u_j) = \frac{S^*}{4(n+1)(n+2)}; \ i \neq j. \end{aligned}$$

On the other hand, M is of constant type λ

$$(17) \qquad K(u_i, u_j) - K'(u_i, u_j) = \lambda, \ i \neq j.$$

From here by summing we obtain $S(u_i)-S'(u_i)=2(n-1)\lambda$; $S-S'=4n(n-1)\lambda$. This equality and (5) give

$$S^*(u_i)=S(u_i)-3(n-1)\lambda/2\,;\; S^*=S-3n(n-1)\lambda. \tag{18}$$

Substituting in the first equalities of (16) we find

$$\frac{4}{n+2}S(u_i)-H(u_i)=\frac{S}{(n+1)(n+2)}+3\,\frac{S-S'}{4n(n-1)}=\mu. \tag{19}$$

From (5) we have

$$K^*(u_i, u_j)=K(u_i, u_j)-\frac{5}{8}\{K(u_i, u_j)-K'(u_i, u_j)\}-\frac{1}{8}\{K(u_i, Ju_j)-K'(u_i, Ju_j)\}.$$

Taking into account (17) we find $K^*(u_i, u_j)=K(u_i, u_j)-3\lambda/4$. This formula, (18) and the second equalities of (16) imply

$$\frac{1}{n+2}\,\frac{S(u_i)+S(u_j)}{2}-K(u_i, u_j)=\frac{S}{4(n+1)(n+2)}-3\,\frac{S-S'}{8n(n^2-1)}=\nu,\quad i\neq j. \tag{20}$$

Now the proposition b) follows from (19), (20) and Theorem 5.

For the inverse let R have the form (12). By Corollary 6 M is of constant type $\lambda=(\mu-4\nu)/3$. Then (5) gives

$$R^*(x, y, z, u)=R(x, y, z, u)-\frac{3}{4}\lambda R_1(x, y, z, u)-\frac{1}{4}\lambda R_2(x, y, z, u),$$

$$S^*(x, y)=S(x, y)-3(n-1)\lambda g(x, y)/2,$$

$$S^*=S-3n(n-1)\lambda.$$

Substituting in (6) we obtain

$$B^*=R-\frac{1}{2(n+2)}R_{S,J}+\nu R_1+\frac{\mu-\nu}{3}R_2=0.$$

Remark. Theorem 9 gives another approach to the problems in [12].

The next theorem is due to Schouten.

Theorem [8]. *The Riemannian manifold M with* dim $M\geq 4$ *is conformally flat if and only if $R(x, y, z, u)=0$ for an arbitrary orthonormal basis of any 4-plane in T_pM for every $p\in M$.*

L. Vanhecke and K. Yano proved an analogous characterization for an AH_1-manifolds with vanishing tensor of Bochner.

Theorem [10]. *Let M be an AH_1-manifold with* dim $M\geq 8$. *The tensor of Bochner is zero if and only if $R(x, y, z, u)=0$ for an arbitrary orthonormal basis of any antiholomorphic 4-plane in T_pM for every $p\in M$.*

We shall prove an analogous property for AH_3-manifolds of constant type.

Theorem 10. *Let M be an AH_3-manifold of constant type and* dim $M\geq 8$. *Then the following conditions are equivalent:*

a) $B^*=0$;

b) *$R(x, y, z, u)=0$ for an arbitrary orthonormal basis of every antiholomorphic 4-plane in T_pM, $p\in M$.*

Proof. Let $B^*=0$. From Theorem 9 it follows that the curvature tensor has the form (12). If $\{x, y, z, u\}$ is an orthonormal basis of any antiholomorphic 4-plane in T_pM, from (12) we obtain $R(x, y, z, u)=0$.

For the inverse let orthonormal quadruple $\{x, y, z, u\}$ span an antiholomorphic 4-plane in T_pM. By our assumption $R(x, y, z, u)=0$. The quadruple

$\{(x+y)/\sqrt{2},\ y,\ z,\ (x-y)/\sqrt{2}\}$ is also an orthonormal basis for an antiholomorphic 4-plane and hence $R((x+y)/\sqrt{2}, y, z, (x-y)/\sqrt{2})=0$. This equality gives

(21) $$R(x, y, z, x)=R(u, y, z, u).$$

The quadruple $\{Jx, y, z, u\}$ span also an antiholomorphic 4-plane and similarly to (21) we have

