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VECTOR TOPOLOGIES FOR THE MULTIPLIER EXTENSIONS OF ADMISSIBLE VECTOR MODULES

ÁRPÁD SZÁZ

In this paper, to provide a satisfactory topological foundation for an abstract convolution calculus, vector topologies for the multiplier extensions of admissible vector modules are defined in a natural way. In particular, the Mikusiński field of operators is given in a natural locally convex topology which is, to some extent, compatible with the sequential convergence of Mikusiński too.

1. Notation and corrigendum to [22]. Throughout this paper \mathcal{B} will always stand for an admissible \mathcal{A} -vector module, and we shall focus our attention to its multiplier extension $\mathfrak{M} = \mathfrak{M}(\mathcal{A}, \mathcal{B})$ which is an $\mathfrak{N} = \mathfrak{N}(\mathcal{A}, \mathcal{B})$ -module [22]. Moreover, we shall mainly use the notation and terminology of the first section of [22], therefore the reader is asked to read it thoroughly.

The paper [22] needs some corrigendum. In Definition 1.1 we forgot to stress that $\mathcal{B} \neq \{0\}$. Definition 1.6 contains two misprints, namely \mathfrak{M} must be replaced by \mathfrak{N} in the curly brackets. The addition in \mathfrak{M} , according to our original notation [21], should be denoted by a fat plus sign.

Finally, we remark that meantime we observed that our construction of the multiplier extension of admissible vector modules greatly resembles that of quotient modules defined by Gabriel topologies [19]. Moreover, we also observed that W. Słowikowski had also initiated an abstract theory for generalized functions [16–18], however it seems not to offer that flexibility and power as ours [21–28, 11].

2. Vector topologies for \mathcal{A} and \mathcal{B} . Before topologizing \mathfrak{N} and \mathfrak{M} , we have to assume that there are topologies associated with \mathcal{A} and \mathcal{B} . The following axioms are patterned after the prime example of the admissible \mathcal{O} -vector module \mathcal{S} [21].

Definition 2.1. Suppose that

(IV) \mathcal{A} is equipped with a vector topology such that the algebra multiplication is separately continuous;

(V) \mathcal{B} is equipped with a vector topology such that the module multiplication is separately continuous;

(VI) the topology of \mathcal{A} is finer than the restriction to \mathcal{A} of the topology of \mathcal{B} .

Remarks 2.2. We shall call such modules admissible vector-topological vector modules. Moreover, if the above vector topologies are locally convex, then we shall simply speak of admissible locally convex vector modules.

All the various convolution vector modules occurring in the theories of Schwartz distributions, Mikusiński operators and other similar generalized functions are locally convex. However, for the sake of greater generality, it seems

reasonable to consider more general vector topologies. The reader, who is interested only in the locally convex case, may replace the expression 'vector topology' by 'locally convex topology' throughout this paper.

Since convolutions in the most important particular cases are separately continuous, but surely not continuous, we were allowed to require only the separate continuity of multiplications in axioms (IV) and (V). (Separate continuity in most of the cases implies a stronger continuity property called hypocontinuity [29].)

The reason why the vector topologies in the above axioms are not supposed to be Hausdorff will be explained by our forthcoming procedure. Namely, we shall use inductive limits of vector topologies, and the Hausdorff property is not, in general, preserved by taking inductive limits [1].

3. Inductive limit decompositions of \mathfrak{N} and \mathfrak{M} . The main difficulty in topologizing \mathfrak{N} and \mathfrak{M} is that their elements have no common domains, and therefore the usual theory of function spaces [10] can not be applied directly. To overcome this difficulty, we shall decompose \mathfrak{N} and \mathfrak{M} into the unions of their more simple subspaces.

Definition 3.1. Let \mathfrak{J} be the family of all ideals I in \mathcal{A} such that I is not a divisor of zero in \mathcal{B} . Moreover, for $I \in \mathfrak{J}$, define $\mathfrak{M}_I = \{F \in \mathfrak{M} : I \subset D_F\}$ and $\mathfrak{N}_I = \{\Phi \in \mathfrak{N} : I \subset \Phi^{-1}(\mathcal{A})\}$.

