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ON NEUMAN'S PROBLEM FOR A CLASS EQUATIONS WITH NONNEGATIVE CHARACTERISTIC FORMS

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In this paper we prove the existence, the uniqueness and the regularity of a classical solution to Neuman's problem for a class of equations of second order with nonnegative characteristic forms. We use the method of elliptic regularization.

1. Introduction and results. After the paper of Keldish [6], the boundary value problems for equations of second order with nonnegative characteristic forms

$$Lu=\sum_{ij=1}^{n} a^{ij}(x)u_{x_ix_j}+\sum_{i=1}^{n} b^i(x)u_{x_i}+c(x)u=f(x),$$

$$\sum_{ij=1}^{n} a^{ij}(x)\xi^i\xi^j\geqq 0 \qquad x\in\Omega,\quad \xi\in\mathbf{R}^n, \tag{1}$$

were investigated by many authors. In 1956 G. Fichera [9], [10] gave the basis to the general boundary value problems for (1). Following Fichera we classify the boundary of the domain Ω into four types $\partial\Omega=\Sigma_1\subset\Sigma_2\cup\Sigma_3\cup\Sigma_4$, according to the sign of function $b(x)=\Sigma_{i=1}^n(b^i(x)-\Sigma_{j=1}^n a_x^{ij}(x)\nu_i$, where ν is the inner normal to the boundary of the domain and

$$\Sigma_3=\{x\in\partial\Omega;\quad \Sigma_{ij=1}^n a^{ij}(x)\nu_i\nu_j>0\},$$

$$\Sigma_2=\{x\in\partial\Omega\setminus\Sigma_3;\quad b(x)<0\},$$

$$\Sigma_1=\{x\in\partial\Omega\setminus\Sigma_3;\quad b(x)>0\},$$

$$\Sigma_0=\{x\in\partial\Omega\setminus\Sigma_3;\quad b(x)=0\}.$$

In the paper of E. Radkevich [7] was investigated the boundary value problem

$$\sum_{ij=1}^{n} a^{ij}(x)\nu_i u_{x_j}+l(x)u=\psi(x)\quad\text{on}\quad\Sigma_3,\quad u(x)=\varphi(x)\quad\text{on}\quad\Sigma_2, \tag{2}$$

for the equation (1). Under the assumption for the regularity of the coefficients of the equation and the boundary value operator, he proved the existence of the weak solution for the boundary value problem (1), (2), when $l(x)<0$, $l(x)-b(x)<0$ on Σ_3 and uniqueness when $c(x)-\Sigma_{i=1}^n b_{x_i}^i(x)+\Sigma_{ij=1}^n a_{x_ix_j}^{ij}(x)\leqq c_1<0$ in $\bar\Omega$. In [5], for Neuman's problem for the equation (1), where the existence of the Lipshitz solution with bounded, in Sobolev's sense, first derivative was proved under the assumption that $c(x) \leq -M$, where M is a large enough positive constant. The question of the existence and the uniqueness of the classical solution to the second initial-boundary value problem for degenerate parabolic equations with coefficient unit before u_t was investigated by G. Fateeva [8], in a cylindrical domain $\Omega \times (0, T)$ with a boundary $\partial\Omega \times (0, T) \in \Sigma_3$. For quasilinear degenerate parabolic equations, in the same paper, was established the validity of the results obtained in the linear case in domains of the type $\Omega \times (0, \delta)$ for $\delta > 0$ small enough.

The object of this paper is to prove the existence and the uniqueness of the classical solution to Neuman's problem for equation (1). Let the bounded domain $\Omega \in \mathbf{R}^n$ with a boundary Γ is $C^{l+3+\lambda}$ smooth diffeomorphic of a ball, l is nonnegative integer, $0 < \lambda < 1$. The operator L is defined with (1) in the domain $\Omega' \supset \bar{\Omega}$. We assume

(i) the operator L is with real coefficients of the class $C^{l+1+\lambda}(\bar{\Omega})$, $a^{ij}(x) \in C^2(\Omega')$;

(ii) $\Sigma_{ij=1}^n a^{ij}(x)\xi^i\xi^j \geq 0$, $x \in \Omega'$, $\xi \in \mathbf{R}^n$, $c(x) < 0$, $x \in \bar{\Omega}$;

(iii) $\Gamma \in \Sigma_3$.

On Γ is defined the boundary value operator B

$$Bu = \sum_{k=1}^{n} \sigma^k(x) u_{x_k} + \sigma(x)u = \varphi(x) \tag{3}$$

with coefficients of $C^{l+2+\lambda}(\Gamma)$. We assume that $\sigma(x) \leq 0$ and the vector field $(\sigma^1, \sigma^2, \ldots, \sigma^n)$ is not tangential to Γ and concludes an acute angle with the inner normal ν. The set of $\bar{\Omega}$ in which the operator L is not strictly elliptic we denote with $\Lambda = \{x \in \bar{\Omega};\ \Sigma_{ij=1}^n a^{ij}(x)\xi^i\xi^j = 0$ for some $\xi \in \mathbf{R}^n$, $\xi \neq 0\}$.

Theorem 1. *Let for the operator L be true* (i)—(iii), *and for the boundary value operator* (3) *and the domain Ω the assumptions above. If $\Lambda \neq \Omega$, then there exists a constant $C_0 < 0$, so that when $c(x) \leq C_0$ in the neighbourhood of Λ, the boundary value problem* (1), (3) *has a unique classical solution of the class $C^l(\bar{\Omega})$.*

Let the domain Ω is $C^{l+3+\lambda}$ smooth diffeomorphic of the domain $\Omega_1 \setminus \bar{\Omega}_2$, where Ω_1, Ω_2 are concentric balls with radii respectively r_1 and r_2. On the boundary of the domain Ω, $\partial\Omega = \Gamma_1 \cup \Gamma_2$ is defined the boundary value operator

$$\begin{aligned} B_1 u &= \sum_{k=1}^{n} \sigma^k(x) u_{x_k} + \sigma(x)u = \varphi(x) \quad \text{on} \quad \Gamma_1, \\ B_2 u &= \sum_{k=1}^{n} \tau^k(x) u_{x_k} + \tau(x)u = \psi(x) \quad \text{on} \quad \Gamma_2, \end{aligned} \tag{4}$$

with coefficients of the class $C^{l+2+\lambda}$. We suppose that the vector fields $(\sigma^1, \sigma^2, \ldots, \sigma^n)$, $(\tau^1, \tau^2, \ldots, \tau^n)$ are not tangential and conclude an acute angle respectively with the inner normal to Γ_1 and Γ_2, $\sigma(x) \leq 0$, $\tau(x) \leq 0$.

