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A NEW BRANCH IN THE THEORY OF SUMS OF INDEPENDENT RANDOM VARIABLES

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1. Introduction. For more than 200 years the central limit theorem (CLT) has been an essential part of the very core of probability theory. The reason is twofold; namely, the statement of the theorem is of eminent practical and theoretical importance, but, moreover, research on more and more general versions of the theorem has demanded new ideas and methods again and again. This way work connected with the CLT has greatly influenced the development and sharpening of the tools now generally used in probability theory. The present survey article shows that this process is going on even in the field of sums of independent random variables (r. v.). Throughout this paper we write for the sake of brevity $X \sim F$ if the r. v. X is subject to the distribution function (d. f.) F. We first consider a sequence $X_1, X_2, \ldots$ of independent identically distributed (i. i. d.) r. v. with $X_1 \sim H$ and put

$$S_n := B_n^{-1}(X_1 + \ldots + X_n) - A_n, \quad n = 1, 2, \ldots \tag{1.1}$$

where $\{A_n\}_1^\infty$ and $\{B_n\}_1^\infty$ $(B_n > 0)$ are certain sequences of numbers. Further we use the notations

$$F_n(x) := \mathsf{P}(S_n < x) = H^{n*}(B_n(x + A_n)), \tag{1.2}$$

$$T_H(x) := 1 - H(x) + H(-x), \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du.$$

Weak convergence of d. f. K_n to a monotone limit function K is denoted by $K_n \to_w K$, while we write $K_n \to_c K$ if, moreover, $K(\infty) - K(-\infty) = 1$ (complete convergence).

Now we are in a position to formulate the following well known version of the CLT due to P. Levy, A. Ja. Khintchine, and W. Feller, see e. g. [3].

Theorem 1.1. *The complete convergence*

$$F_n \to_c \Phi \tag{1.3}$$

holds true if, and only if,

$$\lim_{x\to\infty} [x^2 T_H(x) / \int_{-x}^{x} u^2 dH(u)] = 0. \tag{1.4}$$

Condition (1.4) implies only that large values of X_1 are improbable in a precise sense. Convergence of $\{F_n\}$ to all the other stable d. f. takes place only under very special conditions. This explains why we find d. f. similar to Φ very often in practical situations.

For a long time it has been generally accepted that this theorem is a final step in the long history of the CLT, at least for the simplest model with i. i. d. summands. But in the following section it will be shown that this is not true and that — according to a conjecture of V. M. Zolotarev — conditions necessary and sufficient for (1.3) can be given which are of quite another type than (1.4).

It is now obvious that this idea gave rise to the development of two different new theories; they are, of course, connected, but each of them presents an independent interest and advances knowledge useful for the other.

One of these research directions is connected with Kolmogorov's conjecture. It is devoted to purely analytic problems of a new kind, namely to the continuation of d. f. within appropriate classes. The results now available in this field are contained in [4]—[8], [14], [16]—[19], [21], [22], [25], [26] and are briefly surveyed in [19], see § 2.3 of the present paper.

The other direction is a reconsideration of the theory of sums for r. v. from a new point of view. Its saliant feature is "restricted convergence on F_n", i. e. weak convergence of F_n on a subset $S\subset R_1$, which is assumed to conclude complete convergence. The problem of relative compactness (see 2.2) is an essential part of it. In the present paper we survey the very first results in this field. The reader will feel that the number of unsolved problems is enormous so that we can formulate only a few of them.

2. New versions of the CLT for i. i. d. r. v.

2.1. Zolotarev's idea and the general plan for the proofs of the new limit theorems. It was V. M. Zolotarev who proposed to prove the following new version of the CLT. This theorem was the starting point of the theory to be described in the present paper. Better versions of this result are the theorems 2.4.2 and 2.4.3.

Theorem 2.1.1. *The relations* (1.3) *and* (1.4) *are true if, and only if, we have the weak convergence on a half-line*

$$F_n(x) \to_w \Phi(x), \qquad x<\tau, \tag{2.1.1}$$

or, what is the essential assertion, (2.1.1) *implies* (1.3).

Remark. This means that the restricted convergence (2.1.1) continues to the whole line and that the limit function is a non-defective d. f. which is uniquely defined.

We outline the plan of the proof, since it is typical for all limit theorems of this new type.

Obviously there exists a subsequence $\{n'\}$ of natural numbers and a monotone function F such that

(2.1.2) $$F_{n'} \to_{w} F$$

where F satisfies the condition

(2.1.3) $$F(x)=\Phi(x), \qquad x<\tau.$$

Now inevitably two problems arise.

2.1.1) Is the set $\mathfrak{F}=\{F_n\}_1^\infty$ relatively compact, i. e. does from (2.1.2) and (2.1.3) follow that $F(\infty)=1$?

If this is the case (see § 2.2) then F is an infinitely divisible (i. d.) d. f. This is an immediate consequence of the following result which is a slight extension of a lemma proved in [20].

Lemma 2.1.1. *Let $x_1<x_2$ be two arbitrarily fixed numbers and*

$$f_1:=\limsup_{n\to\infty} F_n(x_1)<\liminf_{n\to\infty} F_n(x_2)=:f_2.$$

Then $\{B_n/\sqrt{n}\}$ is bounded away from zero.

