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ANALYTIC CONTINUATION OF POWER SERIES WITH MEROMORPHIC COEFFICIENTS

W. GAWRONSKI*, R. TRAUTNER

The paper extends the results of the authors published in this journal in 1976. Theorems are proved on the analytic continuation of power series with meromorphic coefficients, behaving locally as a Stieltjes integral transform.

1. Introduction. Extended investigations on the field of analytic continuation have been devoted to power series of the type

$$f(z)=\sum_{n=0}^{\infty}A(n)z^n,\quad \limsup_{n\to\infty}|A(n)|^{1/n}=1, \tag{1}$$

where A is holomorphic in the whole plane $\mathbb{C}$ (e. g. [2] and the references given there) or in an angular region $S_{\alpha,\beta,\varrho}:=\{z\mid -\beta\leqq\arg z\leqq\alpha,\ |z|\geqq\varrho\}$, $0<\alpha$, $\beta<\pi/2$, $\varrho\geqq0$ [1; 2; 5; 6; 9]. Then lower estimates for the domain of analyticity of f depend on growth properties of the interpolation function A in $\mathbb{C}$ or in $S_{\alpha,\beta,\varrho}$ and on α, β. In this paper we are concerned with power series of the type (1), but we admit A to have singularities on the real axis. More precisely, we deal with coefficient functions A behaving locally as a Stieltjes transform, that is

$$A(z)=\int_0^R\frac{d\chi(t)}{z-t}+A_R(z) \tag{2}$$

for every $R>0$, where A_R is analytic in $S_{\alpha,\beta,\varrho}\cap\{|z|\leqq R\}$ and $\int_{n-d_n}^{n+d_n}|d\chi(t)|=0$ for certain $d_n>0$, $n\in\mathbb{N}_0$. We derive a general representation formula for f, from which under certain growth conditions on A we may conclude the analytic continuation of f (theorem 1). The methods are related to and extend those used in [5]. This formula will be fundamental for our further investigations. Its main applications are concerned with power series (1), where A is meromorphic in $S_{\alpha,\beta,\varrho}$.

First, in case when A is meromorphic throughout $S_{\alpha,\beta,\varrho}$ with simple poles on the real axis only we give a general theorem on the singularities of (1) located on $|z|=1$ thereby proving a lower estimate for the length of a connected arc on $|z|=1$ containing at least one singularity (theorem 3). This result essentially is based on density properties of the poles of A.

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In section 4 we give a further important consequence of our basic representation formula. If $\{a_k\}_0^\infty \in l_p$, $1 \leq p < \infty$, and $A(n) = \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{a_k}{n-k}$, $n \in \mathbb{N}_0$, is its Hilbert transform, then we prove that the power series $\Sigma_0^\infty a_k z^k$ and $\Sigma_1^\infty A(n) z^n$ possess the same domain of analyticity in $\mathbb{C}_1^*$ (theorem 5).

For $a \in \mathbb{R}$ we denote throughout $\mathbb{C}_a^* = \{z \in \mathbb{C} \mid \text{if } \operatorname{Re} z \geq a, \text{ then } \operatorname{Im} z \neq 0\}$.

2. The representation formula. Suppose that for every $R \geq \varrho \geq 0$ the interpolation function has the form

$$A(z) = \int_0^R \frac{d\chi(t)}{z-t} + A_R(z) \tag{2}$$

whenever $z \in S_{\alpha,\beta,\varrho} \setminus \operatorname{supp} \chi$, where $\int_0^R |d\chi(t)| < \infty$. Let

$$\operatorname{supp} \chi = \{t \in \mathbb{R} \mid \int_{t-\varepsilon}^{t+\varepsilon} |d\chi(s)| > 0 \quad \text{for every} \quad \varepsilon > 0\}$$

denote the support of χ and

$$A_R(z) \text{ is analytic for } z \in S_{\alpha,\beta,\varrho} \cap \{|z| \leq R\}. \tag{3}$$

Furthermore, we require the following conditions on χ to be satisfied:

$$|\chi(t)| \leq e^{\delta t} \text{ for some } \delta \in \mathbb{R} \text{ and sufficiently large } t \tag{4}$$

and if

$$d_n := \inf\{|t-n| \mid t \in \operatorname{supp} \chi\} > 0 \tag{5}$$

denotes the distance between $n \in \mathbb{N}_0$ and $\operatorname{supp} \chi$, then

$$\liminf_{n \to \infty} d_n^{1/n} =: d > 0 \tag{6}$$

(the latter condition means that the mass function χ is not distributed "too close" to the integers).

