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EPIDEMIC PROCESSES ON RANDOM GRAPHS AND THEIR THRESHOLD FUNCTION

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In this paper we consider random graphs, corresponding to random self-mappings of the finite set $\{1, 2, \ldots, n\}$ in which m points are "infected". We discuss several schemes of an epidemic process, in particular, the case when the infection is delivered inversely to arc direction.

An asymptotic formula is derived for the average number of the cyclic graph's elements which are contained in components of sizes exceeding αn, $0<\alpha<1$. Using this result, we prove that the probability of infecting an essential part of the graph (having the size $O(n)$) tends to 1, whenever $\sqrt{n}=o(m)$.

1. Consider the set $\mathfrak{T}_n$ of all mappings of the finite set $X=\{1, 2, \ldots, n\}$ into itself, which satisfy $Tx\neq x$, $T\in\mathfrak{T}_n$, $x\in X$. There are $(n-1)^n$ different mappings in $\mathfrak{T}_n$. Each mapping $T\in\mathfrak{T}_n$ is a digraph G_T, whose points belong to the set X; the points x and y are joined by an arrow iff $y=Tx$. G_T may consist of disjoined components and each component includes one cycle. We classify the components of G_T corresponding to their size, i. e. to the number of points they consist of. Let an uniform probability distribution on $\mathfrak{T}_n$ be given (each mapping $T\in\mathfrak{T}_n$ has probability $(n-1)^{-n}$). The random mappings just described are the second type mappings studied by B. Harris [1]. We shall consider several schemes of epidemic processes on the random graphs G_T which were treated in the paper of Gertsbakh [2].

Define $T^k x$ to be the k-th iteration of $T\in\mathfrak{T}_n$ on $x\in X$, where k is integer. y is said to be a k-th image of x in T, whenever for some $k>0$, $T^k x=y$. The set of all images (or successors) of x in T is $S_T(x)=\{x, Tx, \ldots, T^{n-1}x\}$ (which need not be distinct elements). y is said to be a k-th inverse of x in T, whenever for some $k\leq 0$, $T^k x=y$. The set of all k-th inverses of x in T is denoted by $T^{(k)}(x)$, and $P_T(x)=\bigcup_{k=-n}^{0} T^{(k)}(x)$ is the set of all inverses (or predecessors) of x.

Let m bacteria be placed at elements $x_1, x_2, \ldots, x_m$, where $x_i\in X$, $i=1, 2, \ldots, m$. All $\binom{n}{m}$ different occupations are equally probable. An inverse epidemic process (IEP) is defined by the infection being delivered from the infected points to all their predecessors. The area which will be infected is the set of all inverses of $x_1, x_2, \ldots, x_m$: $P_T(m)=\bigcup_{i=1}^{m} P_T(x_i)$.

Now imagine that the arc connecting any two vertices x and y $(x, y\in X)$ carries infection in two directions: from x to y if x has been infected first, and conversely. In this way we arrive at the two-sided epidemic process (TEP): the infection is delivered from the infected points "backward" to all their predecessors $P_T(m)$, "forward" to all their successors $S_T(m)$ and again "backward" from each $x \in S_T(m)$ to all its predecessors. The infected area will be $B_T(m) = P_T(m) \cup S_T(m) \cup R_T(m)$, where $R_T(m) = \bigcup_{\{x \in S_T(m)\}} P_T(x)$.

Consider the function $C_I(n): \mathfrak{T}_n \to R^1$ which maps each $T \in \mathfrak{T}_n$ into the integer $|P_T(m)|$. (The symbol $|A|$ stands for the number of distinct elements in the finite set A.) Similarly define $C_T(m)$ as the function which takes value $|B_T(m)|$ at a mapping $T \in \mathfrak{T}_n$.

Let $0<\alpha<1$, and consider the event $\{C_I(m) \geqq \alpha n\}$ which means that the infected area arising from m bacteria exceeds a fixed α-ratio of all elements in X (m bacteria infect an essential part of the population).

Definition. The function $\varphi_I(n)$ is called threshold for IEP if

$$P\{C_I(m) \geqq \alpha n\} \to 1 \quad \text{for } \varphi_I(n) = o(m), \tag{1}$$

$$P\{C_I(m) \geqq \alpha n\} \to 0 \quad \text{for } m = o(\varphi_I(n)), \tag{2}$$

for fixed $\alpha \in (0,1)$ and $n \to \infty$.

