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A CONDITION FOR L_p-INTEGRABILITY OF ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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If $f(z)$ is an entire function of exponential type sk, $0<k<\pi$, $s\geqq 1$, and if under some conditions on $\{\lambda_n\}_{-\infty}^{\infty}$ the series $\sum_{n=-\infty}^{\infty} |f^{(v)}(\lambda_n)|^p$, $p>0$, $v=0, 1, 2, \dots, s-1$, are convergent, then $\int_{-\infty}^{\infty} |f(x)|^p dx$ is convergent too.

Plancherel and Polya [1] proved the following

Theorem 1. *If $f(z)$ is an entire function of exponential type c, $0<c<\pi$, and if the series $\sum |f(n)|^p$, $p>0$ is convergent, then*

$$\int_{-\infty}^{\infty} |f(x)|^p dx < K \sum_{-\infty}^{\infty} |f(n)|^p,$$

where K depends only on p and c (not on f).

In [2] we generalized this theorem. Increasing the type of the function from c to sc, ($0<c<\pi$, $s\geqq 1$ is an integer), we required convergence not only of $\sum |f(n)|^p$, but also of $\sum |f^{(v)}(n)|^p$, $v=1, 2, \dots, s-1$. Under these conditions we established the convergence of $\int_{-\infty}^{\infty} |f(x)|^p dx$. Now we extend the latter theorem, replacing the integers n by complex numbers λ_n.

Theorem 2. *Let $f(z)$ be an entire function of exponential type such that*

$$|f(z)| \leqq A e^{sk|z|}, \tag{1}$$

$A=$ const, $0<k<\pi$, $s\geqq 1$ is an integer. Let $\{\lambda_n\}$ be a sequence of real or complex numbers satisfying the conditions

$$\lambda_0=0, \quad |\lambda_n-n|<L, \quad |\lambda_{n+m}-\lambda_n|\geqq 2\delta>0, \tag{2}$$

where $L=$ const, $\delta=$ const. (It is convenient for our considerations to suppose $L>1$. Evidently, we may do this without any loss of generality.) Let $p>0$ and the series

$$\sum_{n=-\infty}^{\infty} |f(\lambda_n)|^p, \quad \sum_{n=-\infty}^{\infty} |f^{(v)}(\lambda_n)|^p, \quad v=1, 2, \dots, s-1, \tag{3}$$

be convergent. Then

$$\int_{-\infty}^{\infty} |f(x)|^p dx \leqq K_0 \sum_{-\infty}^{\infty} |f(\lambda_n)|^p + K_1 \sum_{-\infty}^{\infty} |f'(\lambda_n)|^p + \dots + K_{s-1} \sum_{-\infty}^{\infty} |f^{(s-1)}(\lambda_n)|^p$$

The constants K_r, $r=0, 1, \dots, s-1$, depend on s, p, L and δ only.

The idea of this theorem aroused from a paper of Boas [3], where he considered the case $s=1$. In the proof of Theorem 2 we employ his method.

For simplicity we shall treat in detail only the case $s=2$. Thus the conditions (1) and (3) take now the form

$$(1') \qquad |f(z)|\leq Ae^{2k|z|}, \quad A=\text{const}, \quad 0<k<\pi.$$

$$(3') \qquad \sum_{n=-\infty}^{\infty}|f(\lambda_n)|^p<\infty, \qquad \sum_{n=-\infty}^{\infty}|f'(\lambda_n)|^p<\infty.$$

In what follows we need two lemmas.