Theorem 2. *Let for the operator L be true* (i)—(iii), *and for the domain Ω and the boundary value operator* (4) *the assumptions above. Then there exists a constant $q_0 < 0$, so that when $c(x) \leq q_0$ in the neighbourhood of Λ, the boundary value problem* (4) *has a unique classical solution of the class $C^l(\bar{\Omega})$.*

Theorem 3. *Let for the operator* L *and the domain* Ω *be true the assumptions in Theorem* 2. *Then there exists a constant* g_0 *so that when* $c(x) \leqq g_0$ *in the neighbourhood of* Λ, *the boundary value problem*

$$B_1 u = \sum_{k=1}^{n} \sigma^k(x) u_{x_k} + \sigma(x) u = \varphi(x) \quad on \ \Gamma_1,$$

$$(5) \qquad B_2 u = u = \psi(x) \quad on \ \Gamma_2,$$

for the equation (1) *has a unique classical solution of the class* $C^l(\bar{\Omega})$.

To prove the theorems above we will use an elliptic regularization of the equation (1)

$$(6) \qquad L_\varepsilon u = Lu + \varepsilon \Delta u, \qquad \varepsilon > 0,$$

where Δ is Laplace's operator.

In part 2 which is the basic in this paper, we prove that the solutions of the boundary value problems (6), (3), (4), (5) and their derivatives up to the order $l+1$ in $\bar{\Omega}$ are uniformly bounded. The proof is done by means of Bernstein's global a priori estimates, because as it is well known from [5], for the equations with a nonnegative characteristic forms, are not true Bernstein's more elementary inner estimates.

In 3 are proved the theorems in 1.

In conclusion we must note, that the results in this paper do not coincide with the results in [7], where only the existence of a weak solution for (1), (2) is proved. As for [8], it is a special case of the equation (1) and the estimates used a priori differ considerably from those in 2.

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2. A priori estimates. In this part we shall need the following inequalities and identities (see [2]): $L(v_1 v_2) = v_1 L v_2 + v_2 L v_1 + 2\Sigma_{ij=1}^{n} a^{ij}(v_1)_{x_i}(v_2)_{x_j} - c v_1 v_2$ for any two functions $v_1 v_2 \in C^2(\Omega)$, $(\Sigma_{ij=1}^{n} a^{ij}\xi^i\eta^j)^2 \leq \Sigma_{ij=1}^{n} a^{ij}\xi^i\xi^j \Sigma_{ij=1}^{n} a^{ij}\eta^i\eta^j$ for any $\xi, \eta \in \mathbf{R}^n$, $(\Sigma_{ij=1}^{n} a_{x_l}^{ij} u_{x_i x_j})^2 \leq M \Sigma_{k=1}^{n} \Sigma_{ij=1}^{n} a^{ij} u_{x_k x_i} u_{x_k x_j}$ for $l = 1, 2, \ldots, n$, where M is a constant which depends on the maximum of the second derivatives of $a^{ij}(x)$ in $\bar{\Omega}$. Besides we shall use the short notations $u_k = u_{x_k}$, $b_{kl}^i = b_{x_k x_l}^i$ and the case of repeated indexes we shall understand an addition from 1 to n. In our further reflexions with M_i, K_i we shall mark the constants which are independent of ε.

Let $u^\varepsilon(x) \in C^{l+3+\lambda}(\bar{\Omega})$ (see [1]) be the solution of (6), (3).

Lemma 1. $\max\limits_{\bar{\Omega}} |u^\varepsilon(x)| \leq K_0$.

Proof. Let $\Phi(x) \in C^{l+3+\lambda}(\bar{\Omega})$ be a function, for which $B\Phi = \varphi$ on Γ. We consider the function $z^\varepsilon(x) = u^\varepsilon(x) - \Phi(x) - N$, where N is a constant, which can fit us. The following inequalities are true: $L_\varepsilon z^\varepsilon = f - L_\varepsilon \Phi - cN \geq 0$ (when N is large enough, depending on f, Φ and the coefficients of the operator L, but independent of ε) and $Bz^\varepsilon = -\sigma(x)N \geq 0$. According to the maximum principle it follows, that if $Z^\varepsilon(x)$ attains a positive maximum in an inner point P_1 of the domain Ω, then $L_\varepsilon z^\varepsilon(P_1) < 0$. If $z^\varepsilon(x)$ attains a positive maximum in a point

$P_2 \in \Gamma$ from theorem 2′ [11] (see also [13]) it follows that $Bz^\varepsilon(P_2)<0$. These inequalities show that $z^\varepsilon(x)$ cannot attain a positive maximum in $\bar{\Omega}$. In the same way is proved that $-u^\varepsilon(x)+\Phi(x)-N$ cannot attain a positive maximum in $\bar{\Omega}$. Thus lemma 1 is proved. By analogy are proved

Lemma 1′. *For the solution $v^\varepsilon(x)$ of* (6), (4) *is true the estimate* $\max_{\bar{\Omega}} |v^\varepsilon(x)| \leq K_0'$, *where K_0' is a constant* (*independent of* ε).

