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SOME ALMOST COMPLEX STRUCTURES AND THEIR ALMOST HOLOMORPHIC FUNCTIONS

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The local existence of the J-almost-holomorphic functions on 4-dimensional almost-complex manifolds (M, J) is examined. Under some natural assumptions for J it is proved that a diffeomorphic to $\mathbf{C}\times\mathbf{R}^2$ local foliation of the underlying manifold M arise.

1. Introduction. Let M be a real $2n$-dimensional differentiable manifold (of class C^∞) equipped with an almost-complex structure J, i. e. for every point $m\in M$ an anti-involutive map $J_m: T_mM\to T_mM$, $J_m^2=-1_{T_mM}$, is given (T_mM denotes the tangent space of M in the point m). By definition J_m depends smoothly by m.

In this paper we shall consider J-almost-holomorphic functions locally on M. This kind of functions are examined first by Eheresmann (non-published result) and later by R. Hermann (1), which give a beautiful differential-geometric proof of the Ehresmann's result. This is a negative result concerning the local existense of the almost-holomorphic functions corresponding to a special almost-complex structure on the 6-dimensional sphere. Here we propose a positive result in the particular case $\dim(M)=4$, $n=2$, under some appropriate restrictions for J.

Suppose that for $\varphi: U\to\mathbf{R}^{2n}$ we have $\varphi(U)=\mathbf{R}^{2n}$, i. e. the diffeomorphisme φ stretches the carte U on the whole model space $\mathbf{R}^{2n}$. Sometimes we shall be interested to stretch the carte U on the unit open ball or the other open subset of the model space. Therefore the investigation of the J-almost-holomorphic functions (briefly: J-AH-functions) $f: U\to\mathbf{C}$ is reduced to the analogous question for the almost-holomorphic function $f\circ\varphi^{-1}: \mathbf{R}^{2n}\to\mathbf{C}$ relative to a naturally determined almost-complex structure on $\mathbf{R}^{2n}$ (the image of J by the diffeomorphisme $\varphi: U\to\mathbf{R}^{2n}$). For this reason we shall consider $\mathbf{R}^{2n}$ as an almost-complex manifold (AH-manifold) which almost-complex structure (AH-structure) is determined by a matrix $J=\|J_q^p(x)\|$, $x\in\mathbf{R}^{2n}$, with globally defined on $\mathbf{R}^{2n}$ real-analytic coefficients. If $\varphi(U)$ is the unit ball or the other open subset of $\mathbf{R}^{2n}$, we consider not only the induced AH-structure from $\mathbf{R}^{2n}$. In the following, for the notation of the vectors $x=(x_1, x_2, x_3, x_4)$ of $\mathbf{R}_4$, matrix $J(x)$, differential d_xf, functions $J_q^p(x)$ etc., we use two kinds of symbols x and y, $x=(x_1, x_2)$, $y=(y_1, y_2)$. So we have (x, y) instead of x, and respectively $J(x, y)$, $d_{(x,y)}f$ and $J_q^p(x, y)$.

By definition, the function $F: \mathbf{R}^4\to\mathbf{C}$ is called a J-AH-function, if for every $(x, y)\in\mathbf{R}^4$ we have

$$d_{(x,y)}F\circ J(x, y)=id_{(x,y)}F. \tag{1.1}$$

If $F=u+iv$, the matrix representation of $d_{(x,y)}F$ is the following

$$\begin{pmatrix} \partial u/\partial x_1 & \partial u/\partial x_2 & \partial u/\partial y_1 & \partial u/\partial y_2 \\ \partial v/\partial x_1 & \partial v/\partial x_2 & \partial v/\partial y_1 & \partial v/\partial y_2 \end{pmatrix}.$$

We receive a matrix differential equation of first order (an over-determined system with non-constant coefficients) for u and v. Our aim is to solve this equation by means of power series or polynomials. Recall that if J is integrable (the tensor of Nijenhuis vanishes), after a suitable transformation of $\mathbf{R}^4$, we receive that (1.1) is the classical Cauchy — Riemann system in $\mathbf{C}^2$.

The main assumption for J is the following: there exist linear J-almost-holomorphic functions. Of course, we consider the case J non-integrable, hence every two linear J-AH-functions are C-linear dependants. We recall that the following lemma (1) is valid. The AC-structure J on the $2n$-dimensional AC-manifold (M, J) is integrable if and only if for every point $m\in M$ there exists a neighbourhood U of m and J-AH-functions $f_j: U\to\mathbf{C}$, $j=1,\ldots,n$, such that the differentials df_j are $\mathbf{C}$-linear independant.

