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AN ANALOGUE OF A GRAY'S THEOREM

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In the presented paper we give an analogue of the well known classical theorem of Bonnet-Meyers in the riemannian geometry and of a theorem of A. Gray in the almost hermitian geometry.

Recently A. Gray has proved the following theorem.

Theorem. *Let M be a connected complete almost hermitian manifold. Assume the holomorphic curvature of M satisfies*

$$(*) \quad H(x) - \|\nabla_x Jx\|^2 \geq \delta > 0$$

for all $x \in M_m$ with $\|x\|=1$ and all $m \in M$. Then M is compact and the diameter of M is not greater than $\pi/\sqrt{\delta}$.

In this paper we give an extended result of the theorem of Gray.

Theorem. *Let M be a connected complete almost hermitian manifold. Assume that for all $x \in M_m$ with $\|x\|=1$ and all $m \in M$ the holomorphic and the sectional curvature of M satisfies*

$$H(x) + K\left(x, \frac{(\nabla_x J)x}{\|(\nabla_x J)x\|}\right) + K\left(x, J \frac{(\nabla_x J)x}{\|(\nabla_x J)x\|}\right)$$

(1)

$$- \|\nabla_x Jx\|^2 - \left\| \nabla_x \frac{(\nabla_x J)x}{\|(\nabla_x J)x\|} \right\|^2 - \left\| \nabla_x J \frac{(\nabla_x J)x}{\|(\nabla_x J)x\|} \right\|^2 \geq 3\delta > 0.$$

Then M is compact and the diameter of M is not greater than $\pi/\sqrt{\delta}$.

Proof. Let $p, q \in M$. Since M is a connected complete manifold there exists a unit speed minimal geodesic σ defined on $[0, b]$ from $p = \sigma(0)$ to $q = \sigma(b)$ [3]; [4]. Let X be the vector field on σ defined by

$$(1) \quad X(t) = \sin(\pi t/b)u(t),$$

where u is a unit vector field on σ orthogonal to the tangent vector field σ' . Then $0 \leq I(X, X) = \int_0^b (\|X^1\|^2 - R(X, \sigma^1, \sigma^1, X)) dt$. Since $X^1 = \Delta_{\sigma'} X = \sin(\pi t/b) \nabla_{\sigma'} X + (\pi \cos(\pi t/b)u)/b$, it follows

$$(3) \quad 0 \leq \int_0^b \left(\frac{\pi^2}{b^2} \cos^2 \frac{\pi t}{b} + \sin^2 \frac{\pi t}{b} (\|\nabla_{\sigma'} u\|^2 - R(u, \sigma^1, \sigma^1, u)) \right) dt.$$

First we put $u_1 = J\sigma'$ and get

$$(4) \quad 0 \leq \int_0^b \left(\frac{\pi^2}{b^2} \cos^2 \frac{\pi t}{b} + \sin^2 \frac{\pi t}{b} (\|\nabla_{\sigma'} J\sigma'\|^2 - H(\sigma')) \right) dt.$$

The field $u_2 = \nabla_{\sigma'} J\sigma' / \|\nabla_{\sigma'} J\sigma'\|$ is also unit and orthogonal to σ' . From (3) we get

$$(5) \quad 0 \leq \int_0^b \left\{ \frac{\pi^2}{b^2} \cos^2 \frac{\pi t}{b} + \sin^2 \frac{\pi t}{b} \left[\left\| V_{\sigma'} J \frac{V_{\sigma'} J \sigma^1}{\|V_{\sigma'} J \sigma^1\|} \right\|^2 - R \left(\frac{V_{\sigma'} J \sigma^1}{\|V_{\sigma'} J \sigma^1\|}, \sigma', \sigma', \frac{V_{\sigma'} J \sigma^1}{\|V_{\sigma'} J \sigma^1\|} \right) \right] \right\} dt.$$

If we put now $J u_2$ in (3) we get

$$(6) \quad 0 \leq \int_0^b \left\{ \frac{\pi^2}{b^2} \cos^2 \frac{\pi t}{b} + \sin^2 \frac{\pi t}{b} \left[\left\| V_{\sigma'} J \frac{V_{\sigma'} J \sigma^1}{\|V_{\sigma'} J \sigma^1\|} \right\|^2 - R \left(J \frac{V_{\sigma'} J \sigma^1}{\|V_{\sigma'} J \sigma^1\|}, \sigma', \sigma^1, J \frac{V_{\sigma'} J \sigma^1}{\|V_{\sigma'} J \sigma^1\|} \right) \right] \right\} dt.$$

We sum (4), (5) and (6):

$$0 \leq \int_0^b \left\{ 3 \frac{\pi^2}{b^2} \cos^2 \frac{\pi t}{b} + \sin^2 \frac{\pi t}{b} \left[\left\| V_{\sigma'} J \sigma' \right\|^2 + \left\| V_{\sigma'} J \frac{V_{\sigma'} J \sigma^1}{\|V_{\sigma'} J \sigma^1\|} \right\|^2 + \left\| V_{\sigma'} J \frac{V_{\sigma'} J \sigma^1}{\|V_{\sigma'} J \sigma^1\|} \right\|^2 - H(\sigma') - K(\sigma', \frac{V_{\sigma'} J \sigma^1}{\|V_{\sigma'} J \sigma^1\|}) - K(\tau', J \frac{V_{\sigma'} J \sigma^1}{\|V_{\sigma'} J \sigma^1\|}) \right] \right\} dt.$$

By the condition (1) we have

$$0 \leq \int_0^b \left(3 \frac{\pi^2}{b^2} \cos^2 \frac{\pi t}{b} - 3\delta \sin^2 \frac{2\pi t}{b} \right) dt$$

and making use of $\int_0^\pi \cos^2 t dt = \int_0^\pi \sin^2 t dt = \pi/2$ we get $0 \leq \pi^2/b^2 - \delta$ which implies $b \leq \pi/\sqrt{\delta}$. Since $b = d(p, q)$ and p, q are arbitrary points of M it follows $d(M) \leq \pi/\sqrt{\delta}$.

Remark 1. If we consider the 3-dimensional space E_x spanned by the vectors $Jx, \frac{(V_x J)x}{\|(V_x J)x\|}, J \frac{(V_x J)x}{\|(V_x J)x\|}$, then $x \perp E_x$ and

$$\rho_{E_x}(x) = H(x) + K(x, \frac{(V_x J)x}{\|(V_x J)x\|}) + K(x, J \frac{(V_x J)x}{\|(V_x J)x\|})$$

is the Ricci curvature of x in respect to E_x [1].

Remark 2. The same conclusion as in the both above theorems holds good if instead of (*) or (†) one of the following two conditions is true:

$$K(x, \frac{(V_x J)x}{\|(V_x J)x\|}) - \left\| V_x \frac{(V_x J)x}{\|(V_x J)x\|} \right\|^2 - \delta > 0;$$

$$K(x, J \frac{(V_x J)x}{\|(V_x J)x\|}) - \left\| V_x J \frac{(V_x J)x}{\|(V_x J)x\|} \right\|^2 \geq \delta > 0.$$

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