Lemma 1. *Let*

$$(4) \qquad G(z)=z\prod_{n=1}^{\infty}(1-z/\lambda_n)(1-z/\lambda_{-n}),$$

where $\{\lambda_n\}$ *is the sequence from Theorem* 2. *Then* $G(z)$ *is an entire function of exponential type* π *and*

$$(4.1) \qquad |G(x+iy)|\leq B(|x|+1)^{4L}, \quad |y|\leq 3L \quad (z=x+iy)$$

$$(4.2) \qquad |G'(\lambda_k)|\geq C(1+|\lambda_k|)^{-4L-1}, \quad (k=\pm 1, \pm 2, \ldots),$$

$$(4.3) \qquad |G''(\lambda_k)|\leq D(1+3L+|\lambda_k|)^{4L}, \quad (k=\pm 1, \pm 2, \ldots),$$

where the constants B, C *and* D *depend only on* L *and* δ.

Lemma 2. *For the points* $z:|z-\lambda_n|>\delta/2$ *and each* $\sigma>0$

$$(4.4) \qquad \exp\{\pi r(|\sin\theta|-\sigma)\}/G(re^{i\theta})=O(1).$$

The assertions of these lemmas with an exception of (4.3) are well-known (see for instance [3]). So we shall prove only (4.3). Since $|\lambda_n-n|<L$, by the Cauchy Integral Formula

$$G''(\lambda_n)=\frac{2!}{2\pi i}\int_{|z-n|=2L}\frac{G(z)}{(z-\lambda_n)^3}\,dz.$$

Estimating $G''(\lambda_n)$ we may apply (4,1), since for the points $z:|z-n|=2L$ we have $|\operatorname{Im} z|\leq 2L$. For the same points $|z-\lambda_n|\geq L$ and $|\operatorname{Re} z|\leq|\lambda_n|+3L$. Hence

$$|G''(\lambda_n)|\leq 4B(|\lambda_n|+3L+1)^{4L}/L^2=D(|\lambda_n|+3L+1)^{4L},$$

where D depends only on L and δ.

Lemmas 1 and 2 enable us to find a proper interpolation formula for the function $f(z)$, that plays an important role in the proof of Theorem 2.

Let $q>0$ be an integer. Consider

$$J_n=\int_{C_n}f(\zeta)(\zeta-z)G^2(\zeta)\zeta^q d\zeta, \quad (n=1, 2, \ldots)$$

along suitably chosen contours C_n. According to the Residue Theorem

$$(5) \qquad J_n=2\pi i(\operatorname{Res} z+\operatorname{Res} 0+\sum_{k\neq 0}\operatorname{Res}\lambda_k).$$

It is easy to find the residues of the integrand at the points $\zeta=z$ and $\zeta=\lambda_k$, thus $\operatorname{Res} z=f(z)/z^qG^2(z)$,

$$\text{Res}\,\lambda_k = -\frac{f(\lambda_k)}{(\lambda_k-z)^2G'^2(\lambda_k)\lambda_k^q} - \frac{qf(\lambda_k)}{(\lambda_k-z)G'^2(\lambda_k)\lambda_k^{q+1}} - \frac{f(\lambda_k)G''(\lambda_k)}{(\lambda_k-z)G'^3(\lambda_k)\lambda_k^q} - \frac{f'(\lambda_k)}{(\lambda_k-z)G'^2(\lambda_k)\lambda_k^q}.$$

We shall evaluate here in detail the residue at the point $\zeta=0$, which is a pole of order $q+2$. Near $\zeta=0$ we have

$$\frac{f(\zeta)}{(\zeta-z)G^2(\zeta)\zeta^q} = -\frac{1}{z}\frac{1}{\zeta^{q+2}}\frac{f(\zeta)}{G^2(\zeta)/\zeta^2}\frac{1}{1-\zeta/z}$$

$$= -\frac{1}{z}\frac{1}{\zeta^{q+2}}\left(\varphi(0)+\varphi'(0)\zeta+\cdots+\frac{\zeta^{(q+1)}(0)}{(q+1)!}\zeta^{q+1}+\cdots\right)$$

$$\times\left(1+\frac{\zeta}{z}+\frac{\zeta^2}{z^2}+\cdots+\frac{\zeta^{q+1}}{z^{q+1}}+\cdots\right),$$

where $\varphi(\zeta)=f(\zeta)\zeta^2/G^2(\zeta)$ is analytic at the point $\zeta=0$. Thus

$$\text{Res}\,0 = -\frac{1}{z}\left(\frac{\varphi(0)}{z^{q+1}}+\frac{\varphi'(1)}{z^q}+\cdots+\frac{\varphi^{(q+1)}(0)}{(q+1)!}\right)$$