For the proof we observe that for all $y>0$ and all x

$$F_n(x+y)-F_n(x)\leqq \sup_x [F_n(x+y)-F_n(x)]=\sup_x \mathsf{P}(x\leqq S_n\leqq x+y)$$

$$=\sup_z \mathsf{P}(z\leqq X_1+\ldots+X_n<z+B_n y).$$

Now by a well known property of concentration functions of sums (see [28, § III. 2] it follows that $F_n(x+y)-F_n(x)\leq C(yB_n+1)/n$, where $c>0$ is an absolute constant. Putting $x=x_1$, $y=x_2-x_1$ we obtain the assertion by an indirect argument.

Now the second problem can be formulated as follows.

2.1.2) Is an i. d. d. f. F with property (2.1.3) uniquely defined?

This problem is treated in § 2.3. Having given an answer in the affirmative to both these questions we have essentially proved theorem 2.1.1. Namely, if the assertion were wrong then there existed a subsequence $\{n'\}$ such that we have (2.1.2) and (2.1.3), but $F\neq\Phi$, which is impossible.

Since [20] was published these two problems have been solved separately for several times under various circumstances. Only in the proof of the CLT quoted in § 4.5 it was possible to treat them in a unique set-up.

2.2. Relative compactness of $\{F_n\}$. As well known (see e. g. [2]) a d. f. F is called stable if for each s there exist constants $Q_s>0$ and P_s such that for all x

(2.2.1) $$F(Q_s x+P_s)=F^{s*}(x), \qquad s=2, 3, \ldots$$

They form the class of all limit d. f. for sums (1.1) of i. i. d. r. v. Turning to problem 2.1.1) we remark that it has not much to do with the normal d. f. Indeed, the classical argument leading from the assumption $F_n\to_c F$ to the equation (2.2.1) can be adapted to our case in which weak convergence of F_n on a half-axis $(-\infty, \tau)$ is assumed. It is an interesting feature of the new theory that certain lemmas of Khintchine and Gnedenko (see [3, § 10]) have to be reformulated. The following lemmas are slightly improved versions of results proved in [17]. Obviously part b) of lemma 2.2.1 concerns a continuation of convergence.

Lemma 2.2.1. a) *Let τ be a real number and let us consider two monotone functions* $\bar{K}:(-\infty, \tau)\to(0,1)$, $\hat{K}:(-\infty,\infty)\to(0,1]$ *with the property*

$$\bar{K}(-\infty)=\hat{K}(-\infty)=0. \tag{2.2.2}$$

For a certain sequence $\{K_k\}$ of d.f. we assume the weak convergence

$$K_k(x)\to_w \bar{K}(x),\qquad x<\tau. \tag{2.2.3}$$

Further we suppose that for certain sequences $\{a_k\}$, $\{b_k\}$ $(b_k>0)$ of numbers we have

$$K_k(b_kx+a_k)\to_w \hat{K}(x),\qquad x\in R_1. \tag{2.2.4}$$

Then $\{a_k\}$ and $\{b_k\}$ are bounded and $\liminf b_k>0$.

b) *Let, moreover, a continuation K of $\bar{K}$ be given by the weak convergence (for a subsequence $\{k'\}$ of natural numbers) $K_{k'}(x)\to_w K(x)$, $x\in R_1$. Then there exist numbers a and b such that in the points of continuity we have $\hat{K}(x)=K(bx+a)$, $x\in R_1$.*

Lema 2.2.2. *Under the assumptions of lemma* 2.2.1 *we have* $a=\lim a_k$, $b=\lim b_k$. If $\hat{K}(x)=\bar{K}(x)$ *for* $bx+a<\tau$, *then* $a=0$, $b=1$.

These lemmas make it possible to obtain the equations (2.2.1) at least on half lines; more precisely we have

Lemma 2.2.3. *Let* $\bar{F}:(-\infty,\tau)\to(0,1)$ *be a monotone function satisfying*

$$\bar{F}(x)>0,\qquad x<\tau,\qquad \bar{F}(-\infty)=0 \tag{2.2.5}$$

and assume the restricted convergence

$$F_n(x)\to_w \bar{F}(x),\qquad x<\tau. \tag{2.2.6}$$

If further a continuation F of $\bar{F}$ is defined by (2.1.2.), *then there exist numbers $Q_s>0$ and P_s such that the equations* (2.2.1) *are true at least on the half-axes $Q_sx+P_s\leq\tau$, $s=2,3,\dots$*

Now we get the following answer to problem 2.2.1), see [20].

Theorem 2.2.1. *Under the assumptions* (2.2.5) *and* (2.2.6) *the set $\mathfrak{F}=\{F_n\}$ is relatively compact and all limit d.f. are i.d.*

Proof. We introduce numbers x_s by the equations $Q_sx_s+P_s=\tau$ and obtain from lemma 2.2.3 $0<F(\tau)=F^{s*}((\tau-P_s)/Q_s)\leq F^{s*}(\infty)\leq F^s(\infty)$, $s=2,3,\dots$ so that $F(\infty)<1$ is impossible. The second assertion follows from lemma 2.1.1.