Lemma 1″. *For the solution $w^\varepsilon(x)$ of* (6), (5) *is true the estimate* $\max_{\bar{\Omega}} |w^\varepsilon(x)| \leq K_0''$, *where K_0'' is constant* (*independent of* ε).

To prove the uniform estimates (independent of ε) for the derivatives of the functions $u^\varepsilon(x)$, $v^\varepsilon(x)$, $w^\varepsilon(x)$ we must restrict the size of the coefficient $c(x)$.

Lemma 2. *Let $\Lambda \neq \bar{\Omega}$. Then there exists a constant $c_1<0$ so that, when $c(x) \leq c_1$ in a neighbourhood of Λ, the following estimates are true*

$$\max_{\bar{\Omega}} |D^\alpha u^\varepsilon(x)| \leq K_1, \qquad |\alpha|=1. \tag{7}$$

Without loss of generality we may assume that Ω is a ball with a radius R and a centre 0, and the operator is strictly elliptic in 0. The concentric to Ω a ball Ω_0 is chosen a sufficiently small radius r, so that the operator L is strictly elliptic in $\bar{\Omega}_0$. The boundary of Ω and Ω_0 we shall denote respectively with Γ and Γ_0. From lemma 1 and Bernstein's inner a priori estimates (see [2]) we obtain (7) in the domain Ω_0. To prove (7) in $\bar{G}$, $G=\Omega \setminus \bar{\Omega}_0$ we make a polar change of the variables. For convenience we preserve the previous mention considering that $x_1, x_2, \ldots, x_{n-1}$ are angular variables and x_n is a radial variable. In the new variables the assumption about the vector field $(\sigma^1, \sigma^2, \ldots, \sigma^n)$ denotes that $\sigma^n(x)<0$ on Γ. We consider the auxiliary function $z_0(x)=-\exp(\delta x_n)$ and $\delta>0$ is chosen so that $\frac{1}{2}c+a^{nn}\delta^2+b^n\delta \leq 0$. This is possible because according to (ii) $c(x)<0$ in $\bar{\Omega}$. Thus we have

$$\begin{aligned} Lz_0 &= (-a^{nn}\delta^2-b^n\delta-c)\exp(\delta x_n) \geq -c/2>0 \quad \text{in} \quad \bar{\Omega}, \\ Bz_0 &= (-\sigma^n\delta-\sigma)\exp(\delta R) \geq -\sigma^n\delta>0 \quad \text{on} \quad \Gamma. \end{aligned} \tag{8}$$

In the following calculations, for convenience, we shall omit the index ε. We define the function

$$z_1(x)=[m_1 \sum_{k=1}^{n-1} u_k^2+2u_n^2+u_n T(u)] \exp((R-x_n)\xi_1)+N_0 z_0+N_0' u^2,$$

where $Tu=4\Sigma_{k=1}^{n-1}\theta^k(x)u_{x_k}+4\theta(x)u-4\varrho(x)$ and $\theta^k(x)$, $\theta(x)$, $\varrho(x) \in C^{l+2+\lambda}(\bar{G})$ are smooth extensions in $\bar{G}$, respectively of the functions σ^k/σ^n, σ/σ^n, φ/σ^n defined on Γ. The positive constant m_1 is chosen so that

$$m_1 \sum_{k=1}^{n-1} u_k^2+2u_n^2+u_n Tu+n_1 \geq \sum_{k=1}^{n} u_k^2,$$

$$2m_1 \sum_{k=1}^{n-1} a^{ij}u_{ki}u_{kj}+4a^{ij}u_{ni}u_{nj}+a^{ij}u_{nj}(Tu)_i \tag{9}$$

$$\geq \frac{3m_1}{2} \sum_{k=1}^{n-1} a^{ij}u_{ki}u_{kj} + 3a^{ij}u_{ni}u_{nj} - M_1 \sum_{k=1}^{n} u_k^2 - M_2,$$

$$m_1 \geq \max(2, (4n^2H_1)^2); \qquad H_1 = \max_{x \in \bar{G},\ 1 \leq k \leq n} |\theta^k(x)|,$$

where M_1 is a constant depending on the coefficients of L, B and Ω (but independent of ε). The positive constants ξ_1, N_0 will be determined for our purpose, so that $z_1(x)$ does not attain a positive maximum in $\bar{G}$. For Bz_1 we have the estimate

$$Bz_1 \geq (-\xi_1\sigma^n) \sum_{k=1}^{n} u_k^2 + 2m_1 \sum_{k=1}^{n-1} u_k((Bu)_{x_k} + [B, \frac{\partial}{\partial x_k}]u)$$

$$+ 4u_nBu_n + Bu_nTu + u_n(T(Bu) + [B, T]u) + N_0Bz_0 + N'_0Bu^2.$$

From the choice of the operator T we have $4u_nBu_n + Bu_nTu = 0$ and from the boundary conditions $(Bu)_{x_k} = (\varphi)_{x_k}$, $T(Bu) = T(\varphi)$, $k = 1, \ldots, n-1$ on Γ. Since $[B, \partial/\partial x_k]$, $[B, T]$ are differential operators of first order, when ξ_1 is large enough depending on the coefficients of the commutators, the inequalities $Bz_1 > 0$ on Γ will be true. According to the maximum principle $z_1(x)$ can not attain a positive maximum on Γ. Let us prove that $z_1(x)$ cannot attain a positive maximum in an inner point of the domain G.