Our purpose is to prove that

$$F(x,y)=\widetilde{F}(\xi+i\eta), \tag{1.2}$$

where $\xi=a_1x_1+a_2x_2$, $\eta=b_1y_1+b_2y_2$, a_1, a_2, b_1, b_2 are real constants uniquely determined by J, and $\widetilde{F}$ is a holomorphic function of one complex variable. This purpose is realized in this paper for polynomial AC-structure J of degree k (all $J^p_q(x)$ are polynomials of $x_1,\ldots,x_n$ of degree k), and F polynomial of degree n, $k\geqq n$. A counterexample for special real analytic J is given. As the third author shows, the assumption $k\geqq n$ can be omitted, but the proof is not exposed here.

The representation (1.2) involve the elementary foliation $\mathbf{C}\times\mathbf{R}^2$, $\xi+i\eta\in\mathbf{C}$, the affine submanifolds $\{(x,y): a_1x_1+a_2x_2=\xi,\ b_1y_1+b_2y_2=\eta\}$ as leafs. Obviously, F is constant on each leaf $\{\xi+i\eta\}\times\mathbf{R}^2$. It is clear that this foliation can be canonically identified with $\mathbf{R}^4$. The diffeomorphism $\varphi^{-1}: \mathbf{R}^4\to U$ carry the considered foliation on the open set U of M. Finally, we see that the existence of a J-AH-functions implies local foliations on the underlying manifold M of the mentioned above type.

2. Existence of linear almost holomorphic functions. Let $J(x,y)$ be a real analytic AC-structure on $\mathbf{R}^4$, i. e.

$$J(x,y)=\sum_{|\alpha|+|\beta|\geqq 1} G_{\alpha\beta}x^\alpha y^\beta+G_0,$$

α and β be multi-indices, $\alpha=(\alpha_1,\alpha_2)$, $\beta=(\beta_1,\beta_2)$, $|\alpha|=\alpha_1+\alpha_2$, $|\beta|=\beta_1+\beta_2$, $G_{\alpha\beta}$ and G_0 be real (4×4)-matrices. We take $G_0=S$ (precisely, after a suitable linear transformation), where

$$S=\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix},\quad E=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\xi+i\eta$ be a linear J-AH-function

$$\xi=a_1x_1+a_2x_2+a_3y_1+a_4y_2,\quad \eta=b_1x_1+b_2x_2+b_3y_1+b_4y_2. \tag{2.1}$$

The almost-holomorphic condition (AH-condition) (1.1) in matrix notation seems as follows: $(a_1, a_2, a_3, a_4)J(x,y)=-(b_1, b_2, b_3, b_4)$, or

$$(2.2)\qquad (a_1, a_2, a_3, a_4)G_{\alpha\beta} = (0, 0, 0, 0) \text{ for all } \alpha, \beta,$$

$$(2.3)\qquad (a_1, a_2, a_3, a_4)S = -(b_1, b_2, b_3, b_4).$$

In view of $J^2(x, y) = -E_4$, E_4 is the unit (4×4)-matrix, we can write

$$(2.2')\qquad (b_1, b_2, b_3, b_4)G_{\alpha\beta} = (0, 0, 0, 0) \text{ for all } \alpha, \beta,$$

$$(2.3')\qquad (b_1, b_2, b_3, b_4)S = (a_1, a_2, a_3, a_4).$$

Evidentely, (2.2) and respectively (2.2′) are infinite systems of homogenous linear equations with four unknowns. For the existence of J-AH-functions it is necessary that this system have at least one common solution, or that $\bigcap_{\alpha\beta} \operatorname{Ker} G_{\alpha\beta} \neq \{0\}$, considering $G_{\alpha\beta}$ as linear operators in $\mathbf{R}^4$.

Lemma 1. *Let $J(x, y)$ be a non-integrable AH-structure on $\mathbf{R}^4$, $J(x, y) = \Sigma_{|\alpha|+|\beta|\geq 1} G_{\alpha\beta}x^\alpha y^\beta + S$. Linear J-AH-functions exist if and only if* $\dim \bigcap_{|\alpha|+|\beta|\geq 1} \operatorname{Ker} G_{\alpha\beta} = 2$.

