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ON SOME DEPENDENCIES BETWEEN FUNCTIONAL FORMS IN FUNCTIONAL PROGRAMMING SYSTEMS

ATANAS A. RADENSKY

In the present paper, some dependencies between functional forms in functional programming systems are demonstrated. It is shown that any functional form can be expressed by using only the forms composition, condition and construction.

1. **Introduction.** In the remarkable paper of John Backus [1] the functional style of programming was opposed to the conventional one. It is shown in the present paper that any functional form in the Bachus' functional programming systems can be expressed by means of only three of the functional forms: composition, condition and construction.

The functional programming systems (FPS) consists of a set Ob of objects, a set F of primitive functions (F : Ob → Ob), a set **F** of functional forms (**F** : F → F) and a set of definitions [1]. We will present briefly some definitions concerning the FPS. Let A be a set comprising strings of digits, some specials symbols, the symbols T (true) and F (false), the empty set $\varnothing$. The elements of A will be referred to as atoms. Any atom is an object. If x_i, $i=1, 2, \ldots$, n, are objects then $\langle x_1, x_2, \ldots, x_n \rangle$ is an object. There exists an object $\perp$ (named "undefined") with the property $\langle \ldots, \perp, \ldots \rangle = \perp$. If $f \in$ F, $x \in$ Ob, then $f:x$ ($f:x$ is an object) is the result of the application of f to x. For any f, $f \in$ F, the equality $f:\perp = \perp$ holds. All primitive functions used in the present paper are shown in Table 1. The expression $p_1 \to e_1;\ p_2 \to e_2; \ldots; p_n \to e_n;\ e_{n+1}:x$ means $e_i:x$ if and only if $1 \leq i \leq n$ and $p_j:x=F$, $j=1, 2, \ldots, i-1$, $p_i:x=T$; it means $e_{n+1}:x$ in any other case. The functional forms (Table 2) are expressions denoting functions. Any definition in the FPS is an expression of the type Def $l \equiv r$, where l is a functional symbol and r is a functional form, and for any x, $x \in$ Ob, the equality $l:x=r:x$ holds.

We will define the usual ordering on functions and equivalence in terms of this ordering.

Let f and g are arbitraty functions, then [1]:

— $f \leq g$ iff for all objects x, either $f:x=\perp$, or $f:x=g:x$.

— $f \equiv g$ iff $f \leq g$ and $g \leq f$.

The so called "recursion theorem" is very useful when proving properties of functional programs.

Let f be a solution of $f \equiv p \to g; Q(f)$ where $Q(k) \equiv h \circ [i, k \circ j]$ for any function $k(p, g, h, i, j$ are given functions). Then according to the recursion theorem [1]:

$$f \equiv p \quad \to g;$$

$$p \circ j \to Q(g);$$

$$\cdots\cdots$$
$$p\circ j^n \to Q^n(g);$$
$$\cdots\cdots$$

where $Q^n(g)$ is $h\circ[i,\ Q^{n-1}(g)\circ j]$, and j^n is $j\circ j^{n-1}$ for $n\geq 2$), and $Q^n(g)\equiv /h\circ[i,\ \circ j, \ldots, i\circ j^{n-1},\ g\circ j^n]$.

The first three functional forms from Table 2 can be used to express any other form according to the following theorems:

Theorem 1. *If f, $f\in$ F, is an arbitrary function and $g1, h1$ are defined by:*

Def $g1 \equiv$ null$\circ$tail $\to$ 1 ; $f\circ[1,\ g1\circ$tail], Def $h1\equiv /f$,

then $g1\equiv h1$.

Table 1

Primitive Functions

Function	Definition
null	null : $x\equiv x=\varnothing \longrightarrow T$; $x\neq\varnothing \longrightarrow F$; $\bot$
tail	tail : $x\equiv x=\langle x_1\rangle \longrightarrow \varnothing$; $x=\langle x_1, x_2, \ldots, x_n\rangle$ & $n\geq 2 \longrightarrow \langle x_2, \ldots, x_n\rangle$; $\bot$
id	id : $x\equiv x$
apndl	apndl : $x\equiv x=\langle y, \varnothing\rangle \longrightarrow y$; $x=\langle y, \langle z_1, z_2, \ldots, z_n\rangle\rangle \longrightarrow \langle y, z_1, z_2, \ldots, z_n\rangle$; $\bot$
not	not : $x\equiv x=T \longrightarrow F$; $x=F \longrightarrow T$; $\bot$
1	1 : $x\equiv x=\langle x_1, \ldots, x_n\rangle \longrightarrow x_1$; $\bot$