(22) $$R(Jx, y, z, Jx)=R(u, y, z, u).$$

From (21) and (22) we find

(23) $$R(x, y, z, x)=R(Jx, y, z, Jx).$$

Using the quadruple $\{x, (y+z)/\sqrt{2}, (y-z)/\sqrt{2}, u\}$ analogously to (23) we obtain

$$R(x, (y+z)/\sqrt{2},\ (y-z)/\sqrt{2},\ x)=R(Jx, (y+z)/\sqrt{2},\ (y-z)/\sqrt{2},\ Jx).$$

From here follows

(24) $$R(x, y, y, x)-R(x, z, z, x)=R(Jx, y, y, Jx)-R(Jx, z, z, Jx).$$

Replacing the vector z with Jz, similarly to (24) we obtain

(25) $$R(x, y, y, x)-R(x, Jz, Jz, x)=R(Jx, y, y, Jx)-R(Jx, Jz, Jz, Jx).$$

Since M is an AH_3-manifold summing (24) and (25) we find

(26) $$R(x, y, y, x)=R(x, Jy, Jy, x)$$

for an arbitrary orthonormal basis $\{x, y\}$ of any antiholomorphic section in T_pM. The vectors $\{(x+y)/\sqrt{2}, (x-y)/\sqrt{2}\}$ span also an antiholomorphic section and hence

$$R((x+y)/\sqrt{2},\ (x-y)/\sqrt{2},\ (x-y)/\sqrt{2},\ (x+y)/\sqrt{2})=R((x+y)\sqrt{2},\ (Jx-Jy)/\sqrt{2},$$
$$(Jx-Jy)/\sqrt{2},\ (x+y)/\sqrt{2}).$$

From here by direct computation we find

$$2R(x, y, y, x)+6R(x, y, Jy, Jx)=R(x, Jx, Jx, x)+R(y, Jy, Jy, y)$$

or

(27) $$2K(x, y)+6K'(x, y)=H(x)+H(y)$$

and with respect to an adapted basis we have

$$2K(u_i, u_j)-6K'(u_i, u_j)=H(u_i)+H(u_j),\ i\neq j;$$
$$2K(u_i, Ju_j)+6K'(u_i, Ju_j)=H(u_i)+H(u_j),\ i\neq j.$$

From here by summing on j it follows that $S(u_i)+3S'(u_i)=(n+2)H(u_i)+\Sigma_{j=1}^{n}H(u_j)$, $i=1, \ldots, n$ or

(28) $$\frac{4}{n+2}S^*(u_i)-H(u_i)=\frac{1}{n+2}\sum_{j=1}^{n}H(u_j),\quad i=1, \ldots, n.$$

Summing on i we obtain $\Sigma_{j=1}^{n}H(u_j)=S^*/(n+1)$ and (28) gets the form

$$\frac{4}{n+2}S^*(u_i)-H(u_i)=\frac{S^*}{(n+1)(n+2)},\quad i=1, \ldots, n.$$

Hence $B^*=0$ [3].

Corollary 11. *Let M be an AH_3-manifold of constant type and* $\dim M \geqq 4$. *The following conditions are equivalent:*

1) $B^* = 0$;

2) $K(x, y) = K(x, Jy)$, *where* $\{x, y\}$ *is an orthonormal basis of any antiholomorphic section in* T_pM, $p \in M$.

Proof. Let $B^* = 0$. From Theorem 9 it follows that the curvature tensor has the form (12). Then 2) follows by direct verification.

Since the condition 2) is (26), the proof of the inverse is the same as in Theorem 10.

Corollary 12. *Let M be an AH_3-manifold of constant type λ and* $\dim M \geqq 4$. *The following conditions are equivalent:*

a) $B^* = 0$

b) $8K(x, y) - H(x) - H(y) = \lambda$, *where* $\{x, y\}$ *is an orthonormal basis of any antiholomorphic section in* T_pM, $p \in M$.

Proof. Let $B^* = 0$. From Theorem 9 and Theorem 5 we have

$$\frac{4}{n+2} S(x) - H(x) = \frac{4}{n+2} S(y) - H(y) = \mu, \quad \frac{1}{n+2} \frac{S(x) + S(y)}{2} - K(x, y) = \nu.$$

From these equalities we find $8K(x, y) - H(x) - H(y) = 2(\mu - 4\nu) = 6\lambda$.