Furthermore we assume that there exists a sequence $\{R_k\}$ of positive numbers tending to infinity such that

$$A(z) \text{ is holomorphic at } z = R_k \tag{7}$$

$$|A(R_k e^{i\varphi})| \leq e^{M \cdot R_k}, \quad -\beta \leq \varphi \leq \alpha \tag{8}$$

and

$$D_k := \inf\{|R_k - n|, \quad n \in \mathbb{N}\} > 0, \tag{9}$$

$$D := \liminf_{k \to \infty} D_k^{1/R_k} > 0. \tag{10}$$

Finally, if

$$h_A(\varphi) = \limsup_{r \to \infty} \frac{\log |A(re^{i\varphi})|}{r} \quad -\beta \leq \varphi \leq \alpha, \quad \varphi \neq 0$$

denotes the growth indicator of A in $S_{\alpha,\beta,\varrho} - \mathbb{R}$ we assume that

(11) $$h_A(-\beta) \quad \text{and} \quad h_A(\alpha) \text{ are finite.}$$

Now we basicly follow the proof of theorem 2 in [5].

Our result below will be a generalization of Cowling's theorem. Without loss of generality we may assume that $\varrho \notin \operatorname{supp}\chi \cap \mathbb{N}_0$. Denote

$$C_k := \{\zeta \mid \arg\zeta = -\beta,\ \varrho \leq |\zeta| \leq R_k\} \cup \{\zeta \mid |\zeta| = R_k,\ -\beta \leq \arg\zeta \leq \alpha\}$$
$$\cup \{\zeta \mid \arg\zeta = \alpha,\ R_k \geq |\zeta| \geq \varrho\} \cup \{\zeta \mid |\zeta| = \varrho,\ \alpha \geq \arg\zeta \geq -\beta\}.$$

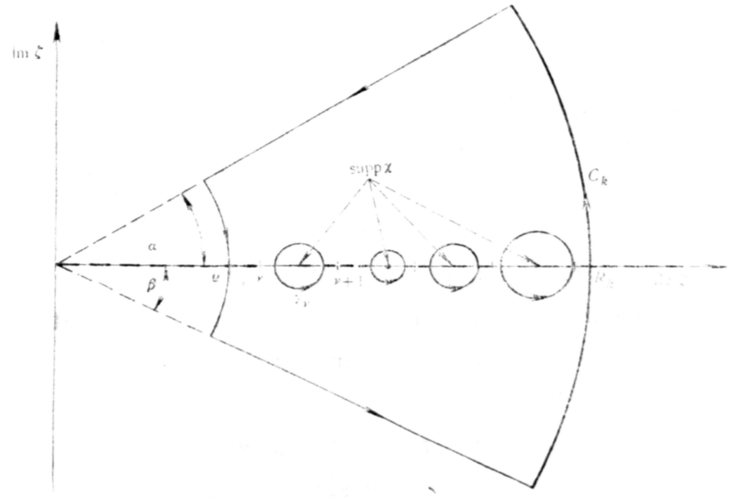

Fig. 1

By (5), it is possible to choose disjoint Jordan curves γ_ν such that the interior of γ_ν contains $\operatorname{supp}\chi \cap (\nu, \nu+1)$. The Jordan curves C_k, γ_ν are assumed to have positive orientation. Then an application of residue calculus yields

(12) $$\int_{C_k} \frac{A(\zeta)e^{\zeta \log z}}{e^{2\pi i\zeta}-1}\,d\zeta = \sum_{\varrho<n<R_k} A(n)z^n + \sum_{\varrho<\nu<R_k} \int_{\gamma_\nu} \Big\{\Big[\int_0^{R_k} \frac{d\chi(t)}{\zeta - t}\Big]\Big/(e^{2\pi i\zeta}-1)\Big\} e^{\zeta \log z}\,d\zeta.$$

Clearly the integration along C_k is possible by (7) and (9). Further $\log z = \log|z| + i\arg z$, $0 \leq \arg z < 2\pi$.

In the last term of (12) the interchange of integration is permitted by continuity arguments (see e. g. [10], p. 25, theorem 15a) and by (5). Again by residue calculus we obtain

$$\int_{C_k} \frac{A(\zeta)e^{\zeta \log z}}{e^{2\pi i\zeta}-1}\,d\zeta = \sum_{\varrho<n<R_k} A(n)z^n + 2\pi i \int_0^{R_k} \frac{e^{t\log z}}{e^{2\pi i t}-1}\,d\chi(t).$$