Now if we knew that $J_n\to 0$, when $n\to\infty$, we would have from (5)

$$f(z) = G^2(z)z^q\sum_{k\neq 0}\text{Res}\,\lambda_k + G^2(z)/z^2P_{q+1}(z), \tag{6}$$

which would give us an interpolation formula for the function $f(z)$. (Here $P_{q+1}(z)=\varphi(0)+\varphi'(0)z+\cdots+\varphi^{q+1}(0)z^{q+1}/(q+1)!$ — i. e. this is the sum of the first $q+2$ terms of the Maclaurin series of $\varphi(z)=z^2f(z)/G^2(z)$). But we avoid the necessity of proving that $J_n\to 0$ and establish (6) going another way.

Let $q\geq 16L+5$. Put

$$H(z) = \sum_{-\infty}^{\infty}{}'\frac{f(\lambda_n)G^2(z)z^q}{(\lambda_n-z)^2G'^2(\lambda_n)(\lambda)_n^q} + \sum_{-\infty}^{\infty}{}'\frac{f(\lambda_n)G^2(z)qz^q}{(\lambda_n-z)G'^2(\lambda_n)\lambda_n^{q+1}}$$
$$+\sum_{-\infty}^{\infty}{}'\frac{f(\lambda_n)G''(\lambda_n)G^2(z)z^q}{(\lambda_n-z)G'^3(\lambda_n)\lambda_n^q} - \sum_{-\infty}^{\infty}{}'\frac{f'(\lambda_n)G^2(z)z^q}{(\lambda_n-z)G'^2(\lambda_n)\lambda_n^q}. \tag{7}$$

where the prime indicates the omission of the term with $k=0$.

It follows from the convergence of the series $\Sigma|f(\lambda_n)|^p$ and $\Sigma|f'(\lambda_n)|^p$ that the sequences $\{f(\lambda_n)\}$ and $\{f'(\lambda_n)\}$ are bounded. Let $|f(\lambda_n)|<C_1$, $|f'(\lambda_n)|<C_2$, $(n=0, \pm 1, \pm 2, \ldots)$. Taking this into account, as well as the special choice of q, we find out that all the series in (7) are uniformly convergent in any bounded domain. Hence $H(z)$ is an entire function. Moreover

$$H(\lambda_n)=f(\lambda_n);\quad H'(\lambda_n)=f'(\lambda_n). \tag{8}$$

The first equation is evident, since $G(z)/z-\lambda_n=G'(\lambda_n)$ for $z=\lambda_n$, and the second can be obtained by considering the Taylor series of $G(z)/z-\lambda_n$ about $z=\lambda_n$.

Now consider the function

$$\psi(z)=\frac{f(z)-H(z)-G^2(z)P_{q+1}(z)/z^2}{z^qG^2(z)}. \tag{9}$$

The equations (8) show, that the points λ_n are not singularities of $\psi(z)$. The point $z=0$ is a zero of order $q+2$ of the function $H(z)$. Having in view what

$P_{q+1}(z)$ is, we see, that $z=0$ is a zero of the same order of the function $(f(z)-G^2(z)P_{q+1})z^{-2}$. The point $z=0$ is also a $q+2$ — fold zero of the denominator of $\psi(z)$. Thus $\psi(z)$ has no singularities, so it is an entire function. Moreover it is of exponential type.

Further, to show that $\psi(z)\equiv 0$, we estimate $\psi(re^{i\theta})$ for large r and θ near $\pm\pi/2$.

