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ON CLASSES OF RANDOM SETS AND POINT PROCESS MODELS

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For a class of random sets called grain-germ-models, which are a generalization of the well-known Boolean model, a formula for the capacity functional is given. Furthermore, for a point process obtained by the Matérn-(2)-thinning procedure starting from a general stationary simple point process formulas for the first and second moment measures are derived.

1. Introduction. In this paper a class of random sets and a class of point processes are studied. Both classes are generalizations of known models which are constructed starting from Poisson point processes. The known random set model is the so-called Boolean model, see [1], which is the union of compact sets, called "grains", located at points, called "germs", of a Poisson Point process. For the more general model, where the germs are points of a stationary simple point process, a formula for the capacity functional is given. The point process model generalizes the second Matérn hardcore process, see [2] and [3], which is obtained by a special dependent thinning of a Poisson point process. For a point process obtained by the same thinning procedure starting from a general stationary simple point process formulas for the first and second moment measures are given.

Let us introduce some notations:

A, A_i, $A(x_i)$	random closed sets (RACS)
T_A	capacity functional of the RACS A
R^d	d-dimensional Euclidian space
$\boldsymbol{F}$	space of all closed subsets of R^d
$\boldsymbol{K}$	space of all compact subsets of R^d
$\boldsymbol{F}_K$	$\{F\in\boldsymbol{F}: F\cap K\neq\emptyset\}$ for $K\in\boldsymbol{K}$
σ_f	σ-field belonging to $\boldsymbol{F}$ generated by the class of all sets $\boldsymbol{F}_K$, $K\in\boldsymbol{K}$
σ_k	σ-field $\sigma_f\cap\boldsymbol{K}$
$A\oplus B$	$\{x+y\in R^d: x\in A,\ y\in B\}$ for A, $B\subset R^d$
$\check{A}$	$\{x\in R^d: -x\in A\}$ for $A\subset R^d$
$\mathfrak{L}^d$	Borel σ-field of R^d
Φ	marked point process in R^d with mark space $\boldsymbol{K}$
$M_{\boldsymbol{K}}$	set of all samples φ of Φ
$\mathfrak{M}_{\sigma_k}$	σ-field corresponding to $M_{\boldsymbol{K}}$
P	distribution of Φ on $(M_{\boldsymbol{K}}, \mathfrak{M}_{\sigma_k})$
Φ'	(non-marked) point process of R^d
M	set of all locally finite counting measures on $(R^d, \mathfrak{L}^d)$
$\mathfrak{M}$	σ-field corresponding to M
P'	distribution of Φ' on $(M, \mathfrak{M})$

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U	set of all bounded measurable functionals $u: R^d \times \boldsymbol{K} \to R^+ = [0, \infty)$ with compact support
V	$\{1-u: u \in U,\ u \leq 1\}$
ν_d	d-dimensional Lebesgue measure
$b(x, r)$	ball with midpoint x and radius r
$[0, 1)^d$	d-dimensional unit cube

2. Grain-Germ-Models. In this section we study a special class of random closed sets (RACS)(see, for general definitions, [1]). Let us understand a RACS as a random variable on a probability space $(\Omega, \boldsymbol{A}, \boldsymbol{P})$ with values in $(\boldsymbol{F}, \sigma_f)$. In the present literature mostly studied example of such RACS is the Boolean model, see [1]. The Boolean model is the union of independent almost sure compact RACS $A(x_i)$, the grains, which are belonging to each point x_i of a Poisson point process Φ' in R^d, that is

$$A = \bigcup_{x_i \in \Phi'} A(x_i). \tag{1}$$

As is well-known from the theory of random sets, the distribution of a RACS A is uniquely determined by its capacity functional T_A: $T_A(K) = \boldsymbol{P}(A \cap K \neq \emptyset)$ $(K \in \boldsymbol{K})$.

For the capacity of a Boolean model A a formula is known (see [1] and section 3 of this paper). But this model is in many cases not more than a first approximation for the description of various phenomena in nature and technology, because it possesses some bad properties (for instance, the Poisson assumption and that the grains $A(x_i)$ can overlap one another). This suggests to investigate model of the form (1), where

(a) Φ' is no Poisson point process and

(b) the grains are not independent.

So we will study in the following models of the form

$$A = \bigcup_{(x_i, A_i) \in \Phi} (A_i + x_i), \tag{2}$$

where Φ is a random marked point process with mark space $\boldsymbol{K}$. Let us call them grain-germ-models.