$$Lz_1 = (I_1 + I_2 + I_3 + I_4)\exp((R - x_n)\xi_1) + N_0Lz_0 + N'_0Lu^2,$$

$$I_1 = (a^{nn}\xi_1^2 - b^n\xi_1)(m_1 \sum_{k=1}^{n-1} u_k^2 + 2u_n^2 + u_nTu + n_1) \leq M_3 \sum_{k=1}^{n} u_k^2 + \mu'_3,$$

$$I_2 = -2 \sum_{j=1}^{n} a^{nj}\xi_1(2m_1 \sum_{k=1}^{n-1} u_ku_{kj} + 4u_nu_{nj} + u_n(Tu)_j + u_{nj}Tu)$$

$$\leq \frac{m_1}{2} \sum_{k=1}^{n-1} a^{ij}u_{ki}u_{kj} + a^{ij}u_{ni}u_{nj} + M_4 \sum_{k=1}^{n} u_k^2 + M'_4,$$

$$I_3 = 2m_1 \sum_{k=1}^{n-1} a^{ij}u_{ki}u_{kj} + 4a^{ij}u_{ni}u_{nj} + 2a^{ij}u_{nj}(Tu)_i$$

$$\geq \frac{3m_1}{2} \sum_{k=1}^{n-1} a^{ij}u_{ki}u_{kj} + 3a^{ij}u_{ni}u_{nj} - M_1 \sum_{k=1}^{n} u_k^2 - M_2,$$

$$I_4 = 2m_1 \sum_{k=1}^{n-1} u_k[-a_k^{ij}u_{ij} - b_k^iu_i - c_ku - \frac{c}{2}u_k + f_k]$$

$$+ (Tu + 4u_n)[-a_n^{ij}u_{ij} - b_n^iu_i - c_nu - cu_n + f_n] + u_nL(Tu)$$

$$\geq -\frac{m_1}{2} \sum_{k=1}^{n-1} a^{ij}u_{ki}u_{kj} - a^{ij}u_{ni}u_{nj}$$

$$- m_1c \sum_{k=1}^{n-1} u_k^2 - 2cu_n^2 + 8c|u_n \sum_{k=1}^{n-1} \theta^ku_k| - M'_5 \sum_{k=1}^{n} u_k^2 - M_6.$$

From the choice of m_1 we have $|8u_n\Sigma_{k=1}^{n-1}\theta^k(x)u_k|\leq(m_1/2)\Sigma_{k=1}^{n-1}u_k^2+u_n^2$. Therefore

$$I_4\geq-\frac{m_1}{2}\sum_{k=1}^{n-1}a^{ij}u_{ki}u_{kj}-a^{ij}u_{ni}u_{nj}-(m_1/2)c\sum_{k=1}^{n-1}u_k^2-cu_n^2-M_5\sum_{k=1}^{n}u_k^2-M_6.$$

We choose N_0 large enough, so that $N_0Lz_0-(M_2+M_6)\exp((R-x_n)\xi_1)\geq 1$ because of (8). When $c(x)\leq c_1$ in the neighbourhood of Λ, where $c_1=-M_1-M_3-M_4-M_5$ will be true the estimate

$$Lz_1\geq\sum_{k=1}^{n}a^{ij}u_{ki}u_{kj}+1>0. \tag{10}$$

The above inequality denotes that $z_1(x)$ cannot attain a positive maximum in an inner point of the domain G. Because of the relation between Cartesian and polar coordinates and from what was proved above, on Γ_0 will be valid $\max_{\Gamma_0}|D^\alpha u|\leq\widetilde{K}_1$, $|\alpha|=1$, in polar coordinates. When N_0 is large enough, the function $z_1(x)$ cannot attain a positive maximum on Γ_0 either, i. e. $z_1(x)$ does not attain a positive in $\bar{G}$. From (9) $\Sigma_{k=1}^{n}u_k^2\leq\bar{K}_1$. This estimate will be true in Cartesian coordinates too. Thus lemma 2 is proved.

Lemma 2′. *There is a constant $q_1<0$, so that when $c(x)\leq q_1$ in the neighbourhood of Λ, for the solution $v^\varepsilon(x)$ of* (6), (4) *we have the following estimates* $\max_{\bar\Omega}|D^\alpha v^\varepsilon(x)|\leq K_1'$, $|\alpha|=1$.

Proof. Without loss of generality we assume that Ω is the domain $\Omega_1\setminus\bar\Omega_2$ where Ω_1, Ω_2 are concentric balls respectively with radii r_1, r_2, boundaries Γ_1, Γ_2 and centre in 0. We make a polar change of the variables and for convenience reserve the previous mentions, so that $x_1, x_2, \ldots, x_{n-1}$ are the angular and x_n is a radial coordinate. In the new variables the assumptions about the vector fields $(\sigma^1, \sigma^2, \ldots, \sigma^n)$, $(\tau^1, \tau^2, \ldots, \tau^n)$ denote that $\sigma^n(x)<0$ on Γ_1 and $\tau^n(x)>0$ on Γ_2. We consider the function

$$Y_0(x)=-\exp(\delta(r_1-x_n)(-x_n+r_2)), \tag{11}$$

where δ is chosen so that $\frac{1}{2}c+a^{nn}\delta^2+b^n\delta\leq 0$. As in lemma 2 we prove that $LY_0>0$ in Ω, $B_1Y_0>0$ on Γ_1, $B_2Y_0>0$ on Γ_2. For convenience we omit the index ε. We consider the function

$$Y_1(x)=[m_1\sum_{k=1}^{n-1}u_k^2+2u_n^2+u_nTu+n_1]\exp((r_1-x_n)(x_n-r_2)\xi_1)+N_0Y_0+N_0'u^2, \tag{12}$$

where $Tv=4\Sigma_{k=1}^{n-1}\theta^k(x)v_{x_k}+4\theta(x)v-4\varrho(x)$ and $\theta^k(x)\in C^{l+2+\lambda}(\bar\Omega)$ are smooth extensions in $\bar\Omega$ of the functions σ^k/σ^n, τ^k/τ^n define respectively on Γ_1 and Γ_2. By analogy θ, $\varrho\in C^{l+2+\lambda}(\bar\Omega)$ are smooth extensions respectively of σ/σ^n, φ/σ^n on Γ_1 and τ/τ^n, ψ/τ^n on Γ_2. The positive constant m_1 is chosen so that we have the inequalities (9) for the solution $v(x)$ of (6), (4). Following with minor changes the scheme of the proof of lemma 2, we have when ξ_1, N_0 are large enough the estimates

$$B_1Y_1>0 \quad\text{on}\quad \Gamma_1, \qquad B_2Y_1>0 \quad\text{on}\quad \Gamma_2$$

$$LY_1\geq\sum_{k=1}^{n}a^{ij}u_{ki}u_{kj}+1 \quad\text{in}\quad \Omega \tag{13}$$

under the assumption that $c(x)\leq q_1<0$ in the neighbourhood of Λ. From (13) it follows that $Y_1(x)$ cannot attain a positive maximum in $\bar{\Omega}$, and from (9) and from the choice of Y_1, we obtain the proof of lemma 2'.