Now making $k \to \infty$ we get for small $|z|$

(13) $$f(z)=Q(z)+\int_C \frac{A(\zeta)e^{\zeta \log z}}{e^{2\pi i\zeta}-1}\,d\zeta-2\pi i\int_0^\infty \frac{e^{t\log z}}{e^{2\pi it}-1}\,d\chi(t),$$

where $Q(z)=\Sigma_{0\leqq n<p}A(n)z^n$ and

(14) $$C:=\{\zeta \mid \arg\zeta=\alpha,\ \infty>|\zeta|\geqq\varrho\}\cup\{\zeta \mid |\zeta|=\varrho,\ \alpha\geqq\arg\zeta\geqq-\beta\}$$
$$\cup\{\zeta \mid \arg\zeta=-\beta,\ \varrho\leqq|\zeta|<\infty\}.$$

To justify (13) we have to show that ($\zeta=R_ke^{i\varphi}$, $z=re^{i\psi}$)

$$I_k:=\int_{-\beta}^{\alpha}\frac{A(R_ke^{i\varphi})\exp(R_ke^{i\varphi}\log z)}{\exp(2\pi iR_ke^{i\varphi})-1}R_kie^{i\varphi}d\varphi\to 0$$

and that the integrals on the right hand side of (13) exist. First, by (9), (10) we have that $|\exp(2\pi iR_ke^{i\varphi})-1|^{-1}\leqq e^{k_1R_k}$, $k_1>0$, $k\in\mathbb{N}$, $-\beta\leqq\varphi\leqq\alpha$. Using (8) we obtain as $k\to\infty$

$$|I_k|\leqq R_ke^{(k_1+M)R_k}\int_{-\beta}^{\alpha}\exp[R_k(\cos\varphi\log r-\psi\sin\varphi)]\,d\varphi\to 0,$$

if $\log r<-[k_1+M+1+2\pi]/\min(\cos\alpha,\ \cos\beta)$.

Next, we consider the integral

(15) $$f_1(z):=\int_C\frac{e^{\zeta\log z}}{e^{2\pi i\zeta}-1}\,d\zeta.$$

Cowling [5] shows that, under (11), it converges and represents an analytic function in the domain bounded by the exponential spirals

$$\{z=re^{i\psi} \mid r<\exp(\psi\tan\alpha-h_A(\alpha)/\cos\alpha),\qquad 0<\psi<2\pi\},$$
$$\{z=re^{i\psi}\}\ r<\exp((2\pi-\psi)\tan\beta-h_A(-\beta)/\cos\beta),\qquad 0<\psi<2\pi\}.$$

Finally we investigate the Laplace-Stieltjes-integral in (13)

(16) $$f_2(z):=\int_0^\infty\frac{e^{t\log z}}{e^{2\pi it}-1}\,d\chi(t).$$

Suppose that $t\in[n-1/2,\ n+1/2]\cap\operatorname{supp}\chi$, $n\in\mathbb{N}$. Then we have by (5) and (6), $(0<\varepsilon<d\leqq 1,\ n\geqq n_0(\varepsilon))$

$$|e^{2\pi it}-1|^{-1}\leqq|t-n|^{-1}\leqq 1/d_n\leqq(d-\varepsilon)^{-n}<\exp[-(t+1/2)\log(d-\varepsilon)]$$

and by (4)

$$\int_0^t\frac{d\chi(\tau)}{e^{2\pi i\tau}-1}=\frac{\chi(t)}{e^{2\pi it}-1}+2\pi i\int_0^t\frac{e^{2\pi i\tau}\chi(\tau)}{(e^{2\pi i\tau}-1)^2}\,d\tau$$
$$=O(\exp[\delta t-t.\log(d-\varepsilon)]+t\exp[\delta t-2t\log(d-\varepsilon)])$$
$$=O(t\exp[t(\delta-2\log(d-\varepsilon))]),\quad \text{as}\quad t\to\infty,$$

where we have assumed that $\chi(0)=0$ w. l. o. g.

This proves, since $\varepsilon>0$ was arbitrary, that (16) converges absolutely if

$\log|z|<-\delta+\log d^2$ and represents an analytic function in the cut disc $\{z \mid |z|<d^2e^{-\delta},\ 0\leq\arg z<2\pi\}$.

In contrary to the case when $A(z)$ is holomorphic in $S_{\alpha,\beta,\varrho}$ [5; 6] the function defined in (15) has a cut along the positive real axis. This is due to the definition of $\log z$ and the fact that the right hand side of (13) contains a sum of two functions of $\log z$.

