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AN ANALOGUE OF A GRAY'S THEOREM

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In the presented paper we give an analogue of the well known classical theorem of Bonnet-Meyers in the riemannian geometry and of a theorem of A. Gray in the almost hermitian geometry.

Recently A. Gray has proved the following theorem.

Theorem. *Let M be a connected complete almost hermitian manifold. Assume the holomorphic curvature of M satisfies*

$$(*)\qquad H(x)-\|(\nabla_x J)x\|^2 \geqq \delta > 0$$

for all $x \in M_m$ with $\|x\|=1$ and all $m \in M$. Then M is compact and the diameter of M is not greater than $\pi/\sqrt{\delta}$.

In this paper we give an extended result of the theorem of Gray.

Theorem. *Let M be a connected complete almost hermitian manifold. Assume that for all $x \in M_m$ with $\|x\|=1$ and all $m \in M$ the holomorphic and the sectional curvature of M satisfies*

$$(1)\qquad \begin{aligned} &H(x)+K\Big(x, \frac{(\nabla_x J)x}{\|(\nabla_x J)x\|}\Big)+K\Big(x, J\frac{(\nabla_x J)x}{\|(\nabla_x J)x\|}\Big) \\ &-\|(\nabla_x J)x\|^2-\Big\|\nabla_x \frac{(\nabla_x J)x}{\|(\nabla_x J)x\|}\Big\|^2-\Big\|\nabla_x J\frac{(\nabla_x J)x}{\|(\nabla_x J)x\|}\Big\|^2 \geqq 3\delta > 0. \end{aligned}$$

Then M is compact and the diameter of M is not greater than $\pi/\sqrt{\delta}$.

Proof. Let $p, q \in M$. Since M is a connected complete manifold there exists a unit speed minimal geodesic σ defined on $[0, b]$ from $p=\tau(0)$ to $q=\sigma(b)$ [3]; [4]. Let X be the vector field on σ defined by

$$(1)\qquad X(t)=\sin(\pi t/b)u(t),$$

where u is a unit vector field on σ orthogonal to the tangent vector field σ', Then $0 \leqq I(X, X)=\int_0^b(\|X^1\|^2-R(X, \sigma^1, \sigma^1, X))dt$. Since $X^1=\Delta_{\sigma'}X=\sin(\pi t/b)\nabla_{\sigma'}X$ $+(\pi\cos(\pi t/b)u/b$, it follows

$$(3)\qquad 0 \leqq \int_0^b\Big(\frac{\pi^2}{b^2}\cos^2\frac{\pi t}{b}+\sin^2\frac{\pi t}{b}(\|\nabla_{\sigma'}u\|^2-R(u, \sigma^1, \sigma^1, k))\Big)dt.$$

First we put $u_1=J\sigma'$ and get

$$(4)\qquad 0 \leqq \int_0^b\Big(\frac{\pi^2}{b^2}\cos^2\frac{\pi t}{b}+\sin^2\frac{\pi t}{b}(\|\nabla_{\sigma'}J\sigma'\|^2-H(\sigma'))\Big)dt.$$

The field $u_2=\nabla_{\sigma'}J\sigma'/\|(\nabla_{\sigma'}J\sigma'\|$ is also unit and orthogonal to σ'. From (3) we get

(5) $$0\leq\int_0^b\left\{\frac{\pi^2}{b^2}\cos^2\frac{\pi t}{b}+\sin^2\frac{\pi t}{b}\left[\left\|\nabla_{\sigma'}\frac{\nabla_{\sigma'}J\sigma^1}{\|\nabla_{\sigma'}J\sigma^1\|}\right\|^2-R\left(\frac{\nabla_{\sigma'}J\sigma^1}{\|\nabla_{\sigma'}J\sigma^1\|},\ \sigma',\ \sigma',\ \frac{\nabla_{\sigma'}J\sigma^1}{\|\nabla_{\sigma'}J_{\sigma'}\|}\right)\right]\right\}dt.$$

If we put now J_{u_2} in (3) we get

(6)
$$0\leq\int_0^b\left\{\frac{\pi^2}{b^2}\cos^2\frac{\pi t}{b}+\sin^2\frac{\pi t}{b}\left[\left\|\nabla_{\sigma'}J\frac{\nabla_{\sigma'}J\sigma^1}{\|\nabla_{\sigma'}J\sigma^1\|}\right\|^2-R\left(J\frac{\nabla_{\sigma}J\sigma^1}{\|\nabla_{\sigma'}J\sigma^1\|},\ \sigma',\ \sigma^1,\ J\frac{\nabla_{\sigma'}J\sigma^1}{\|\nabla_{\sigma'}J\sigma^1\|}\right)\right]\right\}dt.$$

We sum (4), (5) and (6):

$$0\leq\int_0^b\{3\ \frac{\pi^2}{b^2}\cos^2\frac{\pi t}{b}+\sin^2\frac{\pi t}{b}\left[\|\nabla_{\sigma'}J\sigma'\|^2+\left\|\nabla_{\sigma'}\frac{\nabla_{\sigma'}J\sigma'}{\|\nabla_{\sigma'}J\sigma^1\|}\right\|^2\right.$$

$$\left.+\left\|\nabla_{\sigma'}J\frac{\nabla_{\sigma'}J\sigma'}{\|\nabla_{\sigma'}J\sigma^1\|}\right\|^2-H(\sigma')-K\left(\sigma',\frac{\nabla_{\sigma'}J\sigma^1}{\|\Delta_{\sigma'}J\sigma^1\|}\right)-K\left(\tau',J\frac{\nabla_{\sigma'}J_{\tau^1}}{\|\nabla_{\sigma'}J_{\tau^1}\|}\right)\right]\}dt.$$

By the condition (1) we have

$$0\leq\int_0^b\left(3\ \frac{\pi^2}{b^2}\cos^2\frac{\pi t}{b}-3\delta\sin\frac{2\pi t}{b}\right)dt$$

and making use of $\int_0^\pi\cos^2 t dt=\int_0^\pi\sin^2 t\, dt=\pi/2$ we get $0\leq\pi^2/b^2-\delta$ which implies $b\leq\pi\sqrt{\delta}$. Since $b=d(p,q)$ and p, q are arbitrary points of M it follows $d(M)\leq\pi/\sqrt{\delta}$.

Remark 1. If we consider the 3-dimensional space E_x spanned by the vectors Jx, $\frac{(\nabla_x J)x}{\|(\nabla_x)xJ\|}$, $J\frac{(\nabla_x J)x}{\|(\nabla_x J)x\|}$, then $x\perp E_x$ and

$$\varrho_{E_x}(x)=H(x)+K\left(x,\ \frac{(\nabla_x J)x}{\|(\nabla_x J)x\|}\right)+K\left(x,\ J\frac{(\nabla_x J)x}{\|(\nabla_x J)x\|}\right)$$

is the Ricci curvature of x in respect to E_x [1].

Remark 2. The same conclusion as in the both above theorems holds good if instead of (*) or (1) one of the following two conditions is true:

$$K\left(x,\ \frac{(\nabla_x J)x}{\|\nabla_x J)x\|}\right)-\left\|\nabla_x\frac{(\nabla_x J)x}{\|(\nabla_x J)x\|}\right\|^2\geq\delta>0;$$

$$K\left(x,\ J\frac{(\nabla_x J)x}{\|\nabla_x J)x\|}\right)-\left\|\nabla_x J\frac{(\nabla_x J)x}{\|(\nabla_x J)x\|}\right\|^2\geq\delta>0.$$

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