Lemma 2''. *There is a constant $g_1<0$ so that when $c(x)\leq g_1$ in the neighbourhood of Λ, for the solution $w^\varepsilon(x)$ of (6), (5) we have the estimates* $\max_{\bar{\Omega}}|D^\alpha w^\varepsilon(x)|\leq K_1''$, $|\alpha|=1$.

Proof. As in lemma 2' we consider the functions (11), (12), but only in the definition of the operators $T_k w=4\Sigma_{k=1}^{n-1}\theta^k(x)w_{x_k}+4\theta(x)w-4\varrho(x)$ for the coefficients $\theta^k(x)\in C^{l+2+\lambda}(\bar{\Omega})$ we assume that they are smooth extensions of the functions σ^k/σ^n on Γ_1 and a^{nk}/a^{nn} on Γ_2. The functions θ, $\varrho\in C^{l+2+\lambda}(\bar{\Omega})$ are smooth extensions respectively of σ/σ^n, φ/σ^n defined on Γ_1. We will prove that $Y_1(x)$ cannot attain a positive maximum on Γ_2, so that for our purpose we will prove that $(Y_1)_{x_n}>0$ on Γ_2. Without loss of generality we assume that $\psi=0$ on Γ_2. From the boundary conditions we have

$$\frac{\partial Y_1}{\partial x_n}\geq(r_1-r_2)\xi_1\sum_{k=1}^{n}u_k^2+2m_1\sum_{k=1}^{n-1}u_ku_{kn}+4u_nu_{nn}$$

$$+u_n(Tu)_n+u_{nn}Tu+N_0\delta(r_1-r_2)$$

$$\geq(r_1-r_2)\xi_1u_n^2+4u_n[a^{nn}u_{nn}+\sum_{i=1}^{n-1}a^{ni}u_{ni}]/a^{nn}-M_7u_n^2-M_7'+N_0\delta(r-r_2)$$

$$=(r_1-r_2)\xi_1u_n^2-M_7u_n^2-M_7'+4u_n(f-b^nu_n)/a^{nn}+N_0\delta(r_1-r_2)>0$$

on Γ_2, when ξ_1, N_0 are large enough. Our next reflexions are similar to those in lemma 2'.

Lemma 3. *Let $\Lambda\neq\Omega$. Then there exists a constant $c_2\leq c_1$ so that, when $c(x)\leq c_2$ in a neighbourhood of Λ, we have the estimates*

$$\max_{\bar{\Omega}}|D^\alpha u^\varepsilon(x)|\leq K_2,\qquad |\alpha|=2. \tag{23}$$

Proof. The proof of (14) in Ω_0 follows from lemma 1 and Bernstein's inner a priori estimates. To prove (14) in $\bar{G}$ we make a polar change of variables. We use the mentions in lemma 2. We consider the function

$$z_2(x)=[m_2\sum_{kp=1}^{n-1}u_{kp}^2+2\sum_{k=1}^{n}u_{kn}^2+\sum_{k=1}^{n}u_{kn}T_ku+n_2]\exp((R-x_n)\xi_2)+N_1z_1(x),$$

where

$$T_ku=4[\sum_{i=1}^{n-1}\theta^i(x)u_{x_i}+\theta(x)u-\varrho(x)]_{x_k},\quad k=1,\ldots,n-1, \tag{15}$$

$$T_nu=4[-\sum_{i=1}^{n-1}A^{ni}(\sum_{n=1}^{n-1}\theta^ku_{x_k}+\theta u-\varrho)_{x_i}+\sum_{ij=1}^{n-1}A^{ij}u_{x_ix_j} \tag{16}$$

$$+\sum_{i=1}^{n-1}B^iu_{x_i}-B^n(\sum_{k=1}^{n-1}\theta^ku_{x_k}+\theta u-\varrho)+Cu-F].$$

The functions $\theta^k(x)$, $\theta(x)$, $\varrho(x)$ are introduced in lemma 2 and the functions $A^{ij}(x)$, $B_i(x)$, $C(x)$, $F(x)\in C^{l+1+\lambda}(\bar{G})$ are smooth extensions respectively of a^{ij}/a^{nn}, b^i/a^{nn}, c/a^{nn}, f/a^{nn} from Γ to $\bar{G}$. The positive constant m_2 is chosen so that

$$m_2\sum_{kp=1}^{n-1} u_{kp}^2+2\sum_{k=1}^{n} u_{kn}^2+\sum_{k=1}^{n} u_{kn}T_k u+n_2\geq \sum_{ij=1}^{n} u_{ij}^2,$$