2.3. Kolmogorov's conjecture: The continuation theorem for Φ. Now we turn to problem 2.2.2) which was solved in [16] with the aim of solving Zototarev's problem on the CLT. Afterwards it turned out that it was A. N. Kolmogorov who had stated the problem already in the fifties of our century. As we remarked in § 1 it served as a starting-point for the continuation theory of d. f. The exact answer to problem 2.2.2) runs as follows.

Theorem 2.3.1. *If an i. d. d. f. F possesses the property*

(2.3.1) $$F(x)=\Phi(x), \quad x<\tau,$$

then $F\equiv\Phi$.

Soon it was remarked (see [14]) that this very origin of continuation theory leads beyond this theory. The point is that theorem 2.3.1 can be generalized such that we need not assume coincidence of F and Φ in any point. It is just a certain asymptotic behaviour of one tail of F that characterized Φ. More precisely we have

Theorem 2.3.2. *An i. d. d. f. F satisfying* $\lim_{x\to-\infty}(F(x)/\Phi(x))=1$ *coincides with* Φ.

The most general theorem which can be obtained in this direction was found only very recently, see [21]. For its formulation we denote by $\{x_i\}_1^\infty$ and $\{y_i\}_1^\infty$ two denumerable point sets with the property $\lim x_i=\lim y_i=-\infty$

Theorem 2.3.3. *Let F stand for an i. d. d. f. If for* $i\to\infty$ *we have* $(1+o(1))\Phi(x_i)\leq F(x_i)$ *and for all* $c>0$ *we have* $F(y_i)=o(\Phi(y_i)e^{c|y_i|})$, *then it follows that* $F\equiv\Phi$.

This result cannot be improved, see [21]. All theorems quoted in 2.3 are of purely analytic nature and we do not enter a discussion of their proofs, see [19]. All of them yield new versions of the CLT, namely the theorems 2.1.1, 2.4.2 and 2.4.3, resp.

2.4. A general compactness theorem with application to the CLT. The method which in 2.2 led to the compactness of $\{F_n\}$ can be further developed and yields the following result, see [17]. We consider an arbitrary subset $\{F_{n'}\}\subset\{F_n\}$ such that

(2.4.1) $$F_{n'}(x)\to_{\mathrm{w}} F(x), \quad x<\tau,$$

where F is a d. f. We also introduce the notations

$$\underline{F}_k(x):=\liminf_{n'\to\infty}F_{kn'}(x),\ \ \overline{F}_k(x):=\limsup_{n'\to\infty}F_{kn'}(x),\ \ k=1,\ 2,\ldots$$

so that $\underline{F}_1(x)=\overline{F}_1(x)=F(x)$.

Theorem 2.4.1. *Assume, for H fixed,* (2.4.1) *and*

(2.4.2) $$\underline{F}_k(x)>0 \quad \textit{for all} \quad x, \quad \overline{F}_k(-\infty)=0, \quad k=1,\ 2,\ \ldots$$

Let, moreover, there exist two numbers y and $\delta>0$ *such that*

$$\underline{F}_k(y)\geq\delta>0, \quad k=1,\ 2,\ldots,$$

then the set $\{F_n\}$ *is relatively compact and all limit d. f. are i. d.*

This way an interesting asymptotic property of convolutions appears, since relative compactness is equivalent to tightness (see e. g. [1]). Hence by virtue of the compactness theorem we have the following situation. For every $\varepsilon>0$ there exists a number $N(\varepsilon)$ such that

(2.4.3) $$1-F_n(x)+F_n(-x)<\varepsilon \quad \text{for all} \quad x>N(\varepsilon), \quad n=1,\ 2,\ldots$$

But the assumptions concern only the left tails of F_n.

In full generality this theorem has not been applied yet. But putting

$$\underline{F}(x):=\liminf_{n\to\infty}F_n(x), \quad \overline{F}(x):=\limsup_{n\to\infty}F_n(x)$$

we obtain simple criteria (see [17]), e. g.

Criterion 2.4.1. *Let V be an arbitrary d.f. with $V(x)>0$ for all x. If for certain numbers a, b*

$$0<a\leqq\liminf_{x\to-\infty}\underline{F}(x)/V(x)\leqq\limsup_{x\to-\infty}\bar{F}(x)/V(x)\leqq b<\infty,$$

then the set $\{F_n\}$ is relatively compact.

Criterion 2.4.2. *The set $\{F_n\}$ is relatively compact if for all x, $\underline{F}(x)>0$ and $\bar{F}(-\infty)=0$.*

Now we are able to turn to generalizations of theorem 2.1.1 in which convergence of $\{F_n\}$ is not supposed in any point. The first one is an immediate consequence of theorem 2.3.2 and criterion 2.4.1 (see [15]).

Theorem 2.4.2. *The convergence* (1.3) *follows from*

$$\lim_{x\to-\infty}\underline{F}(x)/\Phi(x)=\lim_{x\to-\infty}\bar{F}(x)/\Phi(x)=1.$$

But theorem 2.3.3 and criterion 2.4.2 yield (see [21]).

Theorem 2.4.3. *The convergence* (1.3) *follows from*

$$\underline{F}(x_i)\geqq\Phi(x_i)(1+o(1)),\quad \bar{F}(y_i)=o(\Phi(y_i)e^{c|y_i|})$$

for all $c>0$.