Denote by T_1, T_2, T_3 and T_4 the consequtive sums on the right-hand side of (7). Then

$$
\begin{aligned}
(10)\qquad |\psi(z)| \leq & |f(z)/z^q G^2(z)| + |T_1/z^q G^2(z)| + |T_2/z^q G^2(z)| + |T_3/z^q G^2(z)| \\
& + |T_4/z^q G^2(z)| + |P_{q+1}(z)/z^{q+2}| = U_1+U_2+U_3+U_4+U_5+U_6.
\end{aligned}
$$

In the first place we notice that the inequality $|f(z)|\leq Ae^{2k|z|}$, $(0<k<\pi)$, ogether with the boundedness of the sequences $\{f(\lambda_n)\}$ and $\{f'(\lambda_n)\}$ implies (see [4]) $|f(z)|\leq Ae^{2k|y|}$. ($y=\operatorname{Im} z$). By this inequality and (4.4) we get

$$|U_1|\leq \frac{M}{r^q}\exp\{-2r[(\pi-k)|\sin\theta|-\pi\sigma]\},$$

where M depends on f, θ and σ, while σ is arbitrarily small. Hence U_1 is bounded on the ray $\arg z=\theta$ for large r, if θ is so near to $\pm\pi/2$, that $(\pi-k)|\sin\theta|-\pi\sigma>0$. Moreover, $U_1\to 0$, when $r\to\infty$.

Consider now U_2. Since the sequence $\{f(\lambda_n)\}$ is bounded and $G'(\lambda_n)$ satisfies (4.2),

$$|U_2|\leq \sum_{-\infty}^{\infty}{}' \frac{|f(\lambda_n)|}{|\lambda_n-z|^2|G'^2(\lambda_n)||\lambda_n^q|}\leq R\sum_{-\infty}^{\infty}{}'\frac{(1+|\lambda_n|)^{8L+2}}{|\lambda_n-z|^2|\lambda_n|^q},$$

($R=$const). It follows from the condition $|\lambda_n-n|\leq L$ that $|\operatorname{Im}\lambda_n|\leq L$. But then for any z for which $|\operatorname{Im} z|\geq 2L$ we have $|z-\lambda_n|\geq|y|-|\operatorname{Im}\lambda_n|\geq y-L\geq L$. Hence, if $|\operatorname{Im} z|\geq 2L$,

$$|U_2|\leq\frac{R}{|y|-L}\sum_{-\infty}^{\infty}{}'\frac{(1+|\lambda_n|^{8L+2})}{|\lambda_n|^q}\leq\frac{R}{L}\sum_{-\infty}^{\infty}{}'\frac{(1+|\lambda_n|)^{8L+2}}{|\lambda_n|^q}$$

and this is bounded, since $q\geq 16L+5$. Moreover, we see that $U_2\to 0$, when $|y|\to\infty$.

Proceeding in the same way (using (4.3) when necessary), we find out that U_3, U_4 and U_5 are also bounded for $|y|\geq 2L$ and tend to zero when $|y|\to\infty$. Finally,

$$|U_6|=|P_{q+1}(z)/z^{q+2}|\to 0,$$

because P_{q+1} is a polinomial of degree at most $q+1$.

Thus $\psi(z)$ is an entire function of exponential type, bounded on four rays ($\arg z=\theta$, θ near $\pm\pi/2$), any two consequtive ones of which make an angle of less than π, and hence by a Phragmen-Lindelöf theorem $\psi(z)$ is bounded everywhere and so is a constant. This constant is zero, since as we noticed, all terms in (10) tend to zero for $z=iy$, $y\to\infty$. So $\psi(z)\equiv 0$. Once this is proved, we have from (9)

$$(11)\qquad f(z)=H(z)+G^2(z)P_{q+1}(z)/z^2,$$

which is desired interpolation formula for $f(z)$.