(17) $$2m_2\sum_{kp=1}^{n-1} a^{ij}u_{kpi}u_{kpj}+4\sum_{k=1}^{n} a^{ij}u_{kni}u_{knj}+\sum_{k=1}^{n} a^{ij}u_{knj}(T_k u)_i$$

$$\geq\frac{3}{2}m_2\sum_{kp=1}^{n-1} a^{ij}u_{kpi}u_{kpj}+3\sum_{k=1}^{n} a^{ij}u_{kni}u_{knj}-M_8\sum_{kp=1}^{n} u_{kp}^2-M_9,$$

$$m_2\geq\max(2,\ (4n^3H_2)^2),$$

where H_2 is the maximum of the coefficients before the derivatives of the highest order in T_k. We will prove that $z_2(x)$ cannot attain a positive maximum in the domain G and on boundary Γ, in an appropriate choice of the positive constants ξ_2, N_1. For Bz_2 we have the estimate

$$Bz_2\geq(-\xi_2\sigma^n)\sum_{kp=1}^{n} u_{kp}^2+2m_2\sum_{kp=1}^{n-1} u_{kp}((Bu)_{x_kx_p}$$

$$+[B,\ \frac{\partial^2}{\partial x_k\partial x_p}]u)+4\sum_{k=1}^{n} u_{kn}Bu_{kn}+\sum_{k=1}^{n}(T_k u)Bu_{kn}$$

$$+\sum_{k=1}^{n} u_{kn}(T_k(Bu)+[B,\ T_k]u)+N_1Bz_1.$$

From the choice of T_m and from the boundary conditions, we will have $T_m u+4u_{mn}=0$; $(Bu)_{x_kx_p}=(\varphi)x_kx_p$, $kp=1, 2\ldots, n-1$, on Γ. Because the commutators $[B, \partial^2/\partial x_k\partial x_p]$, $[B, T_k]$ are operators of second order, when ξ_2, N_1 are large enough depending on the coefficients of the commutators, we will have the estimate $Bz_2\geq N_1Bz_1-M_{10}>0$ on Γ. According to the maximum principle $z_2(x)$ cannot attain a positive maximum on Γ. We will prove that $z_2(x)$ doesn't attain a positive maximum in an inner point of the domain G.

$$Lz_2=(I_1+I_2+I_3+I_4)\exp((R-x_n)\xi_2)+N_1Lz_1,$$

$$I_1=(a^{nn}\xi_2^2-b^n\xi_2)(m_2\sum_{kp=1}^{n-1} u_{kp}^2+2\sum_{k=1}^{n} u_{kn}^2+\sum_{k=1}^{n} u_{kn}T_k u)\leq M_{11}\sum_{kp=1}^{n} u_{kp}^2+M'_{11}$$

$$I_2=-2\sum_{j=1}^{n} a^{nj}\xi_2[2m_2\sum_{kp=1}^{n-1} u_{kp}u_{kpj}+4\sum_{k=1}^{n} u_{kn}u_{knj}$$

$$+\sum_{k=1}^{n} u_{kn}(T_k u)_j+\sum_{k=1}^{n} u_{knj}T_k u]\leq\frac{m_2}{2}\sum_{kp=1}^{n} a^{ij}u_{kpi}u_{kpj}$$

$$+\sum_{k=1}^{n} a^{ij}u_{kni}u_{knj}+M_{12}\sum_{kp=1}^{n} u_{kp}^2,$$

$$I_3=2m_2\sum_{kp=1}^{n-1}a^{ij}u_{kpi}u_{kpj}+4\sum_{k=1}^{n}a^{ij}u_{kni}u_{knj}$$

$$+\sum_{k=1}^{n}a^{ij}u_{knj}(T_ku)_i\geqq\frac{3m_2}{2}\sum_{kp=1}^{n-1}a^{ij}u_{kpi}u_{kpj}$$

$$+3\sum_{k=1}^{n}a^{ij}u_{kni}u_{knj}-M_8\sum_{kp=1}^{n}u_{kp}^2-M_9,$$

$$I_4=2m_2\sum_{kp=1}^{n-1}u_{kp}(-a_k^{ij}u_{ptj}-a_p^{ij}u_{kij}-a_{kp}^{ii}u_{ij}$$

$$-b_k^iu_{pi}-b_p^iu_{ki}-cu_{kp}/2)+\sum_{k=1}^{n}(4u_{kn}+T_ku)(-a_k^{ij}u_{nij}$$

$$-a_n^{ij}u_{kij}-a_{kn}^{ij}u_{ij}-b_k^iu_{ni}-b_n^iu_{ki}-cu_{kn}/2)$$

$$+\sum_{k=1}^{n}u_{kn}L(T_ku)O(\sum_{kp=1}^{n}u_{kp}^2)-M'_{11}\geqq-\frac{m_2}{2}\sum_{kp=1}^{n-1}a^{ij}u_{kpi}u_{kpj}$$

$$-\sum_{k=1}^{n}a^{ij}u_{kni}u_{knj}-m_2c\sum_{kp=1}^{n-1}u_{kp}^2-2c\sum_{k=1}^{n}u_{kn}^2$$

$$-8c\,|\sum_{kij=1}^{n-1}\theta^iu_{kn}u_{ij}|-8\,cu_{nn}(-\sum_{ik=1}^{n-1}A^{ni}\theta^ku_{ki}$$

$$+\sum_{ij=1}^{n-1}A^{ij}u_{ij})\quad-M'_{13}\sum_{kp=1}^{n}u_{kp}^2-M_{14}$$

$$\geqq-\frac{m_2}{2}\sum_{kp=1}^{n-1}a^{ij}u_{kpi}u_{kpj}-\sum_{k=1}^{n}a^{ij}u_{kni}u_{knj}$$

$$-(m_2/2)c\sum_{kp=1}^{n-1}u_{kp}^2-c\sum_{k=1}^{n}u_{kn}^2-M_{13}\sum_{kp=1}^{n}u_{kp}^2-M_{14}.$$

When $c(x)\leqq c_2\leqq c_1$ in the neighbourhood of Λ, $c_2\leqq-M_8-M_{11}-M_{12}-M_{13}$ and N_1 is large enough, from the inequalities (10) we have $Lz_2\geqq\Sigma_{kp=1}^{n}a^{ij}u_{kpi}u_{kpj}+1$. From the proof of (14) in Ω_0 and because of the relation between polar and Cartesian coordinates, when N_1 is large enough, $z_2(x)$ doesn't attain a positive maximum on Γ_0. From (17) we have $\Sigma_{kp=1}^{n}u_{kp}^2\leqq K_2$. This estimate will be true in Cartesian coordinates too. Thus lemma 3 is proved.