2.5. The classes $\mathfrak{G}_\tau$. In the continuation theory of i. d. d. f. the following notation plays a considerable role.

Definition 2.5.1. *An i. d. d. f. K belongs to the class $\mathfrak{G}_\tau$ if it possesses the following property: If for an arbitrary i. d. d. f. L we have $K(x)=L(x)(x<\tau)$ then it follows that $K\equiv L$.*

By theorem 2.3.1 this class contains Φ; it is important since we easily recognize that the proof of theorem 2.1.1 outlined in § 2.1 yields also the following result. We need only replace the characterization theorem for Φ by definition 2.5.1.

Theorem 2.5.1. *If $F\in\mathfrak{G}_\tau$, $F(x)>0$ for all x, and $F_n(x)\to_{\mathrm{w}}F(x)$ for $x<\tau$, then we have $F_n\to_{\mathrm{c}}F$.*

A very useful criterion for an i. d. d. f. to be an element of $\mathfrak{G}_\tau$ was given by I. A. Ibragimov in [26].

Theorem 2.5.2. *Let the d. f. K be i. d., $K(x)>0$ for all x, and for all $r>0$*

$$K(x)=O(e^{rx}),\quad x\to-\infty, \tag{2.5.1}$$

then $K\in\mathfrak{G}_\tau$ for all τ.

Note that (2.5.1) is equivalent to the fact that the characteristic function (c. f.) k of K is analytic in the upper half-plan, see [13]. Examples given in [26] show that this supposition is essential.

It is already obvious that subsequences $\{n'\}$ of natural numbers play a considerable role in our context. Therefore the theory of partial attractions outlined in [2, § XVII. 9] and [3, § 47], might be useful. Indeed it provided characterizations of $\mathfrak{G}_\tau$, see [17, § 8.]

3. Convergence of F_n to a positive function of a half line. The results surveyed in 2 make us suppose that assumptions on F_n concerning a half axis only will be sufficient to obtain complete convergence also in the general case, see [17].

We remind the reader of Levy's canonical representation formula according to which every i. d. c. f. is uniquely defined by two numbers a, $\sigma^2 \geqq 0$ and two spectral functions $M:(-\infty, 0) \to [0, \infty)$, $N:(0, \infty) \to (-\infty, 0]$. Therefore we can briefly characterize i. d. d. f. by the notation $F = F(a, \sigma^2, M, N)$.

On the very first sight it seems natural to replace Φ in theorem 2.1.1 by another stable d. f.

$$F_\alpha = F_\alpha(a, 0, c_-/|x|^\alpha, -c_+/x^\alpha), \quad 0<\alpha<2, \quad c_- \geqq 0, \ c_+ \geqq 0, \ c_- + c_+ > 0.$$

But it can be easily derived from [13, § 5.7] that the corresponding c. f. f_α is analytic in the upper half plane if, and only if, $c_- = 0$, so that theorem 2.5.1 can be applied only in this case. It is for this reason that another approach had to be taken and that stable d. f. are not mentioned in the assumptions of

Theorem 3.1.1. *Let $\bar{F}:(-\infty, \tau) \to (0, 1)$ be a monotone function satisfying*

$$\bar{F}(-\infty) = 0, \ \bar{F}(x) > 0, \quad x < \tau, \tag{3.1.1}$$

and assume the restricted convergence

$$F_n(x) \to_{\mathrm{w}} \bar{F}(x), \quad x < \tau, \tag{3.1.2}$$

then there exists a unique stable d. f. F_α such that

$$F_n \to_{\mathrm{c}} F_\alpha. \tag{3.1.3}$$

We outline the proof. The compactness theorem 2.2.1 tells us that the sequence $\{F_n\}_1^\infty$ is relatively compact.

According to the general model for the proofs of such theorems (see § 2.1) in the second step we consider a subsequence $\{F_{n'}\}$ such that we have for a certain i. d. d. f.

$$F_{n'} \to_{\mathrm{c}} F. \tag{3.1.4}$$

The i. d. d. f. F obviously continues $\bar{F}$ to the line, but it is known (§ 2.5) that not all i. d. d. f. F satisfying (3.1.1) belong to $\mathfrak{G}_\tau$. Therefore we have to introduce a new element in the proof outlined in § 2.1. Indeed, we have to carry out the unique continuation of $\bar{F}$ given by (3.1.2); for this purpose we distinguish three different cases described in the reduction theorem quoted below.

Needless to say that the following notion proves useful in our context; for details see [5, 11, 19].

Definition 3.1.1. *A* d. f. *F is called analytic if there exists a strip $S = \{z = x + iy : |y| < b\}$ and a function A analytic in S such that $F(x) = A(x)$ for all x.*

Reduction theorem. *The i. d. d. f. F appearing in* (3.1.4) *belongs either to $\mathfrak{G}_\tau$, or it is an analytic d. f., or it can be characterized by*

$$F_\alpha = F_\alpha(a, 0, c_-/|x|^\alpha, N), \quad c_- > 0, \tag{3.1.5}$$

where the number α, $0<\alpha<1$, is uniquely defined.