Theorem 3.2. (i) With inclusion \mathfrak{J} is a directed set such that $I_1 \cap I_2 \in \mathfrak{J}$ for all $I_1, I_2 \in \mathfrak{J}$.

(ii) For each $I \in \mathfrak{J}$, \mathfrak{N}_I and \mathfrak{M}_I are subspaces of \mathfrak{N} and \mathfrak{M} , respectively such that $K, \mathcal{A} \subset \mathfrak{N}_I, \mathcal{B}, \mathfrak{N}_I \subset \mathfrak{M}_I$, and $\mathfrak{N}_{\mathcal{A}} * \mathfrak{N}_I \subset \mathfrak{N}_I, \mathfrak{M}_{\mathcal{A}} * \mathfrak{M}_I \subset \mathfrak{M}_I$.

(iii) For each $I_1, I_2 \in \mathfrak{J}$, we have $\mathfrak{N}_{I_1} + \mathfrak{N}_{I_2} \subset \mathfrak{N}_{I_1 \cap I_2}, \mathfrak{M}_{I_1} + \mathfrak{M}_{I_2} \subset \mathfrak{M}_{I_1 \cap I_2}$, and $\mathfrak{N}_{I_1} * \mathfrak{N}_{I_2} \subset \mathfrak{N}_{I_1 * I_2}, \mathfrak{M}_{I_1} * \mathfrak{M}_{I_2} \subset \mathfrak{M}_{I_1 * I_2}$.

(iv) For each $I_1, I_2 \in \mathfrak{J}$ such that $I_1 \supset I_2$, we have $\mathfrak{N}_{I_1} \subset \mathfrak{N}_{I_2}$ and $\mathfrak{M}_{I_1} \subset \mathfrak{M}_{I_2}$.

(v) We have $\mathfrak{N} = \bigcup_{I \in \mathfrak{J}} \mathfrak{N}_I$ and $\mathfrak{M} = \bigcup_{I \in \mathfrak{J}} \mathfrak{M}_I$.

Proof. This follows immediately from the corresponding definitions and from some simple facts proved in the first section of [22]. For example, the last inclusion in (iii) follows immediately from the fact that

$$I_1 * I_2 \subset D_F * \Phi^{-1}(\mathcal{A}) \subset \Phi^{-1}(D_F) = D_{F \circ \Phi} \subset D_{F * \Phi}$$

for all $F \in \mathfrak{M}_{I_1}$ and $\Phi \in \mathfrak{N}_{I_2}$.

4. Vector topologies for \mathfrak{N}_I and \mathfrak{M}_I . For each $I \in \mathfrak{J}$, \mathfrak{N}_I and \mathfrak{M}_I may be identified as vector subspaces of the function spaces \mathcal{A}' and \mathcal{B}' , respectively. The most widely used topologies for function spaces is those of pointwise convergences. Therefore, it seems natural to consider \mathfrak{N}_I and \mathfrak{M}_I to be equipped with the corresponding topologies of pointwise convergences on I [10, p. 220].

Definition 4.1. For each $I \in \mathfrak{J}$, equip \mathfrak{N}_I and \mathfrak{M}_I with the coarsest topologies for which the mappings

$$\Phi \longrightarrow \Phi(\varphi) \text{ from } \mathfrak{N}_I \text{ into } \mathcal{A}, \text{ and } F \longrightarrow F(\varphi) \text{ from } \mathfrak{M}_I \text{ into } \mathcal{B},$$

respectively, are continuous for all $\varphi \in I$.