Lemma 3′. *There is a constant $q_2\leqq q_1$ so that when $c(x)\leqq q_2$ in the neighbourhood of Λ, for the solution $v^\varepsilon(x)$ of* (6), (4) *we have the following estimates*

$$\max_{\bar\Omega}|D^\alpha v^\varepsilon(x)|\leqq K'_2,\qquad |\alpha|=2. \tag{18}$$

Proof. We consider the function

(19)
$$Y_2(x)=\exp((r_1-x_n)(x_n-r_2)\xi_2)[m_2\sum_{kp=1}^{n}u_{kp}^2+2\sum_{k=1}^{n}u_{kn}^2+\sum_{k=1}^{n}u_{kn}T_ku+n_2]+N_1Y_1(x)$$

and using a similar argument as in lemma 3, we prove the estimate (18).

Lemma 3''. *There is a constant* $q_2\leq q_1$, *so that when* $c(x)\leq g_2$ *in the neighbourhood of* Λ, *the following estimates are true* $\max_{\overline{\Omega}}|D^\alpha w^\varepsilon(x)|\leq K_2''$, $|\alpha|=2$.

Proof. We consider the auxiliary function $Y_2(x)$, as in (19), (15), (16), so that the functions $\theta^k(x)$, $\theta(x)$, $\varrho(x)$ are introduced in lemma 2''. We will prove that $Y_2(x)$ doesn't attain a positive maximum on Γ_2, so that for our purpose we prove that $(Y_2)_{x_n}>0$ on Γ_2;

$$(Y_2)_{x_n}\geq\xi_2(r_1-r_2)\sum_{k=1}^{n}u_{kn}^2+2m_2\sum_{kp=1}^{n-1}u_{kp}u_{kpn}+4\sum_{k=1}^{n}u_{kn}u_{knn}+\sum_{k=1}^{n}u_{knn}T_ku+\sum_{k=1}^{n}u_{kn}(T_ku)_n+N_1(Y_1)_{x_n}.$$

From the boundary conditions $u_{kp}=0$, $T_ku=0$, $kp=1,2,\dots,n-1$, on Γ_2 hence when ξ_2 is large enough

$$(Y_2)_{x_n}\geq\xi_2(r_1-r_2)\sum_{k=1}^{n}u_{kn}^2+4\sum_{k=1}^{n-1}u_{kn}[u_{knn}+\sum_{i=1}^{n-1}(a^{ni}/a^{nn})u_{kni}]+4u_{nn}[u_{nnn}-\sum_{ki=1}^{n-1}(a^{ni}/(a^{nn})^2)a^{nk}u_{nki}+\sum_{ij=1}^{n-1}(a^{ij}/a^{nn})u_{nij}]-M_{15}'\sum_{k=1}^{n}u_{kn}^2+N_1(Y_1)_{x_n}-M_{16}$$

$$=\xi_2(r_1-r_2)\sum_{k=1}^{n}u_{kn}^2+4\sum_{k=1}^{n-1}(u_{kn}/a^{nn})(\sum_{i=1}^{n}a^{ni}u_{ni})_k+4(u_{nn}/a^{nn})(\sum_{ij=1}^{n}a^{ij}u_{ij})_n-4(u_{nn}/a^{nn})[\sum_{i=1}^{n-1}a^{ni}u_{nni}+\sum_{ki=1}^{n-1}((a^{ni}a^{nk})/a^{nn})u_{nki}]-M_{15}''\sum_{k=1}^{n}u_{kn}^2+N_1(Y_1)_{x_n}-M_{16}$$

$$=\xi_2(r_1-r_2)\sum_{k=1}^{n}u_{kn}^2+4\sum_{k=4}^{n-1}(u_{kn}/a^{nn})(f-b^nu_n)_k+4(u_{nn}/a^{nn})(f-\sum_{i=1}^{n}b^iu_i-cu)_n-M_{15}\sum_{k=1}^{n}u_{nk}^2-M_{16}-4(u_{nn}/a^{nn})\sum_{i=1}^{n-1}a^{ni}[\sum_{k=1}^{n}a^{nk}u_{nk}]_i+N_1(Y_1)_{x_n}>0.$$

On further reflections it follows as in lemma 3.

Lemma 4. *Let* $\Lambda\neq\overline{\Omega}$. *Then there exists a constant* $c_\mu\leq c_{\mu-1}$ *so that when* $c(x)\leq c_\mu$ *in the neighbourhood of* Λ, *for* $\mu\leq l+1$ *we have the following estimates* $\max_{\overline{\Omega}}|D^\alpha u^\varepsilon(x)|\leq K_\mu$, $|\alpha|=\mu$.

Proof. Lemma 4 is proved by induction with the auxiliary function

$$z_\mu(x)=\exp((R-x_n)\xi_\mu)[m_\mu \sum_{|\alpha|=\mu}(D^\alpha_{x'}u)^2$$

$$+2\sum_{\substack{|\alpha+\beta|=\mu,\\ \beta\neq 0}}(D^\alpha_{x'}D^\beta_{x_n}u)^2+\sum_{\substack{|\alpha+\beta|=\mu\\ \beta\neq 0}}T_{\alpha\beta}(u)D^\alpha_{x'}D^\beta_{x_n}u+n_\mu]+N_{\mu-1}z_{\mu-1}.$$