Summarizing the results above we obtain

Theorem 1. *Given the power series*

$$(1)\qquad f(z)=\sum_{n=0}^{\infty}A(n)z^n,\quad \limsup_{n\to\infty}|A(n)|^{1/n}=1.$$

Suppose that for every $R>0$ the interpolation function A has the representation

$$(2)\qquad A(z)=\int_0^R\frac{d\chi(t)}{z-t}+A_R(z),\quad z\in S_{\alpha,\beta,\varrho}\setminus\operatorname{supp}\chi,$$

where $\int_0^R|d\chi(t)|<\infty$, $\chi(0)=0$, and that

$$(3)\qquad A_R \text{ is analytic in } S_{\alpha,\beta,\varrho}\cap\{z \mid |z|\leq R\}\quad (0<\alpha,\ \beta<\pi/2,\ 0\leq\varrho).$$

Assume further that conditions (4)—(11) *are satisfied and that C is defined by* (14). *Then the functions*

$$(15)\qquad f_1(z)=\int_C\frac{A(\zeta)e^{\zeta\log z}}{e^{2\pi i\zeta}-1}\,d\zeta,$$

$$(16)\qquad f_2(z)=\int_0^{\infty}\frac{e^{t\log z}}{e^{2\pi it}-1}\,d\chi(t)$$

define holomorphic functions in the domains

$$G_1^{\alpha,\beta}:=\{z=re^{i\psi} \mid r<\exp(\psi\tan\alpha-h_A(\alpha)/\cos\alpha,$$

$$r<\exp((2\pi-\psi)\tan\beta-h_A(-\beta)/\cos\beta,\quad 0<\psi<2\pi\},$$

$$G_2=\{z=re^{i\psi} \mid r<d^2e^{-\delta},\quad 0<\psi<2\pi\},$$

respectively ($\log z=\log r+i\psi$, $0\leq\psi<2\pi$). *Moreover the representation*

$$(13)\qquad f(z)=Q(z)+\int_C\frac{A(\zeta)}{e^{2\pi i\zeta}-1}e^{\zeta\log z}d\zeta-2\pi i\int_0^{\infty}\frac{e^{t\log z}}{e^{2\pi it}-1}\,d\chi(t),$$

Q being a polynomial, holds in $\{|z|<1\}\cap G_1^{\alpha,\beta}\cap G_2$.

Clearly the right hand side can give the analytic extension of $f(z)$. To this end it remains to replace G_2 by a larger domain, since in general $G_1^{\alpha,\beta}$ is already a nontrivial estimate for the domain of analyticity of f_1. We first state two important consequences of theorem 1 useful for further applications

Theorem 2. *Suppose that the assumptions of theorem* 1 *are satisfied.*

1) *If $h_A(\alpha)\leq 0$, $h_A(-\beta)\leq 0$, then $f(z)$ and $f_2(z)=\int_0^\infty[e^{t\log z}/(e^{2\pi it}-1)]d\chi(t)$ have the same singularities on $\{|z|=1\}$ except for $z=1$ possibly.*

2) *If* $h_A(\varphi)\leqq 0$ *for* $\pi/2-\varepsilon<|\varphi|<\pi/2$, $(0<\varepsilon<\pi/2)$, *then* $f(z)$ *and* $f_2(z)$ *have the same singularities in* $\mathbf{C}_0^*$.

Proof. 1) $h_A(\alpha)\leqq 0$, $h_A(-\beta)\leqq 0$ imply $G_1^{\alpha,\beta}\supseteq\{|z|\leqq 1\}\setminus\{z \mid \arg z=0\}$.

2) Theorem 1 may be applied for any $\pi/2-\varepsilon<\alpha$, $\beta<\pi/2$ and the statement follows from $\lim_{\alpha\to\pi/2,\ \beta\to\pi/2}G_1^{\alpha,\beta}=\mathbf{C}_0^*$.

3. Meromorphic interpolation functions. In this section we assume that the interpolation function A is meromorphic in $S_{\alpha,\beta,\varrho}$ with simple poles on the real axis only. The first result illustrates the influence of the location of the poles of A on the distribution of the singularities of f on $|z|=1$.

Theorem 3. *Suppose that A has the form*

$$A(z)=B(z)/P(z) \tag{17}$$

with

$$B(z) \text{ is holomorphic in } S_{\alpha,\beta,\varrho},\quad 0<\alpha,\ \beta<\pi/2,\ \varrho\geqq 0; \tag{18}$$

$$P(z)=\prod_1^\infty\left(1-\frac{z^2}{\lambda_k^2}\right),\qquad \lambda_k>0. \tag{19}$$