3. The capacity functional T_A of a grain-germ-model A. Let Φ be a marked point process in R^d with mark space $\boldsymbol{K}$.

Definition 1. *The functional $G_P: V \to [0, 1]$, which is given by*

$$G_P(v) = \int_{M_K} \prod_{(x, K) \in \varphi} v(x, K) P(d\varphi) \quad (v \in V)$$

is the generating functional of the point process Φ with corresponding distribution P (see [4]).

For a RACS A of the form (2) the following theorem gives a relation between its capacity T_A and the generating functional G_P of the underlying point process Φ.

Theorem 1. *Let Φ be a simple marked point process of R^d with mark space $\boldsymbol{K}$ and A a grain-germ-model with the underlying point process Φ. For a set $B \in \boldsymbol{K}$ let v_B be the functional defined by $v_B(x, K) = 1 - 1_{\check{K} \oplus B}(x)$, $x \in R^d$, $K \in \boldsymbol{K}$. Then*

$$T_A(B) = 1 - G_P(v_B). \tag{3}$$

Proof. By definition $T_A(B)=\boldsymbol{P}(A\cap B\neq\emptyset)=1-\boldsymbol{P}(A\cap B=\emptyset)$. Because of the construction of A

$$\boldsymbol{P}(A\cap B=\emptyset)=P(\{\varphi\in M_{\boldsymbol{K}}:[\bigcup_{(x,K)\in\varphi}(K+x)]\cap B=\emptyset\})$$

$$=\boldsymbol{P}(\{\varphi\in M_{\boldsymbol{K}}:\text{ for each }(x,K)\in\varphi\text{ holds }(K+x)\cap B=\emptyset\}).$$

Clearly, $(K+x)\cap B=\emptyset$ holds if and only if $x\notin\hat{K}\oplus B$. Thus for the functional $f(\varphi)=\Pi_{(x,K)\in\varphi}(1-1_{\hat{K}\oplus B}(x))$, $\varphi\in M_{\boldsymbol{K}}$, we get: $f(\varphi)=0$ if and only if at least one point $(x,K)\in\varphi$ exists with $x\in\hat{K}\oplus B$, otherwise $f(\varphi)=1$. This gives

$$P(A\cap B=\emptyset)=P(\{\varphi\in M_{\boldsymbol{K}}:f(\varphi)=1\})=E_P f(\Phi)$$

$$=\int_{M_{\boldsymbol{K}}}\prod_{(x,K)\in\varphi}(1-1_{\hat{K}\oplus B}(x))P(d\varphi)=G_P(v_B)$$

and hence formula (3).

Because of this theorem and Choquet's theorem the distribution of a grain-germ-model is uniquely determined by the generating functional of the underlying point process applied to the family of all mappings $\{v_B\}_{B\in\boldsymbol{K}}$.

Examples. 1. If the RACS A is a stationary Boolean model, where the underlying Poisson point process has intensity λ, then for each $x\in R$ the grains $A_x=A(x)-x$ and $A_0=A(0)$ are identically distributed (see [1]). Let ϱ be the corresponding distribution on $(\boldsymbol{F},\sigma_f)$. Then the capacity functional of a stationary Boolean model is, see [1], $T_A(K)=1-\exp(-\lambda E_\varrho\nu_d(A_0\oplus\hat{K}))$.

2. Let us denote by Θ a random measure on $(R^d\times\boldsymbol{K},\mathfrak{L}^d\otimes\sigma_k)$ and by P_Θ the corresponding distribution on the space $(N_{\boldsymbol{K}},\mathfrak{N}_{\sigma_k})$ of all locally finite measures Λ on $(R^d\times\boldsymbol{K},\mathfrak{L}^d\otimes\sigma_k)$ (see [5]). Let Φ be the Cox process (doubly stochastic Poisson process) in the space $R^d\times\boldsymbol{K}$ corresponding to the random measure Θ. Then using formula (3) we obtain

$$T_A(B)=1-\int_{N_{\boldsymbol{K}}}\exp(-\int_{R^d\times\boldsymbol{K}}1_{\hat{K}\oplus B}(x)\Lambda(d(x,K))P(d\Lambda),\qquad B\in\boldsymbol{K}.$$

3. An important quantity for a stationary RACS A is the volume fraction p, $p=E\nu_d(A\cap[0,\ 1)^d)=\boldsymbol{P}(0\in A)=T_A(\{0\})$. Now we give p for a further grain-germ-model. The point process is here a matern cluster process, which can be obtained as follows:

Let each primary point x_i of a stationary Poisson process $\tilde{\Phi}$ in the R^3 (with corresponding distribution $\tilde{P}$ and intensity $\lambda_{\tilde{P}}$) independently on the other points of $\tilde{\Phi}$ generate a random with parameter μ Poisson distributed number of independently identically on the ball $b(x_i,r)$, $r>0$, uniformly distributed secondary points. The union of all these random clusters of secondary points will be denoted by Φ'. Then we obtain after long calculation, see [3], for the volume fraction $p=T_A(\{0\})$ of the stationary RACS

$$A=\bigcup_{x_i\in\Phi'}(B(0,R)+x_i),\quad R<r,$$

$$p=1-\exp(-\lambda_{\tilde{P}}\{4\pi(r-R)^3(1-\exp(-\mu R^3/r^3))/3$$
$$+4\pi\int_{r-R}^{r+R}t^2[1-\exp(\mu(V(t,R)-1))]dt\})$$

with

$$V(t, R)=\{6t^2R^4-16t^3R^3-12(r^2t^2-t^4)R^2+6r^4t^2+16r^3t^3+12r^2t^4-2t^6\}/32t^3r^3.$$

In [3] some other grain-germ-models are studied, but the formulas are very complicated and unsatisfactory.

4. The Matérn-(2)-thinning of a stationary point process in R^d. In his paper [2] Matern studied the following thinning procedure, which yields a hardcore point process with minimal interpoint distance R. Whereas he used it for thinning a Poisson point process, here a general stationary simple point process is thinned.

Let $\tilde{\Phi}$ be a stationary simple point process in R^d with finite intensity $\lambda_{\tilde{P}}$ and $\tilde{P}$ the corresponding distribution on the space $(M, \mathfrak{M})$. First we mark $\tilde{\Phi}$ independently with on (0, 1) uniformly distributed marks, that is, each point x_i of $\tilde{\Phi}$ gets independently on the others a mark $k_i \in (0, 1)$. The marked point process obtained will be denoted by Φ, the corresponding distribution on the space $(M_{(0,1)}, \mathfrak{M}_{\mathfrak{L}^1 \cap (0,1)})$ of all locally finite counting measures on $R^d \times (0, 1)$ by P. Then the following thinning operation is performed:

A point x_i of the process $\tilde{\Phi}$ is retained if no point with a mark less than the mark of x_i is in the ball $b(x_i, R)$, $R>0$; otherwise x_i is eliminated. Let us call the point process $\Phi' \subset \tilde{\Phi}$ obtained in this way the R-Matérn-(2)-thinning of $\tilde{\Phi}$. Because of the independent marking Φ' is also stationary. Let the corresponding distribution on $(M, \mathfrak{M})$ be P'. The following quantities of a point process Ψ in R^d are needed:

n-th moment measure μ_P^n on $(\mathfrak{L}^d)^n$

$$\mu_P^n(B_1\times \dots \times B_n)=\int_M \sum_{x_1 \dots x_n \in \varphi} 1_{B_1\times \dots \times B_n}(x_1, \dots x_n)P(d\varphi),$$

n-th Campbell measure φ_P^n on $(\mathfrak{L}^d)^n \otimes \mathfrak{M}$

$$\varphi_P^n(B\times Y)=\int_M \sum_{x_1, \dots, x_n \in \varphi} 1_{B\times Y}(x_1, \dots, x_n, \varphi)P(d\varphi),\quad B\in(\mathfrak{L}^d)^n,\ Y\in\mathfrak{M},$$

n-th factorial moment measure on $(\mathfrak{L}^d)^n$

$$\alpha_P^n(B_1\times \dots \times B_n)=\int_M \sum_{\substack{x_1, \dots, x_n \in \Phi \\ x_i \neq x_j \text{ for } i\neq j}} 1_{B_1}(x_1) \dots 1_{B_n}(x_n)P(d\varphi),$$

Palm distribution with respect to the points $x_1, \dots, x_n \in R^d$, $P_{x_1, \dots, x_n}$, reduced second moment measure $K(t)$, $t\in R^+$.

For $\varphi \in M$ let $T_x\varphi$ be the translation of φ by $x\in R^d$.