Proof. Set $V = \bigcap_{|\alpha|+|\beta|} G_{\alpha\beta}$, $|\alpha|+|\beta|\geq 1$. According to (2.2), (2.2′), (2.3) and (2.3′), it follows that V is an invariant vector subspace of $\mathbf{R}^4$ under the action of the linear operator S. As S is a complex structure on $\mathbf{R}^4$ $(S^2 = -1)$, it is clear that the dimension of V is an even number, i. e. 2 or 4. When $\dim V = 4$ we have $G_{\alpha\beta} = 0$, $|\alpha|+|\beta|\geq 1$, i. e. $J = S$.

Remark. For $\mathbf{R}^{2n}$ we have $\dim V$ is an even number, two.

According to the above remarks we obtain the following possibilities for the matrices $G_{\alpha\beta}$

(i) $G_{\alpha\beta} = (0)$ for all α, β, i. e. $J = S$,

(ii) there exists a positive integer l such that $G_{\alpha\beta} = (0)$ for all $|\alpha|+|\beta|\geq l+1$, and all $G_{\alpha\beta} \neq 0$ when $|\alpha|+|\beta|\leq l$, i. e. the elements of J are the polynomials of degree l, with non-zero coefficients,

(iii) $G_{\alpha\beta} \neq (0)$ for all α, β,

(iv) $G_{\alpha\beta} \neq (0)$ for finitely many α, β,

(v) $G_{\alpha\beta} \neq (0)$ for infinitely many α, β and both $G_{\alpha\beta} = (0)$ for infinitely many α, β.

Remark that the real and imaginery parts of a linear J-AH-function are $\mathbf{R}$-linear independants. Indeed, if we have $(b_1, b_2, b_3, b_4) = \lambda(a_1, a_2, a_3, a_4)$, $\lambda \in \mathbf{R}$, then the vector (b_1, b_2, b_3, b_4) is an eigenvector of the linear operator with matrix S. From $S^2 = -E$ these eigenvalues are $\pm i$, i. e. λ is not real number.

3. Homogenous almost holomorphic polynomials. Let $P(x_1, x_2, y_1, y_2)$ (we shall write $P(x, y)$) be a homogenous polynomial of degree n. Obviously we can write $P(x, y)$ as:

$$(3.1)\qquad P(x, y) = x_1P_1(x, y) + x_2P_2(x, y) + y_1P_3(x, y) + y_2P_4(x, y),$$

where $P_i(x, y)$ are homogenous polynomials of degree $n-1$, $i = 1, 2, 3, 4$.

First we shall prove

Lemma 2. *Let $P(x, y)$ be an almost holomorphic homogenous polynomial of degree n and $P_i(x, y)$ be the polynomials* (3.1). *Then the polynomials $P_i(x, y)$ are almost holomorphical, too.*

Proof. We must prove that the real and imaginary parts of $P_i(x, y)$ satisfy the AH-condition (1.1). We can write the polynomial $P(x, y) = u(x, y) + iv(x, y)$ as the polynomial of "noncommutative" variables x_1, x_2, y_1, y_2.

(3.2) $$P(x,y)=\sum_{k+l+q+p=n} a_{klqp}x_1^k x_2^l y_1^q y_2^p + i \sum_{k+l+q+p=n} b_{klpq}x_1^k x_2^l y_1^q y_2^p.$$

It is easy to see that the coefficient $a_{klqp}(b_{klqp})$ occurs $\binom{n}{k}\binom{n-k}{l}\binom{n-k-l}{q}\times\binom{n-k-l-q}{p}=\frac{n!}{k!\,l!\,q!\,p!}$ times in (3.2).

Using (1.1) for the polynomial $P(x, y)$, and equalizing the coefficients in front of equal degrees of variables, we obtain (1.1) for a_{klqp} and b_{klqp}. Here we use the fact that the right side of (1.1) is the homogenous polynomial with matrix coefficients of degree $n-1$. ($P(x, y)$ is a homogenous polynomial). Hence the matrix coefficients in the left side in front of monomials of total degree higher $n-1$ vanish. Therefore, when the polynomial $P(x, y)$ is a homogenous, we can consider only these products in the left side of (1.1) which are of degree $n-1$, i. e. the products of $a_{klqp}(b_{klqp})$ with the constants in the structure I. (All the rest products are zero.)