Threshold functions were introduced by Gertsbakh, who posed the problem to find $\varphi_I(n)$ [2].

2. Summary of results. In Section 4 we prove the following asymptitical results, when $n \to \infty$.

The average number of the cyclic elements in indecomposable random mapping is $\sqrt{2n/\pi} + O(1)$.

The average number of the cyclic elements in a random mapping, which are contained in components of sizes exceeding αn, $0<\alpha<1$, is $\sqrt{2n/\pi}\,(\pi/2 - \arcsin\sqrt{\alpha}) + O(1)$.

In the case of IEP, $\varphi_I(n) = \sqrt{n}$. The proof is based on the above asymptotic result.

In the case of TEP, if $m \to \infty$ as $n \to \infty$, and $\alpha \in (0, 1)$, then $P\{C_T(m) \geqq \alpha n\} \to 1$.

The results in Section 4 are based on some preliminary lemmas which are summarized in the next Section 3. Most of them are known.

3. The mapping $T \in \mathfrak{T}_n$ is called indecomposable iff it generates only one cycle. Let $C_{n,k}$ denote the number of indecomposable mappings in $\mathfrak{T}_n$, which have exactly k cyclical elements. For $n \geqq k \geqq 2$ Gertsbakh [2] found that $C_{n,k} = \binom{n}{k} k!\, n^{n-k-1}$. The number of all indecomposable mappings is

$$B_n = \sum_{k=2}^{n} C_{n,k} = (n-1)! \sum_{k=0}^{n-2} \frac{n^k}{k!} = \sqrt{\frac{\pi n}{2}}\, e(n-1)^{n-1}(1+o(1)), n \to \infty. \tag{3}$$

Assign to each indecomposable mapping T the probability B_n^{-1}, and denote by ξ_n the number of cyclic elements in T. The generating function of ξ_n is

$$P_n(x) = \mathrm{E}x^{\xi_n} = \frac{n^{n-1}}{B_n} \sum_{j=2}^{n} \binom{n}{j} j!\, n^{-j} x^j. \tag{4}$$

Consider the probability space $(\mathfrak{T}_n, \mathfrak{B}(\mathfrak{T}_n), P)$, where $\mathfrak{B}(\mathfrak{T}_n)$ is the σ-algebra of subsets of $\mathfrak{T}_n$, and P is the uniform probability measure on $\mathfrak{T}_n$. Let β_m denote the number of components of size $m\,(1 \leqq m \leqq n)$ and λ_m be the number of cyclical elements in these components. Obviously

(5) $$\lambda_m = \xi_{m,1} + \xi_{m,2} + \cdots + \xi_{m,\beta_m},$$

where $\xi_{m,j}$ are independent and identically distributed random variables with generating function $P_m(x)$ (see (4)).

The distribution of $(\beta_2, \beta_3, \ldots, \beta_n)$ for a given $n \geqq 2$ is

(6) $$\mathsf{P}\{\beta_2 = k_2, \ldots, \beta_n = k_n\} = \begin{cases} (n-1)^{-n} M_n(k_2, \ldots, k_n)_1 & \text{if } \Sigma_2^n iki = n, \\ 0 & \text{otherwise.} \end{cases}$$

Here

(7) $$M_n(k_2, \ldots, k_n) = \frac{n!\, B_2^{k_2} \ldots B_n^{k_n}}{(2)!^{k_2} \ldots (n!)^{k} n k_2! \ldots k_n!},$$

and the k's are nonnegative integers. The β's above would be independent if it were not for the condition on $\Sigma\, iki$.

Now define the sequence of generating functions

$$m_0 \equiv 1,\ m_1 \equiv 0,\ m_n(x_2, \ldots, x_n) = \sum_{\substack{k_2, \ldots, k_n \\ \Sigma iki = n}} M_n(k_2, \ldots, k_n) x_2^{k_2} \ldots x_n^{k_n},\ n \geqq 2.$$

Using (5)—(7) and applying Bruno's formula [3, Section 2.8], we obtain a relation between generating functions m_n and P_m.