This way the continuation problem has been essentially reduced, and we need only the following

Continuation theorem. *The i. d. d. f.* (3.1.5) *with α fixed* ($0<\alpha<1$) *are uniquely defined by their values $F_\alpha(x)$, $x<\tau$, where τ is an arbitrary number.*

Theorem 3.1.1 leads us to the

Conjecture. *All stable d. f.* F_α *satisfying* $F_\alpha(x)>0$ *for all* x *belong to* $\mathfrak{G}_\tau$ *for all* τ*.

Why does theorem 3.1.1 not permit such a conclusion?

Let the assumptions of theorem 3.1.1 be satisfied for a certain d. f. H (see (1.2)). Then there might exist another d. f. $\widehat{H}$, a subsequence $\{\widehat{n}\}$ of natural numbers, and sequences $\{A_{\widehat{n}}\}$, $\{B_{\widehat{n}}\}$ such that

$$\widehat{F}_{\widehat{n}}(x):=\widehat{H}^{\widehat{n}*}(B_{\widehat{n}}(x+A_{\widehat{n}}))\to_{\mathrm{c}} G(x),$$

where $G(x)=F_\alpha(x)$, $x<\tau$, but $G\neq F_\alpha$. In view of the uniqueness theorem for stable d. f. (see [7; 18; 25]) it is only clear that G is not stable.

4. The triangular array. 4.1. Statement of the problem. Let $\{X_{nk}\}$ ($n\geq 1$, $1\leq k\leq k_n$, $\lim k_n=\infty$) be a triangular array, i. e. a double sequence of r. v. independent in each row, see [2; 3]. We assume $X_{nk}\sim F_{nk}$ and put $\Sigma_n:=X_{n1}+\ldots+X_{nk_n}-A_n\sim G_n$, $n=1, 2, \ldots$, where $\{A_n\}$ stands for a certain sequence of constants.

As usual the r. v. X_{nk} are said to be infinitely small (i. s.) if

$$\lim_{n\to\infty}\max_{1\leq k\leq k_n}\mathsf{P}(|X_{nk}|\geq\varepsilon)=0 \quad \text{for all} \quad \varepsilon>0. \tag{4.1.1}$$

Recent research on this model was stimulated by the following problem stated by Zolotarev in [25].

Let F stand for an i. d. d. f., while $\mathfrak{S}$ denotes a point set and $F_{\mathfrak{S}}$ the restriction of F to $\mathfrak{S}$. In certain cases F is uniquely defined by $F_{\mathfrak{S}}$ (if $\mathfrak{S}=(-\infty,\tau)$, then this is true for $F_{\mathfrak{S}}\in\mathfrak{G}_\tau$; if F is analytic and $\mathfrak{S}$ is denumerable and bounded this is trivially true).

Now assume that F is uniquely defined by $F_{\mathfrak{S}}$. Further suppose that

$$G_n(x)\to_{\mathrm{w}} F_{\mathfrak{S}}(x), \quad x\in\mathfrak{S}, \tag{4.1.2}$$

where isolated points of $\mathfrak{S}$ are continuity points of $F_{\mathfrak{S}}$ by convention. Does it then follow that $G_n\to_{\mathrm{c}} F$?

Obviously a compactness problem does not arise if $\mathfrak{S}$ is unbounded in either direction. Therefore we will not discuss this case. We focus our attention to three different ways in which this problem has been treated so far, see 4.2, 4.5, 4.6.

4.2. A general compactness criterion. We will denote a median of the r. v. X_{nk} by m_{nk}. Then from results obtained in [23] the following classical compactness condition can be formulated; compare also [24].

Theorem 4.2.1. $\{G_n\}_1^\infty$ *is relatively compact if, and only if,*

$$\sup_n \sum_{k=1}^{k_n}\int_{-\infty}^{\infty}\frac{x^2}{1+x^2}\,dF_{nk}(x+m_{nk})<\infty,$$

* This conjecture has proved wrong by I. V. Ostrovskij; see the material of the 3rd Vilnius Conference on Probability Theory, 1981.

(4.2.1) $$\sup_n \sum_{k=1}^{k_n} P(|X_{nk}-m_{nk}|\geq x)=o(1), \quad x\to\infty,$$

(4.2.2) $$\sup_n \left|\sum_{k=1}^{k_n}\left[m_{nk}+\int_{-\varepsilon}^{\varepsilon} x dF_{nk}(x)\right]-A_n\right|<\infty \quad \textit{for all} \quad \varepsilon>0.$$

For the right understanding of the results concerning the triangular array we remind the reader of the following classical conditions of convergence for G_n.

Lemma 4.2.1. *Let the r. v. X_{nk} be i. s. Then for the convergence $G_n\to_c F(a, \sigma^2, M, N)$ it is necessary and sufficient that for*

(4.2.3) $$\begin{cases}\sum_{k=1}^{k_n} F_{nk}(x+m_{nk})\to_w M(x), & x<0,\\ \sum_{k=1}^{k_n}[F_{nk}(x+m_{nk})-1]\to_w N(x), & x>0,\end{cases}$$

(4.2.4) $$\lim_{\varepsilon\to 0}\liminf_{n\to\infty}\left[\sum_{k=1}^{k_n}\int_{-\varepsilon}^{\varepsilon} x^2 dF_{nk}(x+m_{nk})-\sum_{k=1}^{k_n}\left(\int_{-\varepsilon}^{\varepsilon} x dF_{nk}(x+m_{nk})\right)^2\right]$$
$$=\lim_{\varepsilon\to 0}\limsup_{n\to\infty}[\ \dots\ ]=\sigma^2.$$

Remark. It is easily seen that (4.2.1) is a consequence of (4.2.3) since $\lim_{x\to\infty}(M(-x)-N(x))=0$.

