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ON ZEROS OF THE HYPERGEOMETRIC FUNCTION

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We tabulate the zeros of ${}_2F_1(a, b; c; x)$ for $a=1.0$ (0.5) 5.0, $b=1.0$ (0.5) 5.0 and $c=0.5$ 0.5) 12.0 in the range $-8<x<1$. Some entries have been checked by using analytical formulae.

1. Introduction. The power series

$$\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k\, k!} z^k, \tag{1}$$

where z is a complex variable, a, b and c are parameters which can take arbitrary real or complex values (provided that $c\neq 0, -1, -2, \ldots$) and $(\lambda)_k = \Gamma(\lambda+k)/\Gamma(\lambda)=\lambda(\lambda+1)\ldots(\lambda+k-1)$, $k=1, 2, \ldots, (\lambda)_0=1$, is called the hypergeometric series. The sum of the series (1), that is, the function

$${}_2F_1(a, b; c; z)=\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k\, k!} z^k; \quad |z|<1$$

is called Gauss' hypergeometric function or simply hypergeometric function By an appeal to the analytical continuation, the definition can be extended outside the unit circle, that is, in the plane cut along $[1, \infty)$, the desired formula comes out to be [3; 4]:

$${}_2F_1(a, b; c; z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt,$$

$$\mathrm{Re}(c)>\mathrm{Re}(b)>0, \quad \arg(1-z)<\pi.$$

Hypergeometric function has been studied extensively, and several generalizations of this function have appeared in the literature [5; 7; 8].

The function ${}_2F_1$ has been tabulated by Mackiernan [6], Mathai and Saxena [9], Consul [2] and others. Recently the authors [1] have given a table of ${}_2F_1$ for $a=0.5$ (0.5) 5.0, $b=0.5$ (0.5) 5.0, $c=0.5$ (0.5) 12 and $x=-2.50$ (0.05) 0.95.

In the present paper, we present a table containing zeros of the function ${}_2F_1(a, b; c; x)$ in the interval $-8\leq x<1$ for some suitably chosen fixed values of the parameters. A number of entries of the table have been checked by using analytical formulae.

2. The Zeros. We tabulate the zeros of ${}_2F_1(a, b; c; x)$ for a prescribed set of the parameters a, b and c. We consider, $a=1.0$ (0.5) 5.0, $b=1.0$ (0.5) 5.0 and $c=0.5$ (0.5) 12.0. For each set of prescribed values of a, b and c, the x-axis is swept over all values in the interval $-8.0\leq x<1$, observing the change in sign in the value of ${}_2F_1$ for two consecutive values of x ($\Delta x=0.01$). In the case of a change in sign of the value of ${}_2F_1$, indicating the presence of a zero, we calculate it by an appeal to the secant method obtaining a precision up to 0.5×10^{-8}. Once the whole x-axis is swept over, we print all the zeros, omitting the case in which there are no zeros.

To obtain the numerical values of ${}_2F_1(a, b; c; x)$ first we take into account the nature of the parameters a, b and c. If $c=-m$ ($m=0, 1, 2, \dots$) and a (or b) are not equal to $-l$ ($l=0, 1, 2, \dots$) with $l<m$, ${}_2F_1$ is taken as 10^{70}. Under such circumstances the computer interprets that the function is not defined.

For other values of the parameters, x is analyzed in the following form:

(i) If $x=1$, the value of the function is obtained from

$${}_2F_1(a, b; c; 1)=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}; \quad c-a-b\neq m \quad (m=0, 1, 2, \dots).$$

(ii) If $-0.5<x<1$, then the series

$${}_2F_1(a, b; c; x)=\sum_{k=0}^{p}\frac{(a)_k(b)_k}{(c)_k\,k!}\,x^k$$

s employed and p is chosen in such a way that the truncation error of the series be less than 0.5×10^{-13}.

(iii) If $x\leq-0.5$, we use the transformation

$${}_2F_1(a, b; c; x)=(1-x)^{-a}{}_2F_1(a, c-b; c; x/(1-x))$$

and proceed as in the previous case.

3. Special cases. Consider the function ${}_2F_1(c+n, b; c; -x)$, $b\neq c$. This function has obviously n zeros, as ${}_2F_1(c+n, b; c; -x)=(1+x)^{-n-b}{}_2F_1(-n, c-b; c; -x)$, which is a polynomial of degree n. If we set $n=1$, then the only zero of the function ${}_2F_1(c+1, b; c; -x)$ is $c/(b-c)$ whereas the two zeros of the function ${}_2F_1(c+2, b; c; -x)$ are the two roots of the equation $\frac{(c-b)(c-b+1)}{c(c+1)}x^2+2\frac{c-b}{c}x+1=0$.