Lemma 1. *If* $|z| \leqq e^{-1}$, $z \neq e^{-1}$, *then*

$$\sum_{n=0}^{\infty} m_n(P_2(x_2), \ldots, P_n(x_n)) \frac{z^n}{n!} = \exp\left(\sum_{k=2}^{\infty} \frac{B_k}{k!} P_k(x_k) z^k\right)$$

holds for $|x_k| \leq 1$, $k = 2, 3, \ldots, n$.

Let $B(z) = \Sigma_{n=2}^{\infty} B_n z^n / n!$ be the exponential generating function of the numbers B_n. For $|z| \leqq e^{-1}$ introduce the power series $S(z) = \Sigma_{n=0}^{\infty} n^n z^n / n!$ $(z \neq e^{-1})$ and $\Theta(z) = \Sigma_{n=1}^{\infty} n^{n-1} z^n / n!$. It is well-known [4, Section 7.2] that the function $\Theta(z)$ satisfies the transcendental equations

(8) $$\Theta e^{-\Theta} = z,\ \Theta(ze^{-z}) = z(\Theta(e^{-1}) = 1),$$

and

(9) $$S(z) = [1 - \Theta(z)]^{-1}.$$

Using most general combinatorial results [5] we can obtain

Lemma 2. *For* $|z| \leqq e^{-1}$, $z \neq e^{-1}$, *holds* $\exp\{B(z)\} = S(z) \exp\{-\Theta(z)\}$.

We also need the following inequality, which is independent from the above combinatorial lemmas.

Lemma 3. *Assume that* ξ *is an arbitrary random variable, defined on a probability space* (Ω, U, P) *and* $g(x)$ *is a non-negative, non-increasing Borel-function on the real line. Then for every* $a \geqq 0$ $\mathsf{E} g(\xi) \leqq g(a) \mathsf{P}\{\xi \geqq a\} + a.s.$ $\sup g(\xi) \mathsf{P}\{\xi < a\}$ *holds.*

Proof. The inequality is a consequence of the relations

$$\mathsf{E} g(\xi) = \int_{\{\xi < a\}} g(\xi) + \int_{\{\xi \geqq a\}} g(\xi),\ \int_{\{\xi < a\}} g(\xi) \leqq a.s. \sup g(\xi) \mathsf{P}\{\xi < a\},\ \int_{\{\xi \geqq a\}} g(\xi) \leqq g(a) \times \mathsf{P}\{\xi \geqq a\}.$$

4. Now we shall obtain the asymptotic of $\mathsf{E}\,\xi_n$. We have introduced the random variable ξ_n in Section 3 with the generating functions $P_n(x)$ (see (4)).

Theorem 1. *If $n \to \infty$, then the limit-relation* $\mathsf{E}\,\xi_n = \sqrt{2n/\pi} + O(1)$ *holds.*

Proof. By the relation $j!\, n^{-j-1} = \int_0^\infty e^{-nx} x^j dx$, $j = 0, 1, \ldots$, and by (3) and (4) deriving and putting $x=1$, we obtain

$$P'_n(1) = \frac{n^n}{B_n}\Big[\int_0^\infty (1+x)^n e^{-nx}(nx-1)\,dx - \int_0^\infty (n^2x^2-1)e^{-nx}dx\Big]$$

$$= \frac{n^n}{B_n}\int_0^\infty (1+x)^n e^{-nx}(nx-1)dx - \sqrt{\frac{2}{\pi n}} + o\left(\frac{1}{\sqrt{n}}\right),\quad n \to \infty.$$

To complete the proof it remains to obtain the asymptotic of $I_n = \int_0^\infty (1+x)^n \times e^{-nx}(nx-1)dx$. We shall use the substitution $z^2/2 = x - \log(1+x)$ or $z = x \times (1-(2/3)x + (2/4)x^2 - \cdots)^{1/2}$ (see [6, Section 4.5]). Using the Lagrange's formula we obtain that $I_n = 1 + O(1/\sqrt{n})$ for $n \to \infty$. Now from (3) it follows $P'_n(1) = \sqrt{2n/\pi} + O(1)$, which completes the proof of the theorem.

Furthermore we shall study the asymptotic of the mathematical expectation of the random variable $\mu_{s,n} = \lambda_s + \lambda_{s+1} + \cdots + \lambda_n$, which is equal to the number of cyclical elements in arandom mappings with components' sizes not less than s. The s's will be asymptotically equal to αn for a given $\alpha \in (0,1)$.