Now we put

(4.2.5) $$\underline{G}(x):=\liminf_{n\to\infty} G_n(x), \quad \overline{G}(x):=\limsup_{n\to\infty} G_n(x)$$

and assume that there exist real numbers $a_1, a_2\in\mathfrak{S}$, $a_1<a_2$, such that

(4.2.6) $$\underline{G}(a_2)-\overline{G}(a_1)>0.$$

Theorem 4.2.2. *$\{G_n\}_1^\infty$ is relatively compact for a certain sequence $\{A_n\}_1^\infty$ if, and only if,* (4.2.1) *and* (4.2.6) *are true; in this case A_n satisfies the condition* (4.2.2).

We introduce the concentration function $Q(X,\lambda):=\sup_x P(x\leq X<x+\lambda)$, $\lambda>0$. Then we are able to formulate another version of theorem 4.2.2.

Theorem 4.2.2'. *$\{G_n\}_1^\infty$ is relatively compact for a certain sequence $\{A_n\}_1^\infty$ if, and only if,* (4.2.1) *holds and for some $\lambda>0$ we have* $\inf_n Q(\Sigma_n,\lambda)>0$

We next consider the one-sided condition

(4.2.7) $$\lim_{x\to-\infty}\overline{G}(x)=0.$$

It is trivially satisfied if, for instance, we put $\mathfrak{S}=(-\infty,\tau)$ and assume $F_{\mathfrak{S}}(x)\to 0$, $x\to-\infty$.

Theorem 4.2.3. *$\{G_n\}_1^\infty$ is relatively compact for a certain sequence $\{A_n\}_1^\infty$ if, and only if, we have* (4.2.7), *$\underline{G}(x_0)>0$ for some real numbers x_0 and*

(4.2.8) $$\sup_n \Sigma \mathsf{P}(X_{nk}-m_{nk}\geqq x)=o(1), \qquad x\to\infty.$$

An interesting assertion for the defective case is given by

Theorem 4.2.4. *Let us assume that the* X_{nk} *are i. s. and that we have* $G_n \to_w F$, *where* $F(-\infty)=0$ *and* $0<\alpha:=F(\infty)\leqq 1$. *Then there exists an i. d. d. f.* $\hat{F}$ *such that* $\alpha\hat{F}=F$, $-\log\alpha=\lim_{x\to\infty}\limsup_{n\to\infty} R(n, x)=\lim_{x\to\infty}\liminf_{n\to\infty} R(n, x)$, where $R(n, x):=\Sigma_{k=1}^{k_n}\mathsf{P}(X_{nk}-m_{nk}\geqq x)$.

4.3. The law of large numbers. The compactness theorems in 4.2 are intimately connected with 0—1-laws. This permits us to formulate new versions of the weak law of large numbers, see [23].

First we assume

(4.3.1) $$\liminf_{n\to\infty}\mathsf{P}(|\Sigma_n|<\varepsilon)\geqq\eta>0 \quad \text{for any} \quad \varepsilon>0.$$

Which conditions are necessary and sufficient for the weak law of large numbers if we have (4.3.1)?

Theorem 4.3.1. *Let us assume* (4.3.1). *Then* $\liminf_{n\to\infty}\mathsf{P}(|\Sigma_n|<\varepsilon)=1$ *for any* $\varepsilon>0$ *if, and only if, we have*

(4.3.2) $$\sup_n \sum_{k=1}^{k_n}\mathsf{P}(|X_{nk}-m_{nk}|\geqq a_n x)=o(1), \qquad x\to\infty,$$

for a certain sequence $a_n\to 0$.

As a consequence of theorem 4.3.1 we obtain

Theorem 4.3.2. *Assume* $G_n\to_w F\neq$ *const. and* (4.3.2). *Then either* F *is a unit-step function with* $F(x_0+0)-F(x_0)=1$ *for a certain constant* x_0 *or* F *is a continuous non-defective i. d. d. f.*

4.4. Limit theorems under the assumption of restricted convergence. In 4.1 we mentioned two cases in which continuation theorems for i. d. d. f. $F_{\mathfrak{S}}$ are known. Accordingly we now give limit theorems assuming restricted convergence on a half-axis and on a set with a finite limit point, resp. It is recommendable to compare our theorems with the well-known limit theorems stated in the classical theory, see [3, § 25.].

Theorem 4.4.1. *Assume* $\mathfrak{S}=(-\infty, \tau)$ *and* $F\in\mathfrak{G}_\tau$. *In order that* $G_n\to_c F$ *it is necessary and sufficient that the conditions* (4.1.2) *and* (4.2.1) *are satisfied.*

This theorem is an easy consequence of the compactness condition (4.2.1) and the definition of the class $\mathfrak{G}_\tau$.