For example, for $a=2$, $b=3$ and $c=1$ the only zero is $1/(3-1)=0.5$.

Similarly, for $a=3.5$, $b=4$ and $c=1.5$ we have $\frac{(-2.5)(-1.5)}{(1.5)(2.5)}x^2+2\frac{(-2.5)}{1.5}x+1=x^2-\frac{10}{3}x+1$ whose two zeros are 3 and 1/3.

These simple examples are in complete agreement with our tables of zeros of ${}_2F_1$.

We observe from the tables that for $c\geq a$ (or b) there are no zeros of the function ${}_2F_1(a, b; c; x)$ in the prescribed interval.

$a=1$			$a=1.5$			$a=2$		
b	c	zeros of ${}_2F_1$ $a, b ; c ; z)$		c	zeros of ${}_2F_1$ $(a, b ; c ; z)$	b	c	zeros of ${}_2F_1$ $(a, b ; c ; z$
1	0.5	—2.2767175	1.5	0.5	—0.5000000	2	0.5	—0.2264649
1.5	0.5	—1.0000000	1.5	1	—4.7509196	2	1	—1.0000000
2	0.5	—0.6349132	2	0.5	—0.3333333	2	1.5	—7.2724298
2.5	0.5	—0.4641016	2	1	—2.0000000	2.5	0.5	—5.8284271,—0.1715729
3	0.5	—0.3654317	2.5	0.5	—0.2500000	2.5	1	—0.6666667
3.5	0.5	—0.3012584	2.5	1	—1.2444888	2.5	1.5	—3.0000000
4	0.5	—0.2562127	3	0.5	—0.2000000	3	0.5	—3.4922205,—0.1381247
4.5	0.5	—0.2228639	3	1	—0.8989795	3	1	—0.5000000
5	0.5	—0.1971850	3.5	0.5	—0.1666667	3	1.5	—1.8463790
			3.5	1	—0.7023534	3.5	0.5	—2.4608316,—0.1155999
			4	0.5	—0.1428571	3.5	1	—0.4000000
			4	1	—0.5758195	3.5	1.5	—1.3245553
			4.5	0.5	—0.1250000	4	0.5	—1.8890347,—0.0993957
			4.5	1	—0.4877023	4	1	—0.3333333
			5	0.5	—0.1111111	4	1.5	—1.0299000
			5	1	—0.4228672	4.5	0.5	—1.5284563,—0.0871779
						4.5	1	—0.2857143
						4.5	1.5	—0.8413928
						5	0.5	—1.2813900,—0.0776361
						5	1	—0.2500000
						5	1.5	—0.7107194

$a=2.5$			$a=3$		
b	c	zeros of ${}_2F_1\ (a,\ b\ ;\ c\ ;\ z)$	b	c	zeros of ${}_2F_1\ (a,\ b\ ;\ c\ ;\ z)$
2.5	0.5	−2.8693064, −0.1306936	3	0.5	−1.3075574, −0.0854623
2.5	1	−0.4573617	3	1	−3.7320508, −0.2679492
2.5	1.5	−1.5000000	3	1.5	−0.6901179
3	0.5	−1.8944272, −0.1055728	3	2	−2.0000000
3	1	−7.6514837, −0.3485163	3.5	0.5	−1.0000000, −0.0717968
3	1.5	−1.0000000	3.5	1	−2.4488800, −0.2177866
3	2	−4.0000000	3.5	1.5	−0.5278640
3.5	0.5	−1.4114378, −0.0885622	3.5	2	−1.3333333
3.5	1	−4.5280636, −0.2816560	3.5	2.5	−5.0000000
3.5	1.5	−0.7500000	4	0.5	−0.8101282, −0.0619032
3.5	2	−2.4449678	4	1	−1.8164966, −0.1835034
4	0.5	−1.1237229, −0.0762771	4	1.5	−5.5582720, −0.4277159
4	1	−3.1631295, −0.2363738	4	2	−1.0000000
4	1.5	−0.6000000	4	2.5	−3.0418476
4	2	−1.7459667	4.5	0.5	−5.5152785, −0.6810781, −0.0544081
4.5	0.5	−0.9330127, −0.0669873	4.5	1	−1.4414270, −0.1585730
4.5	1.0	−2.4125356, −0.2030599	4.5	1.5	−3.8589777, −0.3596453
4.5	1.5	−0.5000000	4.5	2	−0.8000000
4.5	2	−1.3531236	4.5	2.5	−2.1651514
5	0.5	−0.7974270, −0.0597159	5	0.5	−4.1475522, −0.5875988, −0.0485331
5	1	−1.9423322, −0.1789130	5	1	−1.1937129, −0.1396204
5	1.5	−0.4285714	5	1.5	−2.9294986, −0.3103311
5	2	−1.1027288	5	2	−0.6666667
			5	2.5	−1.6739791