Further we assume the following conditions to be satisfied

$$d_n:=\inf\{|\lambda_k-n|,\quad k\in\mathbf{N}\}>0,\quad n\in\mathbf{N}_0, \tag{20}$$

$$\liminf_{n\to\infty} d_n^{1/n}=1, \tag{21}$$

$$\lim_{k\to\infty}\frac{k}{\lambda_k}=D, \tag{22}$$

$$\lambda_{k+1}-\lambda_k\geqq c>0,\quad k\in\mathbf{N}, \tag{23}$$

$$h_B(\varphi)\leqq\pi D|\sin\varphi|\quad -\beta\leqq\varphi\leqq\alpha, \tag{24}$$

$$\limsup_{n\to\infty}|B(n)|^{1/n}=1. \tag{25}$$

Then the power series

$$f(z)=\sum_{n=0}^\infty A(n)z^n \tag{1}$$

defines a holomorphic function in the unit disc. Moreover, if

$$\sigma:=\limsup_{k\to\infty}[\log|B(\lambda_k)|]/\lambda_k, \tag{26}$$

then the following statements hold:

a) *If* $\sigma<0$, *then* $z=1$ *is the only singularity of* $f(z)$ *on* $|z|=1$.

b) *If* $\sigma=0$, *then there exists at least one singularity on every subarc of* $|z|=1$ *with length exceeding* $2\pi\min(D, 1)$.

Remarks. 1. Since $d_n\leqq n$ for sufficiently large n, (21) is equivalent to $\lim_{n\to\infty}d_n^{1/n}=1$, that is $d=1$ in (6).

2. From (23) we know that $h:=\liminf_{k\to\infty}(\lambda_{k+1}-\lambda_k)\geqq c$ and thus, by standard theory [8, p. 1], $D\leqq h^{-1}\leqq c^{-1}$.

3. Clearly in the case $D\geqq 1$ assertion b) shall be interpreted as the trivial statement that there is at least one singularity on $|z|=1$.

4. Condition (24) implies that $\sigma \leqq 0$. If (24) can be replaced by the sharper estimate

$$h_B(\varphi) \leqq L\,|\sin\varphi|, \quad -\beta \leqq \varphi \leqq \alpha, \quad L < \pi D, \tag{24'}$$

then by a known density theorem [7, p. 100, theorem XXXII] we have

$$\sigma = \limsup_{k\to\infty} [\log|B(\lambda_k)|]/\lambda_k = \limsup_{r\to\infty} [\log|B(r)|]/r.$$

Further, if $D \leqq 1$, then by the same theorem and (25)

$$0 = \limsup_{n\to\infty} [\log|B(n)|]/n = \limsup_{r\to\infty} [\log|B(r)|]/r.$$

Hence (24′) together with $D \leqq 1$ is sufficient for $\sigma = 0$. The following less trivial example (of course we can choose $B(\lambda_k) = 0$, $k \in \mathbb{N}$, if (24′) does not hold) shows that $\sigma < 0$ may occur when (24′) does not hold.

Define $B(z) := \sin \pi(z + 1/2)/2$ and $\lambda_k = 2k - 1/2 + e^{-k}$, $k \in \mathbb{N}$. Then $D = 1/2$, $h_B(\varphi) = \pi|\sin\varphi|/2$ and conditions (20)—(25) are easily verified. Since

$$B(\lambda_k) = (-1)^k \sin\left(\frac{\pi}{2} e^{-k}\right) = (-1)^k \left(\frac{\pi}{2} e^{-k}\right)(1 + o(1)),$$

it follows that $\sigma = \lim_{k\to\infty} [\log|B(\lambda_k)|]/\lambda_k = -1/2$.

5. If $D = \sigma = 0$, then $|z| = 1$ is the natural boundary for f.

Examples. 1) For the power series $\sum_{n=0}^{\infty} (e^{\sqrt{n}}/\cos \pi\sqrt{n}) z^n$ and $\sum_{n=0}^{\infty} (\tan \pi\sqrt{n}) z^n$ the unit circle is the natural boundary, since $\lambda_k = k^2 - k + 1/4$ and $D = \sigma = 0$.

$$\text{(Choose } P(z) = \prod_1^\infty \left(1 - \frac{z^2}{\lambda_k^2}\right) = \prod_1^\infty \left(1 - \frac{z}{(k-1/2)^2}\right) \prod_1^\infty \left(1 + \frac{z}{(k-1/2)^2}\right)$$

$$= \cos \pi\sqrt{z} \cos h\pi\sqrt{z}).$$

In a similar way the more general power series $\sum_0^\infty (\tan \pi n^\alpha) z^n$, $0 < \alpha < 1$, can be treated.