It is useful to compare it with lemma 4.2.1. In view of (4.1.2) we can drop (4.2.4) and instead of (4.2.3) we need only the weaker condition (4.2.1), see the remark to lemma 4.2.1. In the following we denote by $\mathfrak{S}$ a denumerable and bounded set.

Theorem 4.4.2. *Let* F *be an i. d. d. f. with a Gaussian component* $\sigma^2>0$. *Then* $G_n\to_c F$ *if, and only if, we have* (4.1.2), (4.2.1), *and*

(4.4.1) $$\lim_{\varepsilon\to 0}\liminf_{n\to\infty}\sum_{k=1}^{k_n}\int_{-\varepsilon}^{\varepsilon}x^2 dF_{nk}(x+m_{nk})>0.$$

Note, that (4.4.1) is weaker than (4.2.4), see [3, p. 124]. The next lemma gives a sufficient condition for (4.4.1) which permits an interpretation.

Lemma 4.4.1. *Assume that for any sufficiently small* $\varepsilon>0$ *there exists a sequence* $\{\delta_n(\varepsilon)\}$ *with* $0\leq\delta_n(\varepsilon)\leq\varepsilon$ *and*

$$\limsup_{\varepsilon\to 0}\ \liminf_{n\to\infty}\ \delta_n^2(\varepsilon)\sum_{k=1}^{k_n}\mathsf{P}(\delta_n(\varepsilon)\leq|X_{nk}-m_{nk}|\leq\varepsilon)>0. \tag{4.4.2}$$

Then (4.4.1) *is satisfied.*

Clearly, condition (4.4.2) means that the probability masses of the r. v. X_{nk} are not concentrated too closely to their medians.

Theorem 4.4.3. *Assume* (4.2.1) *and* $G_n\to_{\mathrm{w}}\Psi\neq$ *const.* $(x\in\mathfrak{S})$. *If there exists a real number* η *with* $0<\eta<1$ *and*

$$\liminf_{\varepsilon\to 0}\ \liminf_{n\to\infty}\varepsilon\sum_{k=1}^{k_n}\mathsf{P}(\eta\,.\,\varepsilon\leq|X_{nk}-m_{nk}|\leq\varepsilon)>0, \tag{4.4.3}$$

then we have $G_n\to_{\mathrm{c}}F$ *for a certain i. d. d. f.*

The basic idea of the proof of the theorems 4.4.2 and 4.4.3 is the fact that from (4.4.1) (or (4.4.2)) and (4.4.3), resp., it follows that every limit function of a subsequence $G_{n'}$, is an analytic d. f.

4.5. The general CLT. The general CLT for the triangular array (see [3, § 26]) gives conditions (in terms of F_{nk}) necessary and sufficient that we have

$$G_n\to_{\mathrm{c}}\Phi,\qquad n\to\infty, \tag{4.5.1}$$

see e. g. lemma 4.2.1. The characterization theorem 2.3.3 for Φ makes it possible to derive this convergence from quite different assumptions, where $\{x_i\}_1^\infty$, $\{y_i\}_1^\infty$ have the same meaning as in theorem 2.3.3, see 20. We use the notation (4.2.5).

Theorem 4.5.1. *Let the r. v.* X_{nk} *be i. s. and assume (as* $i\to\infty$*)*

$$(1+o(1))\Phi(x_i)\leq\underline{G}(x_i), \tag{4.5.2}$$

$$\overline{G}(y_i)=o(\Phi(y_i)e^{c|y_i|})\qquad\text{for all}\qquad c>0. \tag{4.5.3}$$

Then (4.5.1) *follows.*

For all the other limit theorems mentioned in the present paper relative compactness of $\{F_n\}$ and $\{G_n\}$, resp., had to be proved separately. But in this case it was possible to obtain both results in a unique set-up. It is theorem 4.2.4 which permits to draw from (4.5.2) and (4.5.3) the conclusion (as $i\to\infty$)

$$a^{-1}(1+o(1))\Phi(x_i)\leq\widehat{F}(x_i),\quad \widehat{F}(y_i)=o(\Phi(y_i)\,e^{c|y_i|})\quad\text{for all}\quad c>0,$$

where $\widehat{F}$ is i. d. Hence theorem 2.3.3 yields $\widehat{F}\equiv\Phi$ whence $a=1$ follows.

4.6. The Lindeberg—Feller theorem. The general assumption adopted in this section is the existence of the variances $\operatorname{Var}X_{nk}=\sigma^2_{nk}$, $k=1,\ldots,k_n$ satisfying

$$\operatorname{Var}\Sigma_n=\sum_{k=1}^{k_n}\sigma^2_{nk}=1,\qquad n=1,2,\ldots \tag{4.6.1}$$

Without loss of generality we may assume for the mathematical expectations $\mathsf{E}X_{nk}=0$; this amounts only to an appropriate choice of $\{A_n\}$. Condition (4.1.1) will be briefly denoted by (I). We introduce the function

$$L_n(\varepsilon):=\sum_{k=1}^{k_n}\int_{|x|\geqq\varepsilon}x^2dF_{nk}(x)$$

and remind the reader of the Lindeberg condition

(L) $$\lim_{n\to\infty}L_n(\varepsilon)=0$$

for all $\varepsilon>0$, which implies that the individual variances σ_{nk}^2 uniformly tend to zero, namely