The coefficient of the monomial $x_1^k x_2^l y_1^q y_2^p$ are $\frac{(k+1)n!}{(k+1)!\,l!\,q!\,p!}a_{k+1lqp}$ in $\frac{\partial u}{\partial x_1}$, $\frac{(l+1)n!}{k!\,(l+1)!\,q!\,p!}a_{kl+1qp}$ in $\frac{\partial u}{\partial x_2}$, $\frac{(q+1)n!}{k!\,l!\,(q+1)!\,p!}a_{klq+1p}$ in $\frac{\partial u}{\partial y_1}$ and $\frac{(p+1)n!}{k!\,l!\,q!\,(p+1)!}\times a_{klqp+1}$ in $\frac{\partial u}{\partial y_2}$. We obtain the coefficients in $\frac{\partial v}{\partial x_i}$ and $\frac{\partial v}{\partial y_i}$ changing a into b. Obviously the coefficients in front of $a_{k+1lqp}, \dots, a_{klqp+1}$ are equal. Hence taking out a common multiplier we obtain for the coefficients a and b the equation

(3.3) $$(a_{k+1lqp}, a_{kl+1qp}, a_{klq+1p}, a_{klqp+1})J(x,y)=(-b_{k+1lqp}, -b_{kl+1qp}, -b_{klq+1p}, -b_{klqp+1}),$$

which is of the type (1.1).

Varying k, l, q, p we obtain some equations of the type (1.1)

Let $P_j(x, y)=u_j(x, y)+iv_j(x, y)$, $j=1, 2, 3, 4$. We shall prove the statement for $P_1(x, y)$. The coefficient a_{k+1lqp} occurs in u_1 as coefficient in front of the monomial $x_1^k x_2^l y_1^q y_2^p$, a_{kl+1qp} — in front of $x_1^{k-1}x_2^{l+1}y_1^q y_2^p$, a_{klq+1p} — in front of $x_1^{k-1}x_2^l y_1^{q+1} y_2^p$, a_{klqp+1} — in front of $x_1^{k-1}x_2^l y_1^q y_2^{p+1}$. The first coefficient occurs $\frac{(n-1)!}{k!\,l!\,q!\,p!}$ times, the second — $\frac{(n-1)!}{(k-1)!\,(l+1)!\,q!\,p!}$ times, the third — $\frac{(n-1)!}{(k-1)!\,l!\,(q+1)!\,p!}$ times and the fourth — $\frac{(n-1)!}{(k-1)!\,l!\,q!\,(p+1)!}$ times. After derivation on x_j and y_j, $j=1, 2$ the above coefficients are multiplied by k, $l+1$, $q+1$, $p+1$, respectively. As we have seen these four numbers are equal. But in $\frac{\partial u_1}{\partial x_j}$, $\frac{\partial u_1}{\partial y_j}$, $j=1, 2$ they are the coefficients in front of $x_1^{k-1}x_2^l y_1^q y_2^p$, and as we know they satisfy (3.3).

As the coefficients in front of the corresponding degrees of $P_1(x, y)$ satisfy (1.1), and all of them are coefficients in $P(x, y)$, we conclude that the polynomial $P_1(x, y)$ is almost holomorphic.

Hence the polynomials $P_j(x, y)$, $j=1, 2, 3, 4$, are almost holomorphic.

We know (Lemma 1) that linear J-AH-functions exist, if and only if, $\dim \bigcap_{\alpha,\beta} \operatorname{Ker} G_{\alpha\beta}=2$. Let the vectiors (a_1, a_2, a_3, a_4) and (a_1', a_2', a_3', a_4') form the fundamental system of solutions of (2.2). We obtain from (2.3) $-(b_1, b_2, b_3, b_4)=(a_1, a_2, a_3, a_4)S$ and $-(b_1', b_2', b_3', b_4')=(a_1', a_2', a_3', a_4')S$. From (2.2′) (b_1, b_2, b_3, b_4)

$=m(a_1, a_2, a_3, a_4)+n(a'_1, a'_2, a'_3, a'_4)$ and $(b'_1, b'_2, b'_3, b'_4)=s(a_1, a_2, a_3, a_4)+t(a'_1, a'_2, a'_3, a'_4)$, where $m, n, s, t \in \mathbf{R}$. By the multiplication with S, we obtain: $1=-m^2-sn$, $1=-t^2-sn$, $0=-n(m+t)$, $0=-s(m+t)$. Since $s\neq 0$, $n\neq 0$ it follows $m=-t$. It is easy to verify that two obtained linear J-AH-functions are **C**-linear dependent. So we obtain unique **C**-independent linear J-AH-function.