We use the mentions introduced in lemmas 2, 3. The operators $T_{\alpha\beta}=T_{\alpha\beta}$ $(x, \partial/\partial x_1, \ldots, \partial/\partial x_{n-1})$ are defined on Γ by the condition $T_{\alpha\beta}(u)=-4D^\alpha_{x'}D^\beta_{x_n}u$, where the derivatives of the type $D^\beta_{x_n}u$ are replaced with their equivalent expressions in which we have derivatives only in the variables $x_1, x_2, \ldots, x_{n-1}$. This is possible from the boundary value operator, the equation (6) and the derivatives of the equation (6) up to the necessary order. In $\bar{G}$ the coefficients of $T_{\alpha\beta}$ are smoothly extended of the class $C^{l+1+\lambda}(\bar{G})$. The positive constant m_μ is chosen so that

$$m_\mu\geq\max(2, (4n^{1+\mu}H_\mu)^2),$$

$$m_\mu\sum_{|\alpha|=\mu}(D^\alpha_{x'}u)^2+2\sum_{\substack{|\alpha+\beta|=\mu\\ \beta\neq 0}}(D^\alpha_{x'}D^\beta_{x_n}u)^2+\sum_{\substack{|\alpha+\beta|=\mu\\ \beta\neq 0}}T_{\alpha\beta}(u)D^\alpha_{x'}D^\beta_{x_n}u+n_\mu\geq\sum_{|\alpha|=\mu}(D^\alpha_x u)^2,$$

$$2m_\mu\sum_{|\alpha|=\mu}a^{ij}(D^\alpha_{x'}D_{x_i}u)(D^\alpha_{x'}D_{x_j}u)+4\sum_{\substack{|\alpha+\beta|=\mu\\ \beta\neq 0}}a^{ij}(D^\alpha_{x'}D^\beta_{x_n}D_{x_i}u)(D^\alpha_{x'}D^\beta_{x_n}D_{x_j}u)$$

$$+\sum_{\substack{|\alpha+\beta|=\mu\\ \beta\neq 0}}a^{ij}(T_{\alpha\beta}u)_{x_i}D^\alpha_{x'}D^\beta_{x_n}D_{x_j}u\geq\frac{3m_\mu}{2}\sum_{|\alpha|=\mu}a^{ij}(D^\alpha_{x'}D_{x_i}u)(D^\alpha_{x'}D_{x_j}u)$$

$$+3\sum_{\substack{|\alpha+\beta|=\mu\\ \beta\neq 0}}a^{ij}(D^\alpha_{x'}D^\beta_{x_n}D_{x_i}u)(D^\alpha_{x'}D^\beta_{x_n}D_{x_j}u)-M_{\mu,1}\sum_{|\alpha|=\mu}(D^\alpha_x u)^2-M_{\mu,2},$$

where H_μ is the maximum of the coefficients before the derivatives of the highest order in $T_{\alpha\beta}$. By analogy we prove

Lemma 4′. *There is a constant* $q_\mu\leq q_{\mu-1}$ *so that when* $c(x)\leq q_\mu$ *in the neighbourhood of* A, *for* $\mu\leq l+1$, *the following estimates are true*: $\max_{\bar{\Omega}}|D^\alpha v^\varepsilon(x)|\leq K'_\mu$, $|\alpha|=\mu$.

Lemma 4″. *There is a constant* $g_\mu\leq g_{\mu-1}$ *so that when* $c(x)\leq g_\mu$ *in the neighbourhood of* A, *for* $\mu\leq l+1$, *the following estimates are true*: $\max_{\bar{\Omega}}|D^\alpha w^\varepsilon(x)|\leq K''_\mu$, $|\alpha|=\mu$.

3. In this paragraph from the uniform estimates which are proved in 2 we will establish the existence and the uniqueness of a solution of the class $C^l(\bar{\Omega})$ of Neuman's problem for the equation (1).

Proof of theorem 1. Let $u_1(x)$, $u_2(x)$ are two classical solutions of (1), (3) and $u(x)=u_1(x)-u_2(x)$. Then $u(x)$ is a solution of the homogeneous boundary value problem

(20) $$Lu=0 \text{ in } \Omega \quad Bu=0 \text{ on } \Gamma.$$

According to the maximum principle it follows that, if $u(x)$ attains a positive maximum in an inner point P_1 of the domain Ω, then $Lu(P_1)<0$ which contradicts (20). If $u(x)$ attains a positive maximum in a point $P_2\in\Gamma$, from theorem 2′ [11] (see also [13]) it follows that $Bu(P_2)<0$, which is impossible because of (20). In the same way it is proved that $-u(x)$ cannot attain a positive maximum in Ω, whence follows the uniqueness of (1), (2). To prove the existence of solution of the class needed, we denote $c_0=\min(c_1, c_2, \dots, c_{l+1})$. When $c(x)\leqq c_0$ in the neighbourhood of Λ, from the a priori estimates in 2, the solutions $u^{\varepsilon}(x)$ of the boundary value problem (6), (3) are uniformly bounded with their derivatives up to the order $l+1$ in $\overline{\Omega}$. Using the Arzela-Ascoly theorem and the diagonalization argument, we can find a sequence $\varepsilon_n\to 0$, so that $D_x^{\alpha}u^{\varepsilon_n}\rightrightarrows D_x^{\alpha}u$ in $\overline{\Omega}$ for $|\alpha|\leqq l$. Therefore letting $\varepsilon_n\to 0$ in (6), (3) we see that $u(x)=\lim_{\varepsilon_n\to 0}u^{\varepsilon_n}(x)$ is a solution of the class $C^l(\overline{\Omega})$ of (1), (3).

The proof of theorems 2, 3 with minor changes follows the scheme of the proof of theorem 1.

Remark. Let Ω be any domain in $\mathbf{R}^n$ with $C^{l+3+\lambda}$ smooth boundary and $\partial\Omega\cap\Lambda$ be situated on a plane part of the boundary of the domain Ω. If the operator L satisfies (i)—(iii), and the boundary value operator satisfies the assumptions in 1, then the conclusions in the theorems 1, 2 and 3 are valid. For our purpose, as in 2 we prove the uniform estimates of the solutions of the regularized problems in the coordinate system with an axis Ox_n coinciding with the direction of the normal to $\partial\Omega\cap\Lambda$. The case when $\partial\Omega\cap\Lambda$ is a sufficiently small part of the boundary is reduced to the one considered above, using a smooth change of the variables, so that the boundary in the neighbourhood of $\partial\Omega\cap\Lambda$ becomes part of a plane.

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