(4.6.2) $$\max_{1\leqq k\leqq k_n}\sigma_{nk}^2\to 0,\qquad n\to\infty.$$

We also introduce the "tail condition"

(T) $$\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{|x|\geqq\varepsilon}dF_{nk}(x)=0.$$

t is easily seen that

$$L_n(\varepsilon)\geqq\varepsilon^2\sum_{k=1}^{k_n}\int_{|x|\geqq\varepsilon}dF_{nk}(x)\geqq\varepsilon^2\mathsf{P}(\max_{1\leqq k\leqq k_n}|X_{nk}|\geqq\varepsilon)$$
$$\geqq\varepsilon^2\max_{1\leqq k\leqq k_n}\mathsf{P}(|X_{nk}|\geqq\varepsilon);$$

hence (L) implies (T) and (I) follows from (T).

For the interpretation of (L) see also [2, § XV. 6].

We now quote from [3, § 21] the

Lindeberg — Feller theorem. *Under the condition* (4.6.1) (L) *is necessary and sufficient for* (I) *and the convergence*

(C) $$G_n\to_c\Phi,\qquad n\to\infty.$$

In another version of this theorem (I) is replaced by (4.6.2), see [2].

Note that condition (4.6.1) is trivially satisfied if we consider a sequence of independent r. v. $X_k\sim F_k$ with variances $\sigma_k^2=\operatorname{Var}X_k$, $k=1,2,\ldots$; we need only to put

$$B_n^2:=\sum_{k=1}^{k_n}\sigma_{nk}^2,\qquad X_{nk}:=X_k/B_n,\qquad 1\leqq k\leqq k_n=n.$$

This theorem gives the impression that (L) is by far a stronger condition than (I), since we have to add (C) to obtain (L) from (I). But from theorem 4.5.1 it is easily seen that this impression is not quite correct; we need only impose the additional assumption (4.6.1) to obtain

Theorem 4.6.1. *Assume* (4.6.1). *Then* (I), (4.5.2) *and* (4.5.3) *are necessary and sufficient for* (L).

For the following two reformulations of the Lindeberg — Feller theorem (see [10]) we again have to treat a compactness problem and a continuation problem separately.

I. *Compactness.* This affair can be easily settled by means of the very simple

L e m m a 4.6.1. *Let* $\mathfrak{F}$ *stand for a set of d. f. F with finite variances* σ_F^2. *If* $\sup_{F\in\mathfrak{F}} \sigma_F^2 < \infty$ *and if the set of all limit functions for sequences* $\{F_n\} \in \mathfrak{F}$ *does not contain the constants* 0 *and* 1, *then* $\mathfrak{F}$ *is relatively compact.*

R e m a r k. Note that the set $\mathfrak{F} = \{\Phi(x+a), \ -\infty < a < \infty\}$ is not relatively compact, but 0 and 1 are limit functions.

Accordingly it is condition (4.6.1) that implies relative compactness provided that convergence to 0 and 1 can be excluded. This is trivially the case in theorems 4.6.2 and 4.6.3.

II. *Continuation.* a) It is a very striking feature of the following result that convergence only in two points is needed. The essential tool (compare [3, § 25 and lemma 4.6.1]) is

L e m m a 4.6.2. *From* (T) *and* (4.6.1) *it follows that all limit functions for* $\{G_n\}$ — *but for the constants* 0 *and* 1 — *are normal.*

But two normal d. f. coinciding on two different points are identical. Hence we immediately obtain

T h e o r e m 4.6.2. *Assume* (4.6.1) *and put* $\mathfrak{S} = \{x_1, x_2\}$, $x_1 \neq x_2$. *Then* (C) *and* (I) *are necessary and sufficient for* (*T*) *and*

$$(C') \qquad G_n(x) \to_{\mathrm{w}} \Phi(x), \qquad x \in \mathfrak{S}.$$

Now the Lindeberg — Feller theorem yields

C o r o l l a r y 4.6.1. *Under* (4.6.1) *the condition* (L) *is equivalent to* (T) *and* (C′).

b) Now we assume that $\mathfrak{S}$ is a set with a finite limit point and adopt conditions guaranteeing that all possible limit d. f. are analytic. For this purpose we introduce the condition

$$(L') \qquad \lim_{\varepsilon\to\infty} \limsup_{n\to\infty} L_n(\varepsilon) < 1$$

being equivalent to (4.4.1) (if (4.6.1) is true).

The key lemma for the case under consideration is

L e m m a 4.6.3. *Assume that* (4.6.1), (I), *and* (L′) *are satisfied.* If $G_{n'} \to_0 G$ *for some subsequence* $\{n'\}$ *of natural numbers then G is i. d. and has a non-degenerated Gaussian component.*

Note that all these i. d. d. f. *G* are analytic so that they are uniquely defined under the assumption of the following

T h e o r e m 4.6.3. *Let* $\mathfrak{S}$ *be a set with a finite limit point. Assume* (4.6.1) *and* (I). *Then* (C) *holds true if, and only if,* (C′) *and* (L′) *are satisfied.*

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