Now we have everything we need to prove Theorem 2. We have to verify that $\int_{-\infty}^{\infty}|f(x)|^p dx<\infty$.

Let m be an integer. Obviously

$$\int_{-\infty}^{\infty}|f(x)|^p dx\leqq\sum_{m=-\infty}^{\infty}\int_{-L}^{L}|f(x+m)|^p dx=\sum_{m=-\infty}^{\infty}\int_{-L}^{L}|f(x+m-\lambda_m+\lambda_m)|^p dx.$$

In view of the inequality $|m-\lambda_m|\leqq L$, for each x belonging to the interval $-L\leqq x\leqq L$ we have $|z|=|x+m-\lambda_m|\leqq 2L$, i. e. z is a point from the square $-2L\leqq x\leqq 2L$, $-2L\leqq y\leqq 2L$. Therefore, if Γ is the boundary of the square,

$$\max_{-L\leqq x\leqq L}|f(x+m)|\leqq\max_{\Gamma}|f(z+\lambda_m)|=\mu_m.$$

From here, on the base of the above inequalities, we get

$$\int_{-\infty}^{\infty}|f(x)|^p dx\leqq 2L\sum_{m=-\infty}^{\infty}\mu_m^p.$$

It is clear now that our task reduces to estimating μ_m. Put $\lambda_n^m=\lambda_{n+m}-\lambda_m$. Clearly $\lambda_0^m=0$; $|\lambda_n^m-n|\leqq 2L$; $|\lambda_n^m-\lambda_k^m|\geqq 2\delta$. The sequence $\{\lambda_n^m\}$ possesses the properties (2) of the sequence $\{\lambda_n\}$ with constants $2L$ and 2δ.

Let $G_m(z)$ denote the function (4) formed with λ_n^m instead of λ_n. Then $G_m(z)$ satisfies (4.1), (4.2) and (4.3) with L replaced by $2L$ and C and D being independent on m. Let $f_m(z)=f(z+\lambda_m)\sin^{q+2}(\eta z)$, where $q\geqq 32L+5$ and $\eta>0$ is such that $(q+2)\eta+2k<2\pi$ and $\eta<1/L$. Then the type of $f_m(z)$ is less than 2π. Besides, since $\sin\eta z$ and $\cos\eta z$ are bounded in the strip $|y|\leqq 2L$ containing all the points λ_n^m, we have

$$|f_m(\lambda_n^m)|\leqq K_1|f(\lambda_{n+m})|=O(1),\quad |f'_m(\lambda_n^m)|\leqq K_1|f'(\lambda_{n+m})|+K_2|f(\lambda_{n+m})|=O(1).$$

It is seen now, that $f_m(z)$ satisfies all the conditions insuring the validity of the interpolation formula (11). In this case $P_{q+1}(z)\equiv 0$, since $f_m(z)$ has a zero of order $q+2$ at the point $z=0$. Thus, applying (11) we obtain

$$f_m(z)=f(z+\lambda_m)\sin^{q+2}(\eta z)=\sum_{n=-\infty}^{\infty}{}' \frac{f(\lambda_{n+m})G_m^2(z)\sin^{q+2}(\eta\lambda_n^m)z^q}{(\lambda_n^m-z)^2G_m'^2(\lambda_n^m)(\lambda_n^m)^p}$$

$$(13)\qquad +\sum_{n=-\infty}^{\infty}{}' \frac{f(\lambda_{n+m})G_m^2(z)\sin^{q+2}(\eta\lambda_n^m)qz^q}{(\lambda_n^m-z)G'^2(\lambda_n^m)(\lambda_n^m)^{q+1}}+\sum_{n=-\infty}^{\infty}{}' \frac{f(\lambda_{n+m})G_m^2(z)\sin^{q+2}(\eta\lambda_n^m)G_m''(\lambda_n^m)z^q}{(\lambda_n^m-z)G_m'^3(\lambda_n^m)(\lambda_n^m)^q}$$