For the inverse let $\{x, y\}$ be an orthonormal basis of any antiholomorphic section in T_pM. Then $8K(x, y) - H(x) - H(y) = 6\lambda$. Using the vectors $\{x, Jy\}$ we have also $8K(x, Jy) - H(x) - H(y) = 6\lambda$ and hence $K(x, y) = K(x, Jy)$. The condition a) follows now from Corollary 11.

The next characterization for the conformally flat Riemannian manifolds is due to R. Kulkarni.

Theorem [7]. *Let M be a Riemannian manifold with* $\dim M \geqq 4$. *M is conformally flat if and only if* $K(x, y) + K(z, u) = K(x, u) + K(y, z)$ *for an arbitrary orthonormal basis* $\{x, y, z, u\}$ *of any 4-plane in* T_pM, $p \in M$.

L. Vanhecke and K. Yano proved an analogous characterization for AH_1-manifolds.

Theorem [10]. *Let M be an AH_1-manifold with* $\dim M \geqq 8$. *The following conditions are equivalent:*

a) $B = 0$;

b) $K(x, y) + K(z, u) = K(x, u) + K(y, z)$, *where* $\{x, y, z, u\}$ *is an orthonormal basis of an arbitrary antiholomorphic 4-plane in* T_pM, $p \in M$.

We shall give an analogous characterization for AH_3-manifolds of constant type.

Theorem 13. *Let M be an AH_3-manifold of constant type and* $\dim M \geqq 8$. *The following conditions are equivalent*

a) $B^* = 0$

b) $K(x, y) + K(z, u) = K(x, u) + K(y, z)$, *where* $\{x, y, z, u\}$ *is an orthonormal basis of an arbitrary antiholomorphic 4-plane in* T_pM, $p \in M$.

Proof. Let $B^* = 0$ and the orthonormal quadruple $\{x, y, z, u\}$ span an antiholomorphic 4-plane in T_pM. From Theorem 9 and Theorem 5 we have

$$K(x, y) + K(z, u) = \frac{1}{2(n+2)} \{S(x) + S(y) + S(z) + S(u)\} + 2\nu,$$

$$K(x, u) + K(y, z) = \frac{1}{2(n+2)} \{S(x) + S(u) + S(y) + S(z)\} + 2\nu,$$

which proves the first implication.

Let now the condition b) be valid and let the orthonormal quadruple $\{x, y, z, u\}$ span an antiholomorphic 4-plane in T_pM. The quadruple $\{(x+z)/\sqrt{2}, y, (x-z)/\sqrt{2}, u\}$ has the same property. Then the following equality

$$R((x+z)/\sqrt{2},\ y, y, (x+z)/\sqrt{2})+R((x-z)/\sqrt{2},\ u, u, (x-z)/\sqrt{2})$$
$$=R((x+z)/\sqrt{2},\ u, u, (x+z)/\sqrt{2})+R(y, (x-z)/\sqrt{2}, (x-z)/\sqrt{2}, y)$$

holds. From this equality by direct computation we find

$$R(x, y, z, y)=R(x, u, z, u), \tag{29}$$

which holds for an arbitrary quadruple $\{x, y, z, u\}$, spanning an antiholomorphic 4-plane. Then (29) is also fulfilled for the quadruple $\{x, (y+u)/\sqrt{2}, z, (y-u)/\sqrt{2}\}$ and hence

$$R(x, y, z, u)+R(x, u, z, y)=0. \tag{30}$$

Changing x and y by y and x, respectively in (30) we obtain

$$R(y, x, z, u)+R(y, u, z, x)=0. \tag{31}$$

Taking into account the first Bianchi identity from (30) and (31) we find $2R(x, y, z, u)-R(y, z, x, u)-R(z, x, y, u)=3R(x, y, z, u)=0$. Now the proposition follows from Theorem 10.

Corollary 14. *Let M be an AH_3-manifold of constant type λ and* dim $M\geq 4$. *The following conditions are equivalent:*

a) $B^*=0$;

b) $K(E^n, p)=c(p)$, *where E^n is an arbitrary antiholomorphic n-plane in T_pM, $p\in M$.*

Proof. Since M is an AH_3-manifold of constant type λ, by a straightforward calculation we find $K(E^n, p)-K'(E^n, p)=n(n-1)\lambda(p)$.
Now the proposition follows from Corollary 3. It is easy to show that

$$c(p)=\frac{n-1}{4(n+1)}S(p)+\frac{3n^2(n-1)}{2(n+1)}\lambda(p).$$

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