Now we shall specify the expression of every homogenous J-AH-polynomial. Since there are not two **C**-linear independent J-AH-functions, we can suppose that every linear homogenous J-AH-polynomial is of the kind $(\gamma+i\delta)(\xi+i\eta)$, where $\xi=ax_1+bx_2+a'y_1+b'y_2$, $\eta=c'x_1+d'x_2+cy_1+dy_2$, $\gamma, \delta, a, b, a', b', c, d, c', d'$ are real constants.

In the case of J block-antidiagonal

$$J=\left(\begin{array}{c|c} 0 & J_{12}(x, y) \\ \hline J_{21}(x, y) & 0 \end{array}\right),$$

where $J_{12}(x, y)$ and $J_{21}(x, y)$ are 2×2 matrices. It is not difficult to see that if $\xi+i\eta$ is J-AH-function, then the same holds for the linear function $ax_1+bx_2+i(cy_1+dy_2)$. We see that in this case we can take $\xi=ax_1+bx_2$, $\eta=cy_1+dy_2$, i. e. we have a "separation of variables x and y". In the secual we suppose always J is a block-antidiagonal structure.

Under this condition we shall prove

Theorem 1. *Every homogenous J-AH-polynomial of degree n is of the kind* $(\gamma+i\delta)(\xi+i\eta)^n$, *where* $\gamma, \delta\in\mathbf{R}$.

Proof. Let $g(x, y)$ be a homogenous J-AH-polynomial of degree 2. According to Lemma 1

$$g(x, y)=x_1(\gamma_1+i\delta_1)(\xi+i\eta)+x_2(\gamma_2+i\delta_2)(\xi+i\eta)+y_1(\gamma_3+i\delta_3)(\xi+i\eta)+y_2(\gamma_4+i\delta_4)(\xi+i\eta) \tag{3.4}$$

Also

$$\begin{aligned} g(x, y)=\ & x_1((a_{11}x_1+a_{12}x_2+a_{13}y_1+a_{14}y_2)+i(b_{11}x_1+b_{12}x_2+b_{13}y_1+b_{14}y_2)) \\ & +\cdots+y_2((a_{41}x_1+a_{42}x_2+a_{43}y_1+a_{44}y_2)+i(b_{41}x_1+b_{42}x_2+b_{43}y_1+b_{44}y_2)). \end{aligned} \tag{3.5}$$

As at least one of the constants a and b is different from zero, we can take $a\neq 0$. Using from (3.5) $a_{ij}=a_{ji}$ and $b_{ji}=b_{ji}$, by equalizing of the coefficients in (3.4), we obtain $\gamma_1 b=\gamma_2 a$, $\delta_2 a=\delta_1 b$, $-\delta_1 c=\gamma_3 a$, $\delta_3 a=\gamma_1 c$, $-\delta_2 d=\gamma_4 a$, $\delta_4 a=\gamma_1 d$. Thus we can find γ_j, δ_j, $j=1, 2, 3, 4$, substitute them in (3.4) and obtain

$$g(x, y)=\frac{1}{a}(\gamma_1+i\delta_1)(\xi+i\eta)(ax_1+bx_2+i(cy_1+dy_2))=\frac{1}{a}(\gamma_1+i\delta_1)(\xi+i\eta)^2.$$

Let $g(x, y)$ be a homogenous J-AH-polynomial of degree n. From Lemma 1, using induction on n, we see that $g(x, y)$ has the representation

$$\begin{aligned} g(x, y)=\ & x_1(\gamma_1+i\delta_1)(\xi+i\eta)^{n-1}+x_2(\gamma_2+i\delta_2)(\xi+i\eta)^{n-1} \\ & +y_1(\gamma_3+i\delta_3)(\xi+i\eta)^{n-1}+y_2(\gamma_4+i\delta_4)(\xi+in)^{n-1}. \end{aligned} \tag{3.6}$$

On the other hand,

$$
\begin{aligned}
g(x, y) = {} & x_1(a_{n000}x_1^{n-1} + (n-1)a_{n-1100}x_1^{n-2}x_2 + (n-1)a_{n-1010}x_1^{n-2}y_1 \\
& + (n-1)a_{n-1001}x_1^{n-2}y_2 + i(b_{n000}x_1^{n-1} + (n-1)b_{n-1100}x_1^{n-2}x_2 + \cdots)) \\
& + x_2(a_{n-1100}x_1^{n-1} + \cdots + i(b_{n-1100}x_1^{n-1} + \cdots)) \\
& + y_1(a_{n-1010}x_1^{n-1} + \cdots + i(b_{n-1010}x_1^{n-1} + \cdots)) \\
& + y_2(a_{n-1001}x_1^{n-1} + \cdots + i(b_{n-1001}x_1^{n-1} + \cdots)).
\end{aligned} \tag{3.7}
$$