$$-\sum_{n=-\infty}^{\infty}{}' \frac{f'(\lambda_{n+m})G_m^2(z)\sin^{q+2}(\eta\lambda_n^m)z^q}{(\lambda_n^m-z)G_m'^2(\lambda_n^m)(\lambda_n^m)^q}\quad\sum_{n=-\infty}^{\infty}{}' \frac{f(\lambda^{n+m})G_m^2(z)(q+2)\eta\sin^{q+1}(\eta\lambda_n^m)\cos(\eta\lambda_n^m)z^q}{(\lambda_n^m-z)G_m'^2(\lambda_n^m)(\lambda_n^m)^q}.$$

In order to estimate μ_m we make use of (13). Beforehand we consider the function $G_m(z)/z-\lambda_n^m$ for $z\in\Gamma$.

Suppose λ_n^m fixed. Let first $z\in\Gamma$, $|z-\lambda_n^m|\geqq 1/2$. Since for any $z\in\Gamma$, $|\operatorname{Im} z|\leqq 2L$, $G_m(z)$ satisfies (4.1) and we have

$$|G_m(z)/z-\lambda_n^m|\leqq 2A(|x|-1)^{8L}\leqq 2A(2L+1)^{8L}.$$

Let now $z\in\Gamma$, but $|z-\lambda_n^m|<1/2$. Applying the Maximum principle for the function $G_m(z)/z-\lambda_n^m$, which is analytic in the whole plane, we get

$$|G_m(z)/z-\lambda_n^m| \leq \max_{|z-\lambda_n^m|=1/2} |G_m(z)/z-\lambda_n^m| \leq 2A(|x|+1)^{8L} \leq 2A(2L+3)^{8L},$$

since taking in view that the distance between λ_n^m and Γ is less than $1/2$, we may conclude, that for any $z: |z-\lambda_n^m|=1/2$ we have $|\operatorname{Im} z| \leq 2L+1$ and so (4.1) is valid again. Thus in both cases treated above

$$(14) \qquad |G_m(z)|z-\lambda_n^m| \leq N, \quad z \in \Gamma,$$

where N depends neither on n nor on m (only on L and δ).

Estimating the terms on the right-hand side of (13), we apply (14), as well as (4.1), (4.2) and (4.3). Besides we have in view the boundedness of z^q, $\sin \eta z$ and $\cos \eta z$ along the contour Γ. Let us notice in addition, that $\min |\sin^{q+2} \eta z|$, $=1/\gamma>0$, $(\eta<1/L)$, and $|\lambda_n^m| \geq \delta$, $n \neq 0$. Thus grouping in (13) the four sums which contain $f(\lambda_{n+m})$, we obtain

$$\mu_m \leq \gamma M \left\{ \sum_{n=-\infty}^{\infty}{}' \frac{|f(\lambda_{n+m})|(1+|\lambda_n^m|)^{24L+3}(1+|\lambda_n^m|+6L)^{8L}}{|\lambda_n^m|^q} + \sum_{n=-\infty}^{\infty}{}' \frac{|f'(\lambda_{n+m})|(1+|\lambda_n^m|)^{16L+2}}{|\lambda_n^m|^q} \right\}$$

($M=$ const depends only on L and δ). Next we have

$$(15) \qquad \mu_m \leq \sum_{n=-\infty}^{\infty} |f(\lambda_{n+m})| a_n^m + \sum_{n=-\infty}^{\infty} |f'(\lambda_{n+m})| b_n^m,$$

where

$$a_n^m = M_1(|\lambda_n^m|+7L)^{32L+3}/|\lambda_n^m|^q; \quad b_n^m = M_2(1+|\lambda_n^m|)^{16L+2}/|\lambda_n^m|^q; \quad (M_1, M_2=\text{const}).$$