Table 2

Functional Forms

Name	Denotation	Definition
composition	$f\circ g$	$(f\circ g) : x\equiv f : (g : x)$
construction	$[f_1, \ldots f_n]$	$[f_1, \ldots, f_n] : x\equiv\langle f_1 : x, \ldots f_n : x\rangle$
condition	$p \longrightarrow f$; g	$(p\longrightarrow f;\ g) : x\equiv p : x=T \longrightarrow f : x$; $p : x=F \longrightarrow g : x$; $\bot$
constant	$\bar{x}$	$\bar{x} : y\equiv y=\bot \longrightarrow \bot$; x
insert	$/f$	$/f : x\equiv x=\langle x_1\rangle \longrightarrow x_1$; $x=\langle x_1, \ldots, x_n\rangle$ & $n\geq 2 \longrightarrow$ $f : \langle x_1, /f : \langle x_2, \ldots, x_n\rangle\rangle$; $\bot$
apply to all	αf	$\alpha f : x\equiv x=\varnothing \longrightarrow \varnothing$; $x=\langle x_1, \ldots, x_n\rangle \longrightarrow \langle f : x_1, \ldots, f : x_n\rangle$; $\bot$
binary to unary	bu f x	(bu f x) : $y\equiv f : \langle x, y\rangle$
while	while p f	(while p f) : $x\equiv p : = T \longrightarrow$ (while p f) : $(f : x)$; $p : x=F \longrightarrow x$; $\bot$

Theorem 2. *If f, $f \in$ F, is an arbitrary function and $g2$, $h2$ are defined by:*

Def $g2 \equiv$ null $\to \varnothing$; apndl$\circ[f\circ 1,\ g2\circ$tail], Def $h2 \equiv \alpha f$,

then $h2 \equiv g2$.

Theorem 3. *If f, $f \in$ F, is an arbitrary function, x, $x \in$ Ob, is an arbitrary object and $g3$, $h3$ are defined by*:

Def $g3 \equiv f\circ[\bar{x},$ id], Def $h3 \equiv$ bu $f\ x$,

then $g3 \equiv h3$.

Theorem 4. *If p, f, $p \in$ F, $f \in$ F, are arbitrary functions and $g4$, $h4$ are difined by*:

Def $g4 \equiv$ not$\circ p \to$ id; $g4\circ f$, Def $h4 \equiv$ while $p f$,

then $g4 \equiv f4$.

2. Proofs of the theorems. 2.1. Proof of Theorem 1.

A. From the recursion theorem [1] and from the definition of $g4$ it follows:

(1) $$g1 \equiv \text{null}\circ\text{tail} \to 1;$$
$$\text{null}\circ\text{tail}^2 \to q_1;$$
$$\dots\dots\dots$$
$$\text{null}\circ\text{tail}^{n+1} \to q_n;$$
$$\dots\dots\dots$$

where $q_1 \equiv f\circ[1,\ 1\circ\text{tail}]$, $q_{n+1} \equiv f\circ[1,\ q_n\circ\text{tail}]$, $n = 1, 2, \dots$

B. We will demonstrate that $h1 \leqq g1$.

B1. Let $x = \langle x_1 \rangle$, then $h1 : x = /f : x = x_1$ (according to the definition of the functional form $/f$). Since null$\circ$tail$: x = T$, it follows from (1) that $g1 : x = 1 : x = x_1 = h1 : x$.

B2. Let $x = \langle x_1, x_2 \rangle$, then $h1 : x = /f : x = f : \langle x_1,\ /f : \langle x_2 \rangle\rangle = f : \langle x_1, x_2 \rangle = f : x$. Consequently $g1 : x = h1 : x$.

B3. Suppose the equality

(2) $$g1 : \langle x_1, \dots, x_{k+1} \rangle = h1 : \langle x_1, \dots, x_{k+1} \rangle$$