Theorem 4.2. (i) For each $I \in \mathfrak{J}$, \mathfrak{N}_I and \mathfrak{M}_I are topological vector spaces such that the topologies of K and \mathcal{A} are finer than the restrictions of the topology of \mathfrak{N}_I to K and \mathcal{A} , respectively; and the topologies of \mathcal{B} and

\mathfrak{N}_I are finer than the restrictions of the topology of \mathfrak{M}_I to \mathfrak{B} and \mathfrak{N}_I , respectively. Moreover, the restriction of the multiplication $*$ to $\mathfrak{N}_{\mathcal{A}} \times \mathfrak{N}_I$ and $\mathfrak{M}_{\mathcal{A}} \times \mathfrak{N}_I$ are separately continuous mappings into \mathfrak{N}_I and \mathfrak{M}_I , respectively.

(ii) For each $I_1, I_2 \in \mathfrak{J}$, the restrictions of the addition $+$ to $\mathfrak{N}_{I_1} \times \mathfrak{N}_{I_2}$ and $\mathfrak{M}_{I_1} \times \mathfrak{M}_{I_2}$ are continuous mappings into $\mathfrak{N}_{I_1 \cap I_2}$ and $\mathfrak{M}_{I_1 \cap I_2}$, respectively; and moreover, the restrictions of the multiplication $*$ to $\mathfrak{N}_{I_1} \times \mathfrak{N}_{I_2}$ and $\mathfrak{M}_{I_1} \times \mathfrak{M}_{I_2}$ are separately continuous mappings into $\mathfrak{N}_{I_1 * I_2}$ and $\mathfrak{M}_{I_1 * I_2}$, respectively.

(iii) For each $I_1, I_2 \in \mathfrak{J}$ such that $I_1 \supset I_2$, the topologies of \mathfrak{N}_{I_1} and \mathfrak{M}_{I_1} are finer than the restrictions of the topologies of \mathfrak{N}_{I_2} and \mathfrak{M}_{I_2} to \mathfrak{N}_{I_1} and \mathfrak{M}_{I_1} , respectively.

Proof. This follows immediately from some well-known properties of projective limit topologies [1; 8] and from the corresponding properties of the topologies of \mathcal{A} and \mathfrak{B} . For example, the separate continuity of the mapping $(F, \Phi) \rightarrow F * \Phi$ from $\mathfrak{M}_{I_1} \times \mathfrak{N}_{I_2}$ into $\mathfrak{M}_{I_1 * I_2}$ follows immediately from that of the mappings

$$(F, \Phi) \rightarrow (F * \Phi)(\varphi * \psi) = F(\varphi) * \Phi(\psi) \quad ((\varphi, \psi) \in I_1 \times I_2)$$

from $\mathfrak{M}_{I_1} \times \mathfrak{N}_{I_2}$ into \mathfrak{B} .

Remarks 4.3. (i) If \mathcal{A} and \mathfrak{B} are locally convex, then \mathfrak{N}_I and \mathfrak{M}_I are also locally convex.

(ii) If \mathfrak{B} is Hausdorff, then \mathfrak{M}_I is also Hausdorff. (In this case, by axiom (VI) in Definition 2.1 and (i) in Theorem 4.2, \mathcal{A} and \mathfrak{N}_I are also Hausdorff.)

(iii) If the continuity of multiplications were required in axioms (IV) and (V) in Definition 2.1, then the continuity of multiplications could be stated in Theorem 4.2.

The above assertions are also immediate consequences of the well-known properties of projective limit topologies.

5. Vector topologies for \mathfrak{N} and \mathfrak{M} . To topologize \mathfrak{N} and \mathfrak{M} , we have used projective limit topologies in the preceding section. From Theorem 3.2, it is quite obvious that to topologize \mathfrak{N} and \mathfrak{M} we shall now use inductive limit topologies [1; 8].

Definition 5.1. Equip \mathfrak{N} and \mathfrak{M} with the finest vector topologies for which the identity mappings of the spaces \mathfrak{N}_I and \mathfrak{M}_I into \mathfrak{N} and \mathfrak{M} , respectively, are continuous for all $I \in \mathfrak{J}$.