Having in mind the binomial development of $\xi^n = (ax_1 + bx_2)^n$, $\xi^{n-1} = (ax_1 + bx_2)^{n-1}$ $(\xi + i\eta)^{n-1}$, and the equality of the coefficients in (3.7) we can calculate the coefficients in (3.6) $\gamma_2 a^{n-1} = \gamma_1 a^{n-2}b$, $\delta_2 a^{n-1} = \delta_1 a^{n-2}b$, $-\gamma_3 a^{n-1} = \delta_1 a^{n-2}c$ $\delta_3 a^{n-1} = \gamma_1 a^{n-2}c$, $-\gamma_4 a^{n-1} = \delta_2 a^{n-2}d$, $\delta_4 a^{n-1} = \gamma_1 a^{n-2}d$. Herefrom, as in the case $\deg g(x, y) = 2$, we obtain $g(x, y) = (\gamma_1 + i\delta_1)(\xi + i\eta)^n/a$.

4. Nonhomogenous almost holomorphic polynomials. In this paragraph we shall consider the case (ii) for $G_{\alpha\beta}$ (p. 2), i e. the structure J is a block-antidiagonal AC-structure, whose elements are real polynomials of degree k. $(G_{\alpha\beta} = (0)$, $|\alpha| + |\beta| \geq k+1)$. In the following we take all the matrices $G_{\alpha\beta}$ equal, i. e. $G_{\alpha\beta} = G_1$. For this class of AC-structures we prove

Theorem 2. *Let J be an AC-structure which elements be real polynomials of degree k. Let $P(x, y)$ be an AH-polynomial of degree $n \leq k$. Then the homogenous parts of $P(x, y)$ are almost holomorphic, too.*

Proof. According to the above note

$$J = G_1 \sum_{\nu=1}^{k} \sum_{|\alpha| + |\beta| = \nu} x^\alpha y^\beta + G_0.$$

We prove the theorem using induction on n — the degree of the polynomial $P(x, y)$. Let $\deg P = n$, $n \leq k$. We can write

$$P(x, y) = \sum_{\nu=0}^{n} \sum_{|\alpha| + |\beta| = \nu} \frac{\nu!\, a_{\alpha\beta}}{k!\, l!\, q!\, p!} x_1^k x_2^l y_1^q y_2^p + i \sum_{\nu=0}^{n} \sum_{|\alpha| + |\beta| = \nu} \frac{\nu!\, b_{\alpha\beta}}{k!\, l!\, q!\, p!} x_1^k x_2^l y_1^q y_2^p,$$

$$\alpha = (k, l),\ \beta = (q, p).$$

On the other hand, $P(x, y) = \sum_{\mu=0}^{n} P_\mu(x, y)$, where $P_\mu(x, y)$ is a homogenous polynomial of degree μ. Now we prove that $P_n(x, y)$ is an J-AH-polynomial. This is executed, when the coefficients of the P_n satisfy the AH-condition (1.1), i. e.

$$(a_{k+1lqp}, a_{kl+1qp}, a_{klq+1p}, a_{klqp+1})G_1 = (0), \text{ where } k + l + q + p = n - 1, \tag{4.1}$$

$$(a_{k+1lqp}, a_{kl+1qp}, a_{klq+1p}, a_{klqp+1})G_0 = -(b_{k+1lqp}, b_{kl+1qp}, b_{klq+1p}, b_{klqp+1}). \tag{4.2}$$