2) Let $f(z) := \sum_1^\infty z^n / \sin \pi(\theta n - \alpha)$, where $\theta, \alpha \in \mathbb{R}$ are rationally independent and $\theta \in (0, 1]$. If α is fixed, then for almost all θ (in the sense of Lebèsgue measure) theorem 2 applies to f for we have

$$\lambda_k = (k + \alpha)/\theta, \quad d_n = \inf_{k\in\mathbb{N}} \left|\frac{k+\alpha}{\theta} - n\right| = \frac{1}{\theta} \|\theta n - \alpha\|,$$

where $\|x\|$ denotes the distance of the real number x to the closest integer. By Khinchin's metric theorem on Diophantine approximation [4, p.121] we have $\|\theta n - \alpha\| \geqq 1/n^2$, if $n \geqq n_0(\theta)$ for almost all θ and thus (21) holds. Hence, by theorem 3, for almost all θ $f(z)$ defines a holomorphic function in $|z| < 1$ and possesses at least one singularity on every connected subarc of $|z| = 1$ the length of which is exceeding $2\pi\theta$ (note that $\sigma = 0$). Actually, if $\alpha = 0$ and $\theta \notin \mathbb{Q}$, then it is known by a different technique [3, p. 71] that the unit circle is the natural boundary (for almost all θ) for this particular series.

In order to prove theorem 3 we need the following

Lemma. *Let $P(z)$ be defined by* (19) *where* $\{\lambda_k\}$ *satisfies* (20)—(23). *Then as* $r \to \infty$ *and for every* $\varepsilon > 0$

(27) $$P(re^{i\varphi}) = O(\exp(\pi D r |\sin\varphi| + \varepsilon r)),$$

(28) $$P'(re^{i\varphi}) = O(\exp(\pi D r |\sin\varphi| + \varepsilon r)),$$

(29) $$[P(re^{i\varphi})]^{-1} = O(\exp(\pi D r |\sin\varphi| + \varepsilon r)) \quad \text{if} \quad |re^{i\varphi} - \lambda_k| \geqq c/4$$

and as $n \to \infty$

(30) $$[P'(\lambda_n)]^{-1} = O(e^{\varepsilon\lambda_n}),$$

(31) $$[P(n)]^{-1} = O(e^{\varepsilon n}).$$

Proof. Assertions (27), (29) and (30) are given in [7, p. 89, theorem XXX] and (28) follows from (27). To show (31) again we use the proof of theorem XXXI in [7, p. 98, (22.29)] to obtain $|P(n)/(n-\lambda_N)| \geqq e^{-\varepsilon n}$, where λ_N makes $|n-\lambda_k|$ a minimum [7, p. 93]. From (20) and (21) we get $1/|n-\lambda_N| \leqq 1/d_n \leqq 1/(1-\varepsilon)^n$ which proves (31).

Proof of theorem 3. Estimates (27) and (31) show $\lim_{n\to\infty} |P(n)|^{1/n} = 1$ and (25) implies that f has radius of convergence unity.

To establish the main part we apply theorem 1. Since $A(z) = B(z)/P(z)$ is meromorphic throughout $S_{\alpha,\beta,\varrho}$ with simple poles at λ_k only, it is of the type required in (2) where $\chi(t)$ reduces to a step function with jumps at λ_k given by the residues of $A(z)$

$$\operatorname*{res}_{z=\lambda_k} A(z) = B(\lambda_k)/P'(\lambda_k).$$

From (24) and (30) it follows that (4) is satisfied. Further from (24) and (29) we obtain

(32) $$h_A(\varphi) \leqq h_B(\varphi) - \pi D |\sin\varphi| \leqq 0, \quad -\beta \leqq \varphi \leqq \alpha, \quad \varphi \neq 0,$$

and moreover, $A(re^{i\varphi}) = O(e^{\varepsilon r})$, $-\beta \leqq \varphi \leqq \alpha$, $r \to \infty$, when $|re^{i\varphi} - \lambda_k| \geqq c/4$. This together with (23) ensures the existence of a sequence $\{R_k\}$ possessing properties (7)—(10). Clearly (20), (21) and (32) imply (5), (6) and (11) respectively. Now, by theorem 1, (13) has the form

(33) $$f(z) = Q(z) + \int_C \frac{A(\zeta)}{e^{2\pi i\zeta} - 1} e^{\zeta \cdot \log z} d\zeta - 2\pi i \sum_1^\infty \frac{B(\lambda_k)}{P'(\lambda_k)} \frac{e^{\lambda_k \cdot \log z}}{e^{2\pi i\lambda_k} - 1}.$$

From (32) we see that the integral on the right hand side defines a holomorphic function in the region

$$G_1^{\alpha,\beta} = \{z = re^{i\psi} \mid r < e^{\psi \tan\alpha}, \quad r < e^{(2\pi-\psi)\tan\beta}\}.$$

Since the sequence $\{k/\lambda_k\}$ is bounded, the abscissa of ordinary and absolute convergence of the Dirichlet series