Theorem 5.2. \mathfrak{M} is an admissible vector-topological \mathfrak{N} -vector module.

Proof. Everything stated here is clear, except perhaps the separate continuity of multiplications. For example, we show that the mapping $(F, \Phi) \rightarrow F * \Phi$ from $\mathfrak{M} \times \mathfrak{N}$ into \mathfrak{M} is separately continuous. For this, by a known property of inductive limit vector topologies [1; 8], we need only to show that, for each $I_1, I_2 \in \mathfrak{J}$, the mapping $(F, \Phi) \rightarrow F * \Phi$ from $\mathfrak{M}_{I_1} \times \mathfrak{N}_{I_2}$ into \mathfrak{M} is separately continuous. However, this follows immediately from (ii) in Theorem 4.2, since the identity mapping of $\mathfrak{M}_{I_1 * I_2}$ into \mathfrak{M} is continuous.

Remark 5.3. In Definition 5.1, \mathfrak{J} may be replaced by any cofinal subset of \mathfrak{J} [10] without changing the topologies of \mathfrak{N} and \mathfrak{M} .

Thus, the special vector-topological properties of \mathfrak{N} and \mathfrak{M} greatly depend on the minimum of cardinal numbers of cofinal subsets of \mathfrak{J} . (Observe that \mathfrak{J} has a finite cofinal subset if and only if $\cap \mathfrak{J} \in \mathfrak{J}$.)

Remark 5.4. If \mathcal{A} and \mathcal{B} are locally convex, then in order that \mathfrak{M} be an admissible locally convex \mathfrak{N} -vector module, we have, in Definition 5.1, to consider \mathfrak{N} and \mathfrak{M} to be equipped with the finest locally convex topologies for which the identity mappings of the spaces \mathfrak{N}_I and \mathfrak{M}_I into \mathfrak{N} and \mathfrak{M} , respectively, are continuous [8]. Namely, incountable inductive limit of locally convex spaces is usually not locally convex [1].

Remark 5.5. The topologies of \mathfrak{N} and \mathfrak{M} are compatible with the Mikusiński-type convergences $\lim_{\mathfrak{N}}$ and $\lim_{\mathfrak{M}}$ [24], respectively, in the sense that they are the finest vector topologies with respect to which the Mikusiński-type convergent nets are still convergent. However, this fact seems to have no particular importance for us.

Remark 5.6. Since the topologies of \mathfrak{N} and \mathfrak{M} do not, in general, inherit the advantageous properties of those of \mathfrak{N}_I and \mathfrak{M}_I (even the convergences in them can not usually be described completely), the reader who does not want to trouble with difficult inductive limit topologies may simply consider \mathfrak{N} and \mathfrak{M} as the union spaces [6] of the spaces \mathfrak{N}_I and \mathfrak{M}_I .

6. An important particular case. One of the most important particular cases is the case when \mathcal{B} has no proper divisors of zero, i. e., for $\varphi \in \mathcal{A}$ and $f \in \mathcal{B}$, we have $f * \varphi = 0$ if and only if $f = 0$ or $\varphi = 0$. Its particular case, when $\mathcal{A} = \mathcal{B}$ is an integral domain, or more specially, a testing function space for Mikusiński operators was studied by several authors.

We shall now consider a slightly more general situation.

Definition 6.1. Let \mathcal{S} be the family of all elements of \mathcal{A} which are not divisors of zero in \mathcal{B} , and suppose that

(VII) $\mathcal{S} \cap I \neq \emptyset$ for all $I \in \mathfrak{J}$.

Remark 6.2. In this case, we have

$$\mathfrak{N} = \left\{ \frac{\psi}{\varphi} : (\varphi, \psi) \in \mathcal{S} \times \mathcal{A} \right\} \text{ and } \mathfrak{M} = \left\{ \frac{f}{\varphi} : (\varphi, f) \in \mathcal{S} \times \mathcal{B} \right\}$$

according to Definition 1.19 in [22].