Theorem 2. *If* $s \sim \alpha n$, $\alpha \in (0,1)$ *and* $n \to \infty$, *then* $\mathsf{E}\,\mu_{s,n} = \sqrt{2n/\pi} \times (\pi/2 - \arcsin\sqrt{\alpha}) + O(1)$.

Proof. By lemma 1, substituting $x_j = 1$ for $j \leq s$ and $x_j = x$ for $j \geq s+1$ we have

$$\sum_{n=0}^{\infty} \psi_n(x)\,\frac{z^n}{n!} = \exp\Big\{B(z) + \sum_{k=s+1}^{\infty} \frac{B_k z^k}{k!}(P_k(x)-1)\Big\},$$

where $\psi_n(x) = m_n(\overbrace{1, \ldots, 1}^{s}, x, \ldots, x)$. Hence for the generating function of $\mu_{s,n}$ it follows

$$\mathsf{E}\,x^{\mu_{s,n}} = (n-1)^{-n}\psi_n(x)$$

$$= \frac{n!}{(n-1)^n}\cdot\frac{1}{2\pi i}\oint \exp\Big\{B(z) + \sum_{k=s+1}^{\infty}\frac{B_k z^k}{k!}(P_k(x)-1)\Big\}\,\frac{dz}{z^{n+1}}.$$

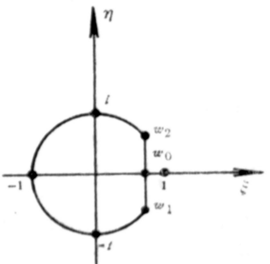

Fig. 1

Substitute in the last integral we^{-w} for z and choose the path of integration C in the plane $w = \xi + i\eta$ as on Fig. 1 (see [7]). Here $w_0 = 1 = 1/\sqrt{s}$, $w_1 = 1 - 1/\sqrt{s} - i\sqrt{1-(1-1/\sqrt{s})^2}$, $w_2 = 1 - 1/\sqrt{s} + i\sqrt{1-(1-1/\sqrt{s})^2}$ By (8), (9) and lemma 2 we obtain

$$\mathsf{E}\,x^{\mu_{s,n}} = \frac{n!}{(n-1)^n}\cdot\frac{1}{2\pi i}\int_C e^{(n-1)w}\exp\Big\{\sum_{k=s+1}^{\infty}\frac{B_k w^k e^{-kw}}{k}(P_k(x)-1)\,\frac{dw}{w^{n+1}}.$$

Deriving and putting $x=1$ we get at once

$$\mathsf{E}\,\mu_{s,n}=\frac{n!}{(n-1)^n}\cdot\frac{1}{2\pi i}\int_C e^{(n-1)w}\sum_{k=s+1}^{\infty}\frac{B_k w^k e^{-kw}}{k!}\,P'_k(1)\frac{dw}{w^{n+1}}\,.$$

It is easy to see that the integral on the unit circle tends to 0, when $n\to\infty$ and $s\sim\alpha n$, $0<\alpha<1$, and the non-zero part of the integral is on the chord $w_1 w_2$:

$$\mathsf{E}\,\mu_{s,n}=\frac{n}{(n-1)^n}\,\frac{1}{2\pi i}\int_{w_1}^{w_2}\frac{e^{(n-1)w}}{k\,!}\sum_{k1=s+}^{\infty}\frac{B_k w^k e^{-kw}}{k\,!}P'_k(1)\,\frac{dw}{w^{n+1}}+o(1).$$

Put in this integral $w=1-v/\sqrt{s}$. For the same values of s it is easy to verify that

$$\frac{n!}{(n-1)^n}\,e^{(n-1)(1-v/\sqrt{s})}\,(1-\frac{v}{\sqrt{s}})^{-n-1}=\sqrt{2\pi n}\,e^{v^2/2\alpha}(1+o(1)). \tag{10}$$

Now we shall study the asymptotic of the sum

$$\sigma_s(v)=\sum_{k=s+1}^{\infty}B_k\left(1-\frac{v}{\sqrt{s}}\right)^k e^{-k(1-v/\sqrt{s})}\,\frac{P'_k(1)}{k!}\,.$$