(34) $$\sum_1^\infty \frac{B(\lambda_k)}{P'(\lambda_k)} \frac{e^{-s\lambda_k}}{e^{2\pi i\lambda_k} - 1}, \quad s = -\log z$$

coincide and may be computed by the formula

(35) $$\sigma_C = \limsup_{k\to\infty}\left(\log\left|\frac{B(\lambda_k)}{P'(\lambda_k)}\cdot\frac{1}{e^{2\pi i\lambda_k}-1}\right|/\lambda_k\right)$$

[8, p. 11, theorem I. 2.5]. From (20), (21) we get $\lim_{k\to\infty}(\log|e^{2\pi i\lambda_k}-1|)/\lambda_k=0$ and (28), (30) imply $\lim_{k\to\infty}(\log|P'(\lambda_k)|)/\lambda_k=0$. Substituting these results in (35) we obtain $\sigma=\sigma_C$, hereby proving that the series (34) represents an analytic function in the slit disc $\{z=re^{i\Psi}\,|\,r<e^{-\sigma},\ 0<\Psi<2\pi\}$. Now we apply theorem XXIX in [7, p. 89] to (34) which shows that on the line $\operatorname{Re} s=\sigma$ there is at least one singularity on every interval with length exceeding $2\pi D$ Since $z=1$ is the only singularity of the integral on the right hand side of (33), this identity and part 1 of theorem 2 complete the proof.

Remarks. 1) If $\sigma<0$, then actually the preceeding proof shows that f can be extended analytically onto $G_1^{\alpha,\beta}\cap\{|z|<e^{-\sigma}\}$.

2) Theorem 3 shows that it is useful to consider interpolation functions $A(z)$ being a "local" Stieltjes transform, because the sum of principal parts of a meromorphic function might not be convergent.

The next theorem is a further consequence of formula (13) stating a correspondence between two power series which "essentially" have the same domain of analyticity.

Theorem 4. *Given a strictly increasing sequence* $\{\lambda_k\}$, $\lambda_k\in\mathbb{N}$, *satisfying* (22) *and* $b\in\mathbb{R}\setminus\mathbb{Z}$ *such that* $\lambda_k+b>0$, $k\in\mathbb{N}$. *Moreover, assume that for all* $\varepsilon>0$ $B(z)$ *is an analytic function in* $S_{\frac{\pi}{2}-\varepsilon,\frac{\pi}{2}-\varepsilon,\varrho}$ *satisfying* (24) *and* (25) *for* $|\varphi|<\frac{\pi}{2}$. *If*

$$P_b(z):=\prod_1^\infty\left(1-\frac{z^2}{(\lambda_k+b)^2}\right),$$

then the power series $\sum_0^\infty\frac{B(n)}{P_b(n)}z^n$ *and* $\sum_1^\infty\frac{B(\lambda_k+b)}{P_b'(\lambda_k+b)}z^{\lambda_k}$ *both converge for* $|z|<1$ *and have the same domain of analyticity in* $\mathbb{C}_1^*$.

Proof. The assumptions of theorems 1 and 3 are easily verified. Hence (33) has the form

$$\sum_0^\infty\frac{B(n)}{P_b(n)}z^n=Q(z)+\int_C\frac{B(\zeta)}{P_b(\zeta)}\,\frac{e^{\zeta\log z}}{e^{2\pi i\zeta}-1}\,d\zeta-2\pi i\sum_1^\infty\frac{B(\lambda_k+b)}{P_b'(\lambda_k+b)}\,\frac{e^{(\lambda_k+b)\log z}}{e^{2\pi i(\lambda_k+b)}-1}$$

and the series on the right hand side reduces to

$$2\pi i\,\frac{e^{b\log z}}{e^{2\pi ib}-1}\sum_1^\infty\frac{B(\lambda_k+b)}{P_b'(\lambda_k+b)}z^{\lambda_k}.$$

Now, by (25) (26), theorem 2, 2) completes the proof.

4. Hilbert transforms. In this section we use the results of section 1 to compare the domain of analyticity of power series $\sum_{k=0}^\infty a_kz^k$, $\{a_k\}\in l_p$, $1\le p<\infty$ and $\sum_0^\infty A(n)z^n$, where $\{A(n)\}$ is the discrete Hilbert transform of $\{a_k\}$. We prove

Theorem 5. *If* $\{a_k\}_0^\infty\in l_p$, $1\le p<\infty$ *and* $A(n)=\sum_{\substack{k=0\\k\neq n}}^\infty a_k/(n-k)$, *then the* *power series $\Sigma_0^\infty a_k z^k$ and $\Sigma_0^\infty A(n)z^n$ converge in $|z|<1$ and have the same domain of analyticity in $\mathbb{C}_1^*$.*