We know that $|\lambda_n^m| \geq \delta$ when $n \neq 0$ and $|n|-2L \leq |\lambda_n^m| \leq |n|+2L$, hence, if $|n|>2L$

$$a_n^m \leq M_1(|n|+9L)^{32L+3}/(|n|-2L)^q; \quad b_n^m \leq M_2(|n|+3L)^{16L+2}/(|n|-2L)^q,$$

and, in the case when $|n| \leq 2L$

$$a_n^m \leq M_1(11L)^{32L+3}/\delta^q; \quad b_n^m \leq M_2(5L)^{16L+2}/\delta^q.$$

Thus we get $a_n^m \leq a_n$; $b_n^m \leq b_n$, $(a_0=0,\ b_0=0)$, where a_n and b_n do not depend on m and by our choice of q $(q \geq 32L+5)$, the series Σa_n and Σb_n are convergent. It follows now (15)

$$\mu_m \leq \sum_{n=-\infty}^{\infty} |f(\lambda_{n+m})| a_n + \sum_{n=-\infty}^{\infty} |f'(\lambda_{n+m})| b_n,$$

or, which is the same

$$(16) \qquad \mu_m \leq \sum_{v=-\infty}^{\infty} |f(\lambda_v)| a_{v-m} + \sum_{v=-\infty}^{\infty} |f'(\lambda_v)| b_{v-m}.$$

We shall treat separately the cases $p>1$ and $p \leq 1$.

Let first $p>1$. In order to complete the proof of Theorem 2 we need the following

Lemma 3. (See [1]). *Let $p>1$ and let the series $\sum_n |x_n|^p$ and $\sum_n b_n = B$, $b_n>0$, be convergent. If $V_m \leq \sum_v b_{v-m}|x_v|$, then $\sum_m |V_m|^p \leq B^p \sum_v |x_v|^p$.*

The inequality (16) implies

$$\mu_m^p \leq 2^p [(\sum_v a_{v-m} |f(\lambda_v)|)^p + (\sum_v b_{v-m} |f'(\lambda_v)|)^p].$$

Now, summing up along m and applying Lemma 3 we get

$$\sum_m \mu_m^p \leq 2^p [(\sum_v a_v)^p \sum_{v=-\infty}^{\infty} |f(\lambda_v)|^p + (\sum_v b_v)^p \sum_{v=-\infty}^{\infty} |f'(\lambda_v)|^p]$$

or

(17) $$\sum_m \mu_m^p \leq 2^p (A^p \sum_v |f(\lambda_v)|^p + B^p \sum_n |f'(\lambda_v)|^p].$$

Let now $p \leq 1$. In this case we chose q in such a way, that $pq \geqq 32pL + 3p + 2$ (then $q \geq 32L+5$ too). By Jensen's inequality we get from (16)

$$\mu_m^p \leq \sum_v a_{v-m}^p |f(\lambda_v)|^p + \sum b_{v-m}^p |f'(\lambda_v)|^p.$$

By the special choice of q the series $\sum_n a_n^p = A_1$ and $\sum_n b_n^p = B_1$ are convergent thus, summing up along m we obtain

(18) $$\sum_m \mu_m^p \leqq A_1 \sum_v |(f\lambda_v)|^p + B_1 \sum_v |f'(\lambda_v)|^p.$$

The inequalities (12), (17) and (18) show that Theorem 2 is true, i. e.

$$\int_{-\infty}^{\infty} |f(x)|^p dx \leq K_1 \sum_{-\infty}^{\infty} |f(\lambda_n)|^p + K_2 \sum_{-\infty}^{\infty} |f'(\lambda_n)|^p,$$

where the constants K_1 and K_2 depend only on L and δ.

The case $s>2$ of Theorem 2 may be handled in the same way on the base of a respective interpolation formula, which could be found out by means of the integral

$$J_n = \int_{C_n} \frac{f(\zeta)d\zeta}{(\zeta - z)G^s(\zeta)\zeta^q}.$$

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