From the easily verifyiable formulas

$$\left(1-\frac{v}{\sqrt{s}}\right)^k e^{kv/\sqrt{s}}=e^{-kv^2/2s}\left(1+O\left(\frac{1}{\sqrt{s}}\right)\right),\ s\to\infty,\quad\frac{B_k e^{-k}}{(k-1)!}=\frac{1}{2}+O\left(\frac{1}{\sqrt{k}}\right),\ k\to\infty,$$

applying theorem 1, we obtain the relations

(11)

$$\sigma_s(v)=\frac{1}{2}\sum_{k=s+1}^{\infty}\frac{1}{k}\,e^{-kv^2/2s}\,P'_k(1)\left(1+O\left(\frac{1}{\sqrt{s}}\right)\right)=\frac{1}{2}\sqrt{\frac{2s}{\pi}}\sum_{k=s+1}^{\infty}\left(\frac{k}{s}\right)^{-\frac{1}{2}}e^{-kv^2/2s}\,\frac{1}{s}+O(1).$$

We may consider the above sum as a Riemann's integral sum of the function $t^{-1/2}e^{-tv^2/2}$ in $[1,\infty)$. Therefore

$$\sigma_s(v)=\sqrt{\frac{2s}{\pi}}\,\frac{1}{2}\int_1^{\infty}\frac{e^{-tv^2/2}}{\sqrt{t}}\,dt+O(1)=\sqrt{\frac{2s}{\pi}}\,\frac{1}{v}\,K(v^2)+O(1), \tag{12}$$

where

$$K(x)=\int_x^{\infty}\frac{e^{-u/2}}{\sqrt{u}}\,du.$$

Now (10) — (12) imply

$$\mathsf{E}\,\mu_{s,n}=\frac{\sqrt{n}}{2\pi i}\int_{1-i\infty}^{1+i\infty}e^{v^2/2\alpha}\,\frac{1}{v}\,K(v^2)dv+O(1).$$

Putting again in the last integral $v^2=p$ and simplifying we obtain that

$$\mathsf{E}\,\mu_{s,n}=\frac{1}{2\pi i}\sqrt{\frac{n}{2}}\int_{\Pi}e^{p/2\alpha}\,L(p)dp+O(1),$$

where $L(p)=p^{-1/2}\int_{1/2}^{\infty}(e^{-pt}/\sqrt{t})\,dt$, and the path of integration Π is the one corresponding to the substitution parabola. Obviously $L(p)$ is the Laplace transform of the convolution $f_1(t)*f_2(t)$, where

$$f_1(t)=(\pi t)^{-1/2}, \quad f_2(t)=\begin{cases} t^{-1/2}, & \text{when } t\geq 1/2,\\ 0, & \text{when } t<1/2.\end{cases}$$

The well-known properties of the Laplace-transform give

$$\mathsf{E}\,\mu_{s,n}=\sqrt{\frac{n}{2\pi}}\int_{1/2}^{1/2\alpha}\frac{d\tau}{\sqrt{\tau(1/2\alpha-\tau)}}+O(1)=\sqrt{\frac{2n}{\pi}}\left(\frac{\pi}{2}-\arcsin\sqrt{\alpha}\right)+O(1).$$

Consider now the IEP, described in Section 1. We shall find the threshold function $\varphi_I(n)$. Remembering the structure of a mapping $T\in\mathfrak{T}_n$, we find that m bacteria will infect a fixed α-ratio of all elements in X if there exist some bacteria which are placed in cyclic elements belonging to components whose sizes are not less than $s\sim\alpha n$. Let $\eta_{m,s}$ be the number of bacteria satisfying the above condition. According to the formula of total probability for the distribution of $\eta_{m,s}$, we obtain

$$\mathsf{P}\{\eta_{m,s}=k\}=\sum_{l=2}^{n}\left[\binom{l}{k}\binom{n-l}{m-k}\Big/\binom{n}{m}\right]\mathsf{P}\{\mu_{s,n}=l\}. \tag{13}$$

Theorem 3. $\varphi_I(n)=\sqrt{n}$.

Proof. It suffices to verify the condition (1) for $\sqrt{n}=o(m)$ and $m/n\to 0$. The proof of condition (2) is contained in the paper of Gertsbakh [2].