First we shall prove (4.1). We consider the coefficient in front of the monomials x_1^{k+n-1} and $x_1^{k+n-2}x_2$, in the AH-condition. They are $n(a_{n000}, a_{n-1100}, a_{n-1010}, a_{n-1001})G_1 = (0)$ in front of x_1^{k+n-1} and $n(a_{n000}, a_{n-1100}, a_{n-1010}, a_{n-1001})G_1 + 2\binom{n}{2}(a_{n-1100}, a_{n-2200}, a_{n-2110}, a_{n-2101})G_1 = (0)$ in front of $x_1^{k+n-1}x_2$. (Here we use essentially the fact that $\deg J = k$). Therefore $(a_{n-1100}, a_{n-2200}, a_{n-2110}, a_{n-2101})G_1 = (0)$. In this way we obtain $(a_{n-l+1l-100}, a_{n-ll00}, a_{n-ll-110}, a_{n-ll-101})G_1 = (0)$ and changing x_1 into y_1 and y_2 — $(a_{n-l+10l-10}, a_{n-l1l-10}, a_{n-l0l0}, a_{n-l0l-11})$

$G_1=(0)$ and $(a_{n-l+100l-1}, a_{n-l10l-1}, a_{n-l01l-1}, a_{n-l00l})G_1=(0)$. If we consider the monomial $x_1^{n+k-3}x_2y_1$, receive $(a_{n-2110}, a_{n-3210}, a_{n-3120}, a_{n-3111})G_1=(0)$. In this way, gradually increasing the degree of x_2, y_1, y_2 in the monomial $x_1^{n+k-4}x_2y_1y_2$ we obtain $(a_{k+1lqp}, a_{kl+1qp}, a_{klq+1p}, a_{klqp+1})G_1=(0)$ as the coefficient in front of the monomial $x_1^{2k}x_2^ly_1^qy_2^p$, $k+l+q+p=n-1$, i. e. (4.1).

Now we shall prove (4. 2). This equation occurs in the AH-condition for the polynomial P (as a fragment), which are obtained by equalization of the coefficients in front of the monomials of degree $n-1$. The matrix S_0 do not occur in the equations which are obtained by the monomials of degree from $k+n-1$ to n, as the polynomials $\partial u/\partial x_j$, $\partial u/\partial y_j$, $j=1,2$, are of degree $n-1$.

Now if we consider the monomial x_1^{k+n-2}, as above mentioned we obtain $(a_{n-1000}, a_{n-2100}, a_{n-2010}, a_{n-2001})G_1=(0)$. With similar reasonings we receive

$$(4.3)\qquad (a_{k+1lqp}, a_{kl+1qp}, a_{klq+1p}, a_{klqp+1})G_1=(0), \text{ where } k+l+q+p=\nu, \quad \nu=1, \dots, n-1.$$

Now we equalize the coefficients in AH-condition for the P in front of the $x_1^kx_2^ly_1^qy_2^p$, $k+l+q+p=n-1$ and obtain

$$\frac{n!}{k!\,l!\,q!\,p!}(a_{k+1lqp}, a_{kl+1qp}, a_{klq+1p}, a_{klqp+1})G_0$$

$$+\frac{(n-1)!}{k!\,l!\,q!\,p!}(a_{klqp}, a_{k-1l+1qp}, a_{k-1lq+1p}, a_{k-1lqp+1})G_1+\dots+(a_{1000}, a_{0100}, a_{0010}, a_{0001})G_1$$

$$=-\frac{n!}{k!\,l!\,q!\,p!}(b_{k+1lqp}, b_{kl+1qp}, b_{klq+1p}, b_{klqp+1}).$$

From (4.3) we receive $(a_{k+1lqp}, a_{kl+1qp}, a_{klq+1p}, a_{klqp+1})G_0=-(b_{k+1lqp}, b_{kl+1qp}, b_{klq+1p}, b_{klqp+1})$, i. e. the $P_n(x, y)$ is an J-AH-polynomial. By induction we can conclude that the homogenous parts of the J-AH-polynomial $P(x, y)$ are almost holomorphic, too.

Let the elements of J be polynomials of degree k and J is such a AC-structure as in Theorem 1, then follows

Corollary 1. *Let $P(x, y)$ be an J-AH-Polynomial of degree $n\leq k$. Then $P(x, y)$ has the representation:*

$$P(x, y)=c_0(\xi+i\eta)^n+c_1(\xi+i\eta)^{n-1}+\dots+c_{n-1}(\xi+i\eta)+c_n=\widetilde{P}(\xi+i\eta).$$

5. The other cases. In this paragraph we shall discuss the cases (iii), (iv) and (v) for the structure J, obtained in p. 2. For the cases (iii) and (iv) we state the following

Conjecture. *Let J be an AC block-antidiagonal structure. Every AH-polynomial $P(x, y)$ has the representation $P(x, y)=\widetilde{P}(\xi+i\eta)$.*

For the case (v) we give the following counter-example. Let

$$J(x, y)=\begin{pmatrix} 0 & 0 & 1/(1+x_1) & 0 \\ 0 & 0 & 0 & 2 \\ -1-x_1 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \end{pmatrix}.$$

J is real analytic, when $|x_1|<1$. The generating linear AH-function is $\xi+i\eta=x_2+i2y_2$. Now we consider the following function $f(x,y)=x_1^2-2x_1-i2y_1$. It is easy to verify that this function is AH-function. Obviously $f(x,y)$ can not represent as function of $\xi+i\eta$.

