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FACTORIZATION OF LINEAR OPERATORS, MAPPING (DF)-SPACES INTO (F)-SPACES

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The main question of the paper is the following: In which rate global properties of an operator acting between locally convex spaces are determined by its behaviour on the bounded subsets? A statement of this type is Grothendieck's Theorem: A linear continuous operator mapping a (DF)-spaces into a Banach space is (weakly) compact, iff it maps bounded subsets into relatively (weakly) compact sets. The paper generalizes this statement to other properties then (weakly) compactness, in fact, to such properties, which can be formulated in the language of operator ideals (Theorems A and B). Overmore, the results of this paper yield simple criteria for the factorization of operators through Banach spaces, for the idempotency of ideals and statements concerning the structure of the system of bounded subsets of metrizable locally convex spaces.

1. Introduction. In the recent years the theory of ideals of operators acting on Banach spaces was used to investigations of locally convex spaces. This is not very surprising, because problems of the theory of locally convex spaces (as nuclearity, compactness etc.) were one of the springs of the theory of operator ideals. Fundamental results in this field were obtained by A. Grothendieck during the fifties and by A. Pietsch during the sixties. After this only a few new results were known. Now, it seems, that the reach theory of operator ideals put a new light to the theory of the locally convex spaces, especially of (F)- and (DF)-spaces. Here we use the theory of operator ideals to answer the question, in which kind global properties of a linear operator acting on certain locally convex spaces are determined by its behaviour on the bounded subsets.

Let us recall the definition of operator ideals. Only for the sake of simplicity we assume in all the following the completeness of the considered locally convex spaces. If E, F are locally convex spaces, then by $\mathfrak{L}(E, F)$ we denote the set of all linear continuous operators from E into F and by $\mathfrak{F}(E, F)$ we donote the subset of all finite dimensional operators.

1.1. Definition (see [6]). *Let* $\mathbf{C}$ *be a class of locally convex spaces containing all finite dimensional subspaces of the members of* $\mathbf{C}$. *An operator ideal* $\mathfrak{A}$ *over* $\mathbf{C}$ *is a class of linear continuous operators acting between the spaces of* $\mathbf{C}$ *such that the following holds:*

(i) *The components* $\mathfrak{A}(E, F):=\mathfrak{A}\cap\mathfrak{L}(E, F)$ *are linear subspaces of* $\mathfrak{L}(E, F)$ *containing* $\mathfrak{F}(E, F)$ *for all* $E, F\in\mathbf{C}$.

(ii) *If* $R\in\mathfrak{L}(E, E_0)$, $T\in\mathfrak{A}(E_0, F_0)$, $Q\in\mathfrak{L}(F_0, F)$ *then* $QTR\in\mathfrak{A}(E, F)$ *for all* $E, E_0, F, F_0\in\mathbf{C}$.

For $\mathbf{C}$ we will consider the class BAN of all Banach spaces and the class CLCS of all complete locally convex spaces and we will call the ideals BAN-ideals and CLCS-ideals, respectively.

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To obtain statements of the type of G r o t h e n d i e c k's Theorem mentioned in the summary we give the following preliminary definition. Let $\mathcal{A}$ be a BAN-ideal and E, F locally convex spaces. An operator $T \in \mathfrak{L}(E, F)$ is said to have the *weak $\mathcal{A}$-property*, if for all Banach spaces B_1, B_2 and all operators $R \in \mathfrak{L}(B_1, E)$, $Q \in \mathfrak{L}(F, B_2)$ the product $QTR: B_1 \longrightarrow E \longrightarrow F \longrightarrow B_2$ belongs to $\mathcal{A}(B_1, B_2)$. The operator T is said to have the *strong $\mathcal{A}$-property*, if there are Banach spaces B_1, B_2 and a factorization

$$\begin{array}{ccc} E & \xrightarrow{\;T\;} & F \\ {\scriptstyle R}\downarrow & & \uparrow{\scriptstyle Q} \\ B_1 & \xrightarrow{\;T_0\;} & B_2 \end{array}$$