holds for every k, $1 \leqq k \leqq n-1$. Let $x = \langle x_1, \dots, x_{n+1} \rangle$. From the above assumption: $h1 : \langle x_2, \dots, x_{n+1} \rangle = g1 : \langle x_2, \dots, x_{n+1} \rangle$. Besides null$\circ$tail$: \langle x_2, \dots, x_{n+1} \rangle = \dots =$ nnll$\circ$tail$^{n-1} : \langle x_2, \dots, x_{n+1} \rangle = F$, null$\circ$tail$^n : \langle x_2, \dots, x_{n+1} \rangle = T$, hence $g1 : \langle x_2, \dots, x_{n+1} \rangle = q_{n-1} : \langle x_2, \dots, x_{n+1} \rangle$. In addition $h1 : x = /f : x = f : \langle x_1,\ h1 : \langle x_2, \dots, x_{n+1} \rangle\rangle$, according to the definition of $/f$. Since: null$\circ$tail$: x = \dots =$ null$\circ$tail$^n : x = F$, null$\circ$tail$^{n+1} : x = T$, it follows that $g1 : x = q_n : x = f\circ[1,\ q_{n-1}\circ\text{tail}] : x = f : \langle x_1,\ q_{n-1} : \text{tail} : x \rangle = f : \langle x_1,\ q_{n-1} : \langle x_2, \dots, x_{n+1} \rangle\rangle = f : \langle x_1,\ q1 : \langle x_2, \dots, x_{n+1} \rangle\rangle = f : \langle x_1,\ h1 : \langle x_2, \dots, x_{n+1} \rangle\rangle = h1 : x$, consequently $h1 : x = g1 : x$. Inductively it follows that the equality (2) holds for any k, $k \geqq 1$.

B4. Points B1, B2, B3 and the definition of $/f$ imply $h1 \leqq g1$.

C. We will demonstrate that $g1 \leqq h1$.

Let x, $x \in$ Ob be such an object for which $g1 : x \neq \bot$. Therefore there exists an integer n such, that null$\circ$tail$: x = \dots =$ null$\circ$tail$^{n-1} : x = F$, null$\circ$tail$^n : x = T$. Consequently $x = \langle x_1, \dots, x_n \rangle$ and from points B2 and B3 it follows $g1 \leq h1$.

D. The equality $g1 \equiv h1$ follows from the relations $h1 \leqq g1$ and $g1 \leq h1$ [1].

2.2. Proof of Theorem 2. A. The recursion theorem [1] and the definition of $g2$ give:

$$(3)\qquad \begin{aligned} g2 \equiv \text{null} &\to \bar{\varnothing};\\ \text{null}\circ\text{tail} &\to q_1(\bar{\varnothing});\\ &\ldots\ldots\\ \text{null}\circ\text{tail}^n &\to q_n(\bar{\varnothing});\\ &\ldots\ldots \end{aligned}$$

with $q_1(\bar{\varnothing}) = \text{apndl}\circ[f\circ 1,\ \bar{\varnothing}]$, $q_n(\bar{\varnothing}) = \text{apndl}\circ[f\circ 1,\ q_{n-1}(\bar{\varnothing})\circ\text{tail}]$, $n=2, 3, \ldots$.

B. We will demonstrate that $h2 \leq g2$.

B1. Let $x=\varnothing$, then $h2 : x = \alpha f : x = \varnothing$ (according to the definition of the functional form αf). Since $\text{null} : x = T$, then $g2 : x = \bar{\varnothing} : x = \varnothing = h2 : x$.

B2. Let $x = \langle x_1\rangle$, then $h2 : x = \alpha f : x = \langle f : x_1\rangle$ (according to the definition of αf). In view of the fact that $\text{null} : x = F$, $\text{null}\circ\text{tail} : x = T$ it follows $g2 : x = q_1(\bar{\varnothing}) : x = \text{apndl}\circ[f\circ 1,\ \bar{\varnothing}] : x = \text{apndl} : \langle f : x_1,\ \varnothing\rangle = \langle f : x_1\rangle = h2 : x$. Consequently $g2 : x = h2 : x$.

B3. Suppose the equality:

$$(4)\qquad g2 : \langle x_1, \ldots, x_k\rangle = h2 : \langle x_1, \ldots, x_k\rangle$$

holds for every k, $1 \leq k \leq n-1$. Let $x = \langle x_1, \ldots, x_n\rangle$. From the above assumption: $h2 : \langle x_2, \ldots, h_n\rangle = g2 : \langle x_2, \ldots, x_n\rangle$. Besides that we have $\text{null} : \langle x_2, \ldots, x_n\rangle = \text{null}\circ\text{tail} : \langle x_2, \ldots, x_n\rangle = \ldots = \text{null}\circ\text{tail}^{n-2} : \langle x_2, \ldots, x_n\rangle = F$, $\text{null}\circ\text{tail}^{n-1} : \langle x_2, \ldots, x_n\rangle = T$, hence $g2 : \langle x_2, \ldots, x_n\rangle = q_{n-1}(\bar{\varnothing}) : \langle x_2, \ldots, x_n\rangle$. In addition $h2 : x = \alpha f : x = \langle f : x_1, \ldots, f : x_n\rangle$, according to the definition of αf. Since $\text{null}\circ\text{tail} : x = \ldots = \text{null}\circ\text{tail}^{n-1} : x = F$, $\text{null}\circ\text{tail}^n : x = T$, then $g2 : x = q_n(\bar{\varnothing}) : x = \text{apndl}\circ[f\circ 1\ q_{n-1}(\bar{\varnothing})\circ\text{tail}] : x = \text{apndl} : \langle f\circ 1 : x,\ q_{n-1}(\bar{\varnothing})\circ\text{tail} : x\rangle = \text{apndl} : \langle f : x_1,\ q_{n-1}(\bar{\varnothing}) : \langle x_2, \ldots, x_n\rangle\rangle = \text{apndl} : \langle f : x_1, g2 : \langle x_2, \ldots, x_n\rangle\rangle = \text{apndl} : \langle f : x_1, \alpha f : \langle x_2, \ldots, x_n\rangle\rangle = \text{apndl} : \langle f : x_1, \langle f : x_2, \ldots, f : x_n\rangle\rangle = \langle f : x_1, f : x_2, \ldots, f : x_n\rangle = h2 : x$. Consequently $h2 : x = g2 : x$.