On the other hand, in this case, the family of all ideals I_φ of \mathcal{A} generated by $\varphi \in \mathcal{S}$ form a cofinal subset of \mathfrak{J} . Thus, by Remark 5.3, we have to consider only the subspaces

$$\mathfrak{N}_\varphi = \mathfrak{N}_{I_\varphi} \text{ and } \mathfrak{M}_\varphi = \mathfrak{M}_{I_\varphi} \quad (\varphi \in \mathcal{S})$$

of \mathfrak{N} and \mathfrak{M} , respectively. Moreover, for each $\varphi \in \mathcal{S}$, \mathfrak{N}_φ and \mathfrak{M}_φ are algebraically and topologically isomorphic to \mathcal{A} and \mathcal{B} , respectively.

Theorem 6.3. The topologies of \mathfrak{N} and \mathfrak{M} are finer than the quotient topologies on \mathfrak{N} and \mathfrak{M} for the mappings

$$(\varphi, \psi) \longmapsto \frac{\psi}{\varphi} \text{ from } \mathcal{S} \times \mathcal{A} \text{ into } \mathfrak{N}, \text{ and } (\varphi, f) \longmapsto \frac{f}{\varphi} \text{ from } \mathcal{S} \times \mathcal{B} \text{ into } \mathfrak{M},$$

respectively.

Proof. Let $\mathcal{S}_{\mathfrak{M}}$ be the topology of \mathfrak{M} as given in Definition 5.1, and denote $\mathcal{S}'_{\mathfrak{M}}$ the finest topology on \mathfrak{M} for which the mapping $(\varphi, f) \longmapsto \frac{f}{\varphi}$ from the product space $\mathcal{S} \times \mathcal{B}$ into \mathfrak{M} is continuous. We have to show that $\mathcal{S}'_{\mathfrak{M}} \subset \mathcal{S}_{\mathfrak{M}}$, i. e., the identity mapping of $(\mathfrak{M}, \mathcal{S}'_{\mathfrak{M}})$ onto $(\mathfrak{M}, \mathcal{S}_{\mathfrak{M}})$ is continuous. For this, we need only to show that for $\varphi \in \mathcal{S}$, the identity mapping of \mathfrak{M}_φ

into $(\mathfrak{M}, \mathcal{S}'_{\mathfrak{M}})$ is continuous. However, this is quite obvious since the identity mapping of \mathfrak{M}_φ into $(\mathfrak{M}, \mathcal{S}'_{\mathfrak{M}})$ is the composition of the mappings $\frac{f}{\varphi} \rightarrow (\varphi, f)$ from \mathfrak{M}_φ into $\mathcal{S} \times \mathcal{B}$ and $(\varphi, f) \rightarrow \frac{f}{\varphi}$ from $\mathcal{S} \times \mathcal{B}$ into $(\mathfrak{M}, \mathcal{S}'_{\mathfrak{M}})$ and the above two mappings are continuous. (The continuity of the mapping $\frac{f}{\varphi} \rightarrow (\varphi, f)$ from \mathfrak{M}_φ into $\mathcal{S} \times \mathcal{B}$ follows at once the fact that it is the composition of the continuous mappings $\frac{f}{\varphi} \rightarrow f = \frac{f}{\varphi}(\varphi)$ from \mathfrak{M}_φ into \mathcal{B} and $f \rightarrow (\varphi, f)$ from \mathcal{B} into $\mathcal{S} \times \mathcal{B}$.)

The corresponding assertion for \mathfrak{N} can be proved quite similarly.

Remark 6.4. The above quotient topologies on \mathfrak{N} and \mathfrak{M} are usually not vector topologies since \mathcal{S} is not a vector space.

A quotient topology for the field of Mikusiński operators was greatly utilized by T. K. Boehme in [4].

It is an interesting question when the above quotient mappings are open for the quotient topologies, or for the original topologies of \mathfrak{N} and \mathfrak{M} .

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Department of Mathematics
University of Debrecen
H-4010 Debrecen, Hungary

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