of T with operators $R \in \mathfrak{L}(E, B_1)$, $Q \in \mathfrak{L}(B_2, F)$ and $T_0 \in \mathcal{A}(B_1, B_2)$. By $\mathcal{A}^w$ and $\mathcal{A}^s$ we denote the classes of all operators having the weak or the strong $\mathcal{A}$-property, respectively. Of course, the strong $\mathcal{A}$-property implies the weak $\mathcal{A}$-property, $\mathcal{A}^s \subseteq \mathcal{A}^w$. The main problem consists in finding of sufficient conditions, such that the weak $\mathcal{A}$-property implies the strong one. Theorems of his type are called "factorization theorems". This will be done in the next section. Let us supply the above definition by the following consideration. By a CLCS-*extension* of a BAN-ideal $\mathcal{A}$ we mean a CLCS-ideal $\mathcal{A}^{\text{ext}}$ satisfying the equation $\mathcal{A}^{\text{ext}}(B_1, B_2) = \mathcal{A}(B_1, B_2)$ for all Banach spaces B_1, B_2.

1.2. P r o p o s i t i o n. *The smallest and the greatest* CLCS-*extension of a* BAN-*ideal $\mathcal{A}$ are the classes $\mathcal{A}^s$ and $\mathcal{A}^w$, respectively.*

We omit the simple proof and refer to [6, 29.5]. By this proposition, $\mathcal{A}^w = \mathcal{A}^s$ would imply the uniqueness of CLCS-extensions. At last we give a simple but more interior criteria for the weak and strong $\mathcal{A}$-property, respectively. Let E be a locally convex space. By $\mathfrak{B}(E)$ we donote the directed from below systtem of all bounded, absolutely convex subsets of E. By $\mathfrak{U}(E)$ we denote the directed from above system of all absolutely convex neighbourhoods of zero of E. Let $A \in \mathfrak{B}(E)$ and $U \in \mathfrak{U}(E)$ be given. By p_A and p_U we denote their gauge functionals. Let us define the linear spaces $E(A) = \cup_{n=1}^{\infty} nA$ and $E/U = E/p_U^{-1}(0)$, which can be normed by $\|x\|_A = p_A(x)$ and $\|\widehat{x}\|_U = p_U(x)$, respectvely. Their completions we denote by E_A and E_U, respectively. There are canonical linear continuous mappings $C_A : E_A \longrightarrow E$ and $C_U : E \longrightarrow E_U$, which are defined by $C_A x = x$ and $C_U x = \widehat{x}$, respectively. The product of these operators we denote by $C_{AU} = C_U C_A : E_A \longrightarrow E_U$. Futhermore, if $A, B \in \mathfrak{B}(E)$ with $A \subset \varrho B$ and $U, V \in \mathfrak{U}(E)$ with $V \subset \varrho U$ for some $\varrho > 0$, then there exist canonical mappings $C_{AB} : E_A \longrightarrow E_B$ and $C_{VU} : E_V \longrightarrow E_U$. These mappings are unique defined by the equations $C_B C_{AB} = C_A$ and $C_{VU} C_V = C_U$.

1.3. P r o p o s i t i o n. *Let $\mathcal{A}$ be a* BAN-*ideal. An operator $T \in \mathfrak{L}(E, F)$ belongs to $\mathcal{A}^w$ if and only if for all $A \in \mathfrak{B}(E)$ and all $U \in \mathfrak{U}(F)$ the product $C_U T C_A$ belongs to $\mathcal{A}(E_A, F_U)$. The operator T belongs to $\mathcal{A}^s$ if and only if there are $U \in \mathfrak{U}(E)$, $A \in \mathfrak{B}(F)$ and $T_0 \in \mathcal{A}(E_U, F_A)$ such that $T = C_A T_0 C_U$.*

We omit the simple proof.

2. Factorization theorems for closed ideals. In this section we answer our main problem, to find sufficient conditions for $\mathcal{A}^w = \mathcal{A}^s$, in the case, when $\mathcal{A}$ is a closed BAN-ideal. A BAN - ideal $\mathcal{A}$ is called *closed*, if for all Banach spaces B_1, B_2 the ideal components $\mathcal{A}(B_1, B_2)$ are closed in $\mathfrak{L}(B_1, B_2)$ with respect to the uniform operator topology. At first, we have to correct our problem, because