We use the following lemmas (see [6]):

L e m m a 1. *For all measurable functionals* $f: R^{dn}\times M \to R^+$

$$\int_M \sum_{a_n \in \varphi^n} f(a_n, \varphi)P(d\varphi)=\int_{R^{dn}\times M} f(a_n, \varphi)\varphi_P^n(d(a_n, \varphi)).$$

L e m m a 2. *If Ψ is a stationary point process with intensity λ_P, then*

$$\varphi_P^n(d(x_1, \dots, x_n, \varphi))=P_{x_1, \dots, x_n}(d\varphi)\mu_P^n(d(x_1, \dots, x_n))$$
$$=P_{0,x_2-x_1, \dots, x_n-x_1}(dT_{x_1}\varphi)\mu_P^n(d(x_1, \dots, x_n)).$$

Let us first compute the intensity measure $\mu^1_{P'}$ of Φ', $\mu^1_{P'}(B)=E\Phi'(B)$, $B\in\mathfrak{L}^d$. Because of the construction of the thinning we get

$$\mu^1_{P'}(B)=\int_{M_{(0,1)}}\sum_{(x,k)\in\varphi}1_B(x)1_Y(x,k,\varphi)P(d\varphi)$$

with $Y=\{(x,k,\varphi)\in R^d\times(0,1)\times M_{(0,1)}:\varphi(b(x,R)\times(0,1))=\varphi(b(x,R)\times[k,1))\}$. For the marking happens independently, we obtain

$$\mu^1_{P'}(B)=\int_M\sum_{x\in\varphi}1_B(x)P\,(\text{"each point of }\varphi\text{ in }b(x,R)\text{ has a greater mark than }x\text{"})$$

$$\widetilde{P}(d\varphi)=\int_M\sum_{x\in\varphi}1_B(x)\int_0^1(1-k)^{\varphi(b(x,R))-1}dk\widetilde{P}(d\varphi).$$

Using lemma 1 we get

$$\mu^1_{P'}(B)=\int_{R^d\times M}1_B(x)\int_0^1(1-k)^{\varphi(b(x,R))-1}\,dk\varphi^1_{\widetilde{P}}(d(x,\varphi))$$

and by lemma 2 and integration with respect to k

$$\mu^1_{P'}(B)=\lambda_{\widetilde{P}}\int_{B\times M}\varphi(b(x,R))^{-1}\widetilde{P}_0(dT_x\varphi)\nu_d(dx).$$

Substituting $\psi=T_x\varphi$ and using $T_{-x}\psi(b(x,R))=\psi(b(0,R))$ we obtain

$$\mu^1_{P'}(B)=\lambda_{\widetilde{P}}\int_B\int_M\psi(b(0,R))^{-1}\,\widetilde{P}_0(d\psi)\nu_d(dx=\lambda_{\widetilde{P}}E_{\widetilde{P}_0}\widetilde{\Phi}(b(0,R))^{-1}\nu_d(B).$$

In particular, this says that the Matérn-(2)-thinned process Φ' has the intensity $\lambda_{P'}=\lambda_{\widetilde{P}}E_{\widetilde{P}_0}\widetilde{\Phi}(b(0,R))^{-1}$. If especially $\widetilde{\Phi}$ is a stationary Poisson process with intensity $\lambda_{\widetilde{P}}$, we get (see also [2]):

$$E\Phi'(B)=\nu_d(B)/\nu_d(b(0,R))\{1-\exp(-\lambda_{\widetilde{P}}\nu_d(b(0,R)))\}.$$

Let us now compute the second moment measure of Φ'. For any point process Ψ in R^d we have $\mu^2_P(B_1\times B_2)=\mu^1_P(B_1\cap B_2)+\alpha^2_P(B_1\times B_2)$. For the thinned process Φ' is stationary we get $\mu^2_{P'}(B_1\times B_2)=\lambda_{P'}\nu_d(B_1\cap B_2)+\alpha^2_{P'}(B_1\times B_2)$. Because of the construction of Φ' we obtain for $\varkappa=\alpha^2_{P'}(B_1\times B_2)$

$$\varkappa=\int_{M_{(0,1)}}\sum_{(x_1,k_1),(x_2,k_2)\in\varphi}\{1_{F\cap B_1\times B_2}(x_1,x_2)1_Y(x_1,x_2,k_1,k_2,\varphi)\}P(d\varphi)$$

with $F=\{(x_1,x_2)\in R^d\times R^d:|x_1-x_2|>R\}$ and

$$Y=\{(x_1,x_2,k_1,k_2,\varphi):\varphi(b(x_1,R)\times(0,1))=\varphi(b(x_1,R)\times[k_1,1)),$$
$$\varphi(b(x_2,R)\times(0,1))=\varphi(b(x_2,R)\times[k_2,1))\}.$$