6. An example of AC-structure without linear J-AH-functions. Using paragraph 2 it is not difficult to describe the coefficients J_q^p of the all almost-complex structures J which admit linear J-AH-functions.

Now we give an example of AC-structure which coefficients are polynomials of degree 2. This AC-structure do not admit the linear AH-functions.

Let J be the block-antidiagonal AC-structure with the following coefficients:

$$J_3^1=\frac{2}{3}f^2(x,y)-f(x,y)+1,\quad J_4^1=\frac{2}{3}f^2(x,y)+f(x,y),\ J_3^2=\frac{1}{3}f^2(x,y),$$

$$J_4^2=\frac{1}{3}f^2(x,y)+f(x,y)+1,$$

where $f(x,y)=ax_1+bx_2+cy_1+dy_2$, $a,b,c,d\in\mathbf{R}$, $f(x,y)\neq 0$. The other coefficients of J ($J_1^3, J_2^3, J_3^4, J_4^4$) are uniquely determinate by the above given $J_3^1, J_4^1, J_3^2, J_4^2$, from the condition $J_{12}J_{21}=-E$.

Let $\xi+i\eta$ be a linear J-AH-functions $\xi=c_1x_1+c_2x_2+d_1y_1+d_2y_2$, $\eta=a_1x_1+a_2x_2+b_1y_1+b_2y_2$. Using AH-condition (1.1), we obtain

$$\frac{\partial\xi}{\partial y_1}=J_3^1\frac{\partial\eta}{\partial x_1}+J_3^2\frac{\partial\eta}{\partial x_2},\quad \frac{\partial\xi}{\partial y_2}=J_4^1\frac{\partial\eta}{\partial x_1}+J_4^2\frac{\partial\eta}{\partial x_2},\quad \frac{\partial\xi}{\partial x_1}=J_1^3\frac{\partial\eta}{\partial y_1}+J_1^4\frac{\partial\eta}{\partial y_2},$$

$$\frac{\partial\xi}{\partial x_2}=J_2^3\frac{\partial\eta}{\partial y_1}+J_2^4\frac{\partial\eta}{\partial y_2}.$$

By equalizing of the coefficients in front of equal degrees of x and y, we receive

$$2a_1+a_2=0,\ a_1=0,\ d_1=a_1,\ \text{from where } a_1=a_2=d_1=0,$$

$$2a_1+a_2=0,\ a_1+a_2=0,\ d_2=a_2,\ \text{from where } d_2=0,$$

$$b_2-b_1=0,\ b_1=0,\ c_1=-b_1\ \text{from where } b_1=b_2=c_1=0,$$

$$b_1-b_2=0,\ b_1+b_2=0,\ c_2=-b_2,\ \text{from where } c_2=0.$$

Therefore do not exist nonzero linear J-AH-functions.

7. A class of uniform limits of J-AH-functions. Let us consider a set $AHP_J(\mathbf{R}^4)$ of J-AH-polynomial $P(x,y)$ with the representation (1.2), $P(x,y)=\tilde{P}(\xi+i\eta)$, $\xi=a_1x_1+a_2x_2$, $\eta=b_1y_1+b_2y_2$, $a_1,a_2,b_1,b_2\in\mathbf{R}$.

It is not difficult to prove (with the help of some L^2-estimation of Hörmander (2), ch. 5) that a Wierstrass type theorem for J-AH-functions holds (3), i. e. if $\{f_n\}$ is uniformely convergent on compact subsets of $\mathbf{R}^4$ sequence of J-AH-functions then the limit $f=\lim_n f_n$ is also a J-AH-function.

As consequence we obtain that the above mentioned representation holds for the closure $\overline{AHP_J(\mathbf{R}^4)}$ of $AHP_J(\mathbf{R}^4)$. It is interesting to know if every real-analytic J-AH-function is an uniform limit of J-AH-polynomials.

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