Since

$$\{\eta_{m,s}\geq 1\}\subset\{C_I(m)\geq\alpha n\}, \tag{14}$$

it suffices to find a lower bound for the probability

$$\mathsf{P}\{\eta_{m,s}\geq 1\}=1-\mathsf{P}\{\eta_{m,s}=0\}. \tag{15}$$

By (13) it follows, that

$$\mathsf{P}\{\eta_{m,s}=0\}=\sum_{l=2}^{n}\binom{n-l}{m}\binom{n}{m}^{-1}\mathsf{P}\{\mu_{s,n}=l\}.$$

Since

$$\binom{n-l}{m}\binom{n}{m}^{-1}=\left(1-\frac{m}{n}\right)\left(1-\frac{m}{n-1}\right)\cdots\left(1-\frac{m}{n-l+1}\right)<\left(1-\frac{m}{n}\right)^l,$$

then

$$\mathsf{P}\{\eta_{m,s}=0\}\leq\mathsf{E}\,(1-m/n)^{\mu_{s,n}}. \tag{16}$$

Let $c_\alpha=\sqrt{2/\pi}\,(\pi/2-\arcsin\sqrt{\alpha})$ and $g(x)=(1-m/n)^x$. Applying lemma 3 for the non-increasing function $g(x)$ and for the constant $c_\alpha\sqrt{n}-n/m>0$ with the restriction $m/n\to 0$, we obtain

$$\begin{aligned}\mathsf{E}\,(1-m/n)^{\mu_{s,n}}&\leq(1-m/n)^{c_\alpha\sqrt{n}}(1-m/n)^{-n/m}\,\mathsf{P}\{\mu_{s,n}\geq c_\alpha\sqrt{n}-n/m\}\\&\quad+\text{a.s.}\sup g(\mu_{s,n})\,\mathsf{P}\{\mu_{s,n}<c_\alpha\sqrt{n}-n/m\}\\&<\exp(3-c_\alpha m/\sqrt{n})+\mathsf{P}\{c_\alpha\sqrt{n}-\mu_{s,n}>n/m\}.\end{aligned} \tag{17}$$

On the other hand, using the Markov inequality and the result of theorem 2, we find that

(18) $$\mathsf{P}\left\{c_\alpha\sqrt{n}-\mu_{s,n}>\frac{n}{m}\right\}\leqq\frac{m}{n}\,\mathsf{E}\,(c_\alpha\sqrt{n}-\mu_{s,n})=O\left(\frac{m}{n}\right)\to 0.$$

Now combining (14) — (18) we obtain theorem 3.

Finally, we shall consider the TEP. In TEP one bacterium will kill each component in which it is placed. In order to obtain information about the asymptotic of $\{PC_T(m)\geqq \alpha n\}$ we introduce, as in theorem 3, the random variable $\xi_{m,s}$, equal to the number of bacteria, which are placed in components with sizes not less than s for the same values of s. Let $\lambda_{s,n}$ be the number of vertices which participate in components whose sizes are not less than s. For the TEP Gertsbakh has proved that for each function $m=\delta(n)$ such that $m\to\infty$ as $n\to\infty$, there is a positive probability of infecting an essential part of the graph. We shall prove the following

Theorem 4. *If* $m\to\infty$, *as* $n\to\infty$, *then for fixed* $\alpha\in(0,1)$, $\mathsf{P}\{C_T(m)\geqq\alpha n\}\to 1$.

Proof. As in the proof of theorem 3

(19) $$\{\xi_{m,s}\geqq 1\}\subset\{C_T(m)\geqq\alpha n\}$$

and

(20) $$\mathsf{P}\{\xi_{m,s}\geqq 1\}=1-\mathsf{P}\{\xi_{m,s}=0\}.$$

Using the formula for the total probability, we find that

(21) $$\mathsf{P}\{\xi_{m,s}=0\}=\sum_{l\geqq s}\binom{n-l}{m}\binom{n}{m}^{-1}\mathsf{P}\{\lambda_{s,n}=l\}$$

$$\leqq\binom{n-s}{m}\binom{n}{m}^{-1}\mathsf{P}\{\lambda_{s,n}\geqq s\}<\left(\frac{n-s}{n-m}\right)^m.$$

Suppose that $m\leqq\beta n$ for some $\beta\in(0,\alpha)$. Then

(22) $$((n-s))/(n-m))^m\leqq((1-\alpha)/(1-\beta))^m\to 0,\ m\to\infty.$$

Relations (19) — (22) yield the theorem.

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