$a=3.5$			$a=4$		
b	c	zeros of ${}_2F_1\,(a,\ b,\ ;\ c\ ;\ z)$	b	c	zeros of ${}_2F_1\,(a,\ b\ ;\ c\ ;\ z)$
3.5	0.5	−4.6627997, −0.7767234, −0.0603769	4	0.5	−2.9493445, −0.5243514, −0.0449733
3.5	1	−1.6862449, −0.1776686	4	1	−7.8729833, −1.0000000, −0.1270167
3.5	1.5	−4.5916501, −0.4083499	4	1.5	−2.0623664, −0.2734241
3.5	2	−0.9238253	4	2	−5.4494897, −0.5505103
3.5	2.5	−2.6000000	4	2.5	−1.1580831
4	0.5	−4.3119411, −0.6359638, −0.0520951	4	3	−3.0000000
4	1	−1.2872939, −0.1500871	4.5	0.5	−2.2398288, −0.4464627, −0.0395661
4	1.5	−3.0000000, −0.3333333	4.5	1	−5.0712662, −0.8186166, −0.1101172
4	2	−0.7084974	4.5	1.5	−1.5724165, −0.2319141
4	2.5	−1.6666667	4.5	2	−3.5491933, −0.4508067
4	3	−6.0000000	4.5	2.5	−0.8898991
4.5	0.5	−3.1654671, −0.5387198, −0.0458131	4.5	3	−2.0000000
4.5	1	−1.0414186, −0.1299441	4.5	3.5	−7.0000000
4.5	1.5	−2.2182458, −0.2817542	5	0.5	−1.8044908, −0.3889163, −0.0353206
4.5	2	−6.5852750, −0.5751717	5	1	−3.7094163, −0.6933851, −0.0971986
4.5	2.5	−1.2500000	5	1.5	−1.2707912, −0.2014159
4.5	3	−3.6377316	5	2	−2.6180340, −0.3819660
5	0.5	−2.4916638, −0.4674521, −0.0408841	5	2.5	−7.6102789, −0.7234823
5	1	−6.4736754, −0.8745556, −0.1145813	5	3	−1.5000000
5	1.5	−1.7559289, −0.2440711	5	3.5	−4.2329857
5	2	−4.5510518, −0.4843292			
5	2.5	−1.0000000			
5	3	−2.5830052			

$a=4.5$			$a=5$		
b	c	zeros of ${}_2F_1(a, b; c; z)$	b	c	zeros of ${}_2F_1(a, b; c; z)$
4.5	0.5	—1.7373073,—0.3815672,—0.0348199	5	0.5	—5.1125000,—1.1730290,—0.2918305,—0.0277671
4.5	1	—3.4587973,—0.6755656,—0.0955608	5	1	—2.0302160,—0.4925584,—0.0746123
4.5	1.5	—1.2215730,—0.1971894	5	1.5	—3.9663867,—0.8258862,—0.1494901
4.5	2	—2.4369860,—0.3713011	5	2	—1.4422001,—0.2695842
4.5	2.5	—6.3062430,—0.6937570	5	2.5	—2.8106446,—0.4705541
4.5	3	—1.3926877	5	3	—7.1622777,—0.8377223
4.5	3.5	—3.5000000	5	3.5	—1.6275246
5	0.5	—7.5486322,—1.4202766,—0.3333333,—0.0310912	5	4	—4.0000000
5	1	—2.6207704,—0.5756034,—0.0844145			
5	1.5	—5.8284271,—1.0000000,—0.1715729			
5	2	—1.8562525,—0.3159091			
5	2.5	—4.0971675,—0.5694991			
5	3	—1.0717968			
5	3.5	—2.3333333			

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