Proof. For $\delta\in\mathbb{R}$ define $A_\delta(z):=\Sigma_{k=0}^\infty a_k/(z-k-\delta)$. Conditions (4)–(7), 9), (10) are directly verified for $0<|\delta|\leq 1/8$. Further by Hölder's inequality $1/p+1/q=1)$

$$|A_\delta(z)|\leq(\sum_0^\infty |a_k|^p)^{1/p}(\sum_0^\infty \frac{1}{|z-k-\delta|^q})^{1/q}$$

(the second sum has to be replaced by $\sup_{k\in\mathbb{N}_0}|z-k-\delta|^{-1}$, if $q=\infty$), which yields

(36) $A_\delta(z)$ is bounded and continuous on $\{(z,\delta)\,|\,\inf_{k\in\mathbb{N}_0}|z-k|\geq 1/4, |\delta|\leq 1/8\}$

with bounded modulus of continuity.
This implies (8) and that

$$\limsup_{r\to\infty}\frac{\log|A_\delta(re^{i\varphi})|}{r}=h_{A_\delta}(\varphi)\leq 0, \tag{37}$$

exists uniformly for $|\delta|\leq 1/8$, provided $\varphi\neq 0$. Hence, in particular (11) is satisfied for any $0<\alpha$, $\beta<\pi/2$. Applying now theorem 1 to $f_\delta(z):=\Sigma_1^\infty A_\delta(n)z^n$ we get

$$f_\delta(z)=\int_C\frac{A_\delta(\zeta)e^{\zeta\log z}}{e^{2\pi i\zeta}-1}d\zeta-2\pi i\sum_{k=0}^\infty\frac{a_k}{e^{2\pi i\delta}-1}e^{(k+\delta)\log z}$$
$$=:B_\delta(z)-2\pi i\frac{e^{\delta\log z}}{e^{2\pi i\delta}-1}\sum_{k=0}^\infty a_k z^k,$$

where C is chosen as in theorem 1 with $\varrho=1/2$, and $0<\alpha$, $\beta<\pi/2$ arbitrarily. So we get

$$\frac{1}{2}(f_\delta(z)+f_{-\delta}(z))=\frac{1}{2}(B_\delta(z)+B_{-\delta}(z))-\pi i\left(\frac{e^{\delta\log z}}{e^{2\pi i\delta}-1}+\frac{e^{-\delta\log z}}{e^{-2\pi i\delta}-1}\right)\sum_{k=0}^\infty a_k z^k.$$

Letting now δ tend to zero we obtain

$$\lim_{\delta\to 0}\frac{1}{2}(B_\delta(z)+B_{-\delta}(z))=\int_C\frac{A_0(\zeta)e^{\zeta\log z}}{e^{2\pi i\zeta}-1}d\zeta:=B(z)$$

the interchange of lim and $\int$ being justified by (36).

From (37) and the arguments used in the proof of theorem 2, part 2, we get

(38) $B(z)$ is analytic in $\mathbb{C}_0^*$.

Furthermore, we have uniformly with respect to z in any compact subset in $\mathbb{C}_0^*$

$$\frac{e^{\delta\log z}}{e^{2\pi i\delta}-1}+\frac{e^{-\delta\log z}}{e^{-2\pi i\delta}-1}=\frac{\log z-\pi i}{\pi i}+O(\delta)\quad\text{as}\quad\delta\to 0.$$

Again by application of Hölder's inequality we see that

$$A_\delta(n)+A_{-\delta}(n)=\sum_{\substack{k=0\\k\neq n}}^{\infty} a_k\left(\frac{1}{n-k-\delta}+\frac{1}{n-k-\delta}\right)$$

is continuous in δ and bounded uniformly in $n\in\mathbb{N}_0$ and δ, $|\delta|\leq 1/8$, which implies

$$\lim_{\delta\to 0}\frac{1}{2}(A_\delta(n)+A_{-\delta}(n))=\sum_{\substack{k=0\\k\neq n}}^{\infty}\frac{a_k}{n-k}=A(n)$$

and

$$\lim_{\delta\to 0}\frac{1}{2}(f_\delta(z)+f_{-\delta}(z))=\sum_{n=1}^{\infty}A(n)z^n,\quad |z|<1.$$

Combining these results we obtain $\Sigma_{n=1}^{\infty}A(n)z^n=B(z)+(\pi i-\log z)\Sigma_{k=0}^{\infty}a_k z^k$. Now from (38) the statement follows and the proof is complete.

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Abteilung für Mathematik, Universität Ulm
D-7900 Ulm/Donau BRD

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