Analogously to the computation of the intensity measure we get because of independent marking

$$\varkappa=\int_M\sum_{x_1,x_2,\in\varphi}1_{F\cap B_1\times B_2}(x_1,x_2)\int_0^1\int_0^1f(x_1,x_2,k_1,k_2,\varphi)dk_2\,dk_1\,\widetilde{P}(d\varphi)$$

with

$$f(x_1, x_2, k_1, k_2, \varphi)=(1-k_1)^{\varphi(b(x_1, R)\setminus b(x_2, R))-1}$$
$$\times(1-k_2)^{\varphi(b(x_2, R)\setminus b(x, R))-1}(1-\max\{k_1, k_2\})^{\varphi(b(x_1, R)\cap b(x_2, R))}.$$

Integration gives

$$\varkappa=\int\limits_M \sum_{x_1, x_2\in\varphi} 1_F(x_1, x_2)1_{B_1}(x_1)1_{B_2}(x_2)g(x_1, x_2, \varphi)\widetilde{P}(d\varphi),$$

where

$$g(x_1, x_2, \varphi)=\frac{1}{\varphi(b(x_1, R)\cup b(x_2, R))}\left\{\frac{1}{\varphi(b(x_1, R))}+\frac{1}{\varphi(b(x_2, R))}\right\}.$$

With lemma 1 and lemma 2 and because of the definition of the set F

$$\varkappa=\int\limits_{B_1\times B_2} 1_F(x_1, x_2)\int\limits_M g(x_1, x_2, \varphi)\widetilde{P}_{0,x_2-x_1}(dT_{x_1}\varphi)\alpha^2_{\widetilde{P}}(d(x_1, x_2))$$

and, finally, by the substitution $T_{x_1}\varphi=\psi$ we obtain

$$\alpha^2_{P'}(B_1\times B_2)=\int\limits_{B_1\times B_2} 1_F(x_1, x_2)\int\limits_M g(0, x_2-x_1, \psi)\widetilde{P}_{0, x_2-x_1}(d\psi)\alpha^2_{\widetilde{P}}(d(x_1, x_2)).$$

If the underlying point process is additionally isotropic, then the term

$$1_F(x_1, x_2)\int\limits_M g(0, x_2-x_1, \psi)\widetilde{P}_{0,x_2-x_1}(d\psi)\alpha^2_{\widetilde{P}}(d(x_1, x_2))$$

only depends on $|x_2-x_1|$ and by introduction of the reduced second moment measure $\widetilde{K}(t)$, $t\in R^+$ (see [7; 8])

$$\alpha^2_{P'}(B_1\times B_2)=\int\limits_{B_1}\int\limits_R^\infty E_{\widetilde{P}_{0,\mathbf{t}}}g(0, \mathbf{t}, \widetilde{\Phi})\sigma_t^{B_2}(x_1)\lambda^2_{\widetilde{P}}d\widetilde{K}(t)\nu_d(dx_1),$$

where $\sigma_t^{B_2}(x_1)$ is for a sphere with midpoint x_1 and radius t the portion o that part of the surface which is contained in B_2, $\widetilde{K}(t)$, $t\geqq 0$, is the reduced second moment measure of $\widetilde{\Phi}$ and $\mathbf{t}=(t, 0, 0, \ldots, 0)$.

If especially $\widetilde{\Phi}$ is a stationary Poisson process, we get (see also [2])

$$\mu^2_{P'}(B_1\times B_2)=\frac{\nu_d(B_1\cap B_2)}{\nu_d(b(0, R))}\{1-\exp(-\lambda_{\widetilde{P}}\nu_d(b(0, R)))\}$$
$$+2\int\limits_{B_1}\int\limits_R^\infty\left\{\frac{1}{\lambda_{\widetilde{P}}\nu_d(b(0, R)}\left[\frac{1}{v}=\frac{\exp(-\lambda_{\widetilde{P}}\nu_d(b(0, R)))}{z}\right]+\frac{\exp(-\lambda_{\widetilde{P}}v)}{\lambda_{\widetilde{P}}zv}\right\}$$
$$\times\sigma_t^{B_2}(x_1)d\nu_d(b(0, 1))t^{d-1}\nu_1(dt)\nu_d(dx_1)$$

with $v=\nu_d(b(0, R)\cup b(\mathbf{t}, R))$ and $z=\nu_d(b(\mathbf{t}, R)\setminus b(0, R))$.

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