Inductively it follows that the equality (4) holds for any k, $k \geq 1$.

B4. Points B1, B2, B3 and the definition of αf imply $h2 \leq g2$.

C. We will demonstrate that $g2 \leq h2$. Let x, $x \in \text{Ob}$, be such an object that $g2 : x$ is defined. Consequently $x = \varnothing$ or $x = \langle x_1, \ldots, x_n\rangle$, $n \geq 1$. From points B1 and B3 it follows $g2 : x = h2 : x$, so $g2 \leq h2$.

D. The equality $g2 \equiv h2$ follows from the relations $h2 \leq g2$ and $g2 \leq h2$.

2.3. Proof of Theorem 3. Let $y, y \in \text{Ob}$ is an arbitrary object, then $h3 : y = (\text{bu}\ f\ x) : y = f : \langle x,\ y\rangle$ and $g3 : y = f\circ[\bar{x},\ \text{id}] : y = f : \langle \bar{x} : y,\ \text{id} : y\rangle = f : \langle x : y, y\rangle$. If $y = \perp$, then $h3 : y = f : \langle x,\ \perp\rangle = f : \perp = \perp$ and $g3 : \perp = f : \langle \bar{x} : \perp,\ \perp\rangle = f : \perp = \perp$. If $y \neq \perp$ then $g3 : y = f : \langle x,\ y\rangle$. Consequently $g3 \equiv g3$.

2.4. Proof of Theorem 4. A. Let f_i, $i = 0, 1, 2, \ldots$ are functions which are defined in the following way:

$$\begin{aligned} &\text{Def } f_0 \equiv \perp,\\ &\text{Def } f_n \equiv \text{not}\circ p \to \text{id};\\ &\qquad\quad \text{not}\circ p\circ f \to \text{id}\circ f;\\ &\qquad\quad \ldots\ldots\\ &\qquad\quad \text{not}\circ p\circ f^{n-1} \to \text{id}\circ f^{n-1};\ \perp,\quad \text{where}\quad n = 1, 2, \ldots \end{aligned}$$

For any h, $h \in$ F, let denote $E(h) \equiv$ notop $\rightarrow$ id ; h of. It is obvious that $E(f_0) \equiv E(\perp) \equiv$ notop $\rightarrow$ id ; $\perp$of $\equiv$ notop $\rightarrow$ id ; $\perp \equiv f_1$. Suppose that for $k = 1, 2, \ldots, n-1$, $n > 1$, the equality $f_k \equiv E(f_{k-1})$ holds. Therefore $E(f_n) \equiv$ notop $\rightarrow$ id ; f_no$f \equiv$ notop $\rightarrow$ id; (notop $\rightarrow$ id ; notopof $\rightarrow$ idof ; ...; notopo$f^{n-1} \rightarrow$ idof^{n-1}; $\perp$)of $\equiv$ not o $p \rightarrow$ id ; notopof $\rightarrow$ id of; ...; notopo$f^n \rightarrow$ idof^n; $\perp \equiv f_{n+1}$. Inductively it follows that $E(f_n) \equiv f_{n+1}$ for $n = 0, 1, 2, \ldots$ The extension theorem [1] gives:

(5) $g4 \equiv$ notop $\rightarrow$ id ;

notopof $\rightarrow$ idof ;

. . . .

notopo$f^n \rightarrow$ idof^n ;

. . . .

B. On the basis of (5), the rest of the proof of Theorem 4 is the same as the proofs of Theorems 1 and 2. In a similar way as with Theorems 1 and 2 the relations $g4 \leq h4$ and $h4 \leq g4$ can be demonstrated and since $g4 \equiv h4$.

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Centre for Mathematics and Mechanics
1090 Sofia P.O.Box, 373

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