$\mathcal{A}^w \neq \mathcal{A}^s$ for all BAN-ideals. This can be seen as follows. Let E be an infinite dimensional Banach space and E_σ its associated weak topologized space. The identity map $1: E_\sigma \longrightarrow E_\sigma$ belongs of course to $\mathcal{A}^w$ for all $\mathcal{A}$. But if this mapping would belong to $\mathcal{A}^s$ too, then there would be a bounded weak neighbourhood. Therefore, E_σ would be normed. This is possible only for finite dimensional E, a contradiction. Therefore, we correct our question as follows:

Find sufficiently large classes $\mathfrak{K}_1$, $\mathfrak{K}_2$ of (complete) locally convex spaces and general assumptions on the ideal $\mathcal{A}$, such that $\mathcal{A}^w(E, F) = \mathcal{A}^s(E, F)$ for all $E \in \mathfrak{K}_1$ and $F \in \mathfrak{K}_2$.

It will be seen, that the classes of (F)- and (DF)-spaces or the larger class of the quasinormable spaces are very convenient to answer the problem. Let us recall the definitions.

2.1. Definition ([3, 14. III. 1, Lemma 6]). *A locally convex space E is called quasinormable, if for every neighbourhood $U \in \mathfrak{V}(E)$ there is a $V \in \mathfrak{V}(E)$ such that for every $\varepsilon > 0$ there is a bounded set $M_\varepsilon \subset E$ satisfying the inclusion $V \subset \varepsilon U + M_\varepsilon$.*

2.2. Definition ([3, p. 64]). *A locally convex space E is called a* (DF)*-space, if it has a countable increasing fundamental system of bounded convex sets (B_n) and if the intersection $V = \bigcap_{n=1}^{\infty} U_n$ of every countable system of neighbourhoods $U_n \in \mathfrak{V}(E)$ is a neighbourhood of zero, assumed that V absorbs each bounded subset of E.*

It is known and easy to see, that each (DF)-space is a quasinormable space. An (F)-space is a complete locally convex space E containing a countable basis of neighbourhoods of zero. The intersection of the classes of the complete (DF)- and (F)-spaces coincides with the class of all Banach spaces. Not every (F)-space is quasinormable.

2.3. Definition ([6]). *A* BAN*-ideal $\mathcal{A}$ is called injective, if for arbitrary Banach spaces B_i $(i = 1, 2, 3)$ and operators $T \in \mathfrak{L}(B_1, B_2)$, $S \in \mathcal{A}(B_1, B_3)$ the in equality $\|Tx\| \leq \|Sx\|$ for all $x \in B_1$ implies $T \in \mathcal{A}(B_1, B_2)$.*

The ideal $\mathcal{A}$ is called surjective, if $T \in \mathfrak{L}(B_1, B_3)$, $S \in \mathcal{A}(B_2, B_3)$ and $T(U_1) \subset S(U_2)$ implies $S \in \mathcal{A}(B_1, B_3)$, where U_i denotes the unit ball in B_i $(i = 1, 2)$.

By using standard arguments one can see, that the injectivity and the surjec tivity, respectively, of an ideal $\mathcal{A}$ means, that the property $T \in \mathcal{A}$ is invariant under changing the domain and the image space of T, respectively. The ideal $\mathfrak{K}$ of all compact operators and the ideal $\mathfrak{W}$ of all weakly compact operators are both, surjective and injective. Now we are ready to state our theorems.

Theorem A. *Let $\mathcal{A}$ be a closed surjective ideal. For all quasinormable complete spaces E and all Banach spaces B we have $\mathcal{A}^w(E, B) = \mathcal{A}^s(E, B)$.*

Theorem B. *Let $\mathcal{A}$ be a closed, injective and surjective ideal. For all complete* (DF)*-spaces E and all (F)-spaces F we have $\mathcal{A}^w(E, F) = \mathcal{A}^s(E, F)$.*

The proof of the theorems requires some propositions of the theory of BAN-ideals, on the one hand, and some insight to the structure of quasinormable spaces, on the other hand. We start with the first one.

The *cartesian l_p-product* $(1 \leq p \leq \infty)$ of a sequence $(X_n)_{n \in \mathbb{N}}$ of Banach spaces is the Banach space

$$X = l_p((X_n)_{n \in \mathbb{N}}) = \{x = (x_n)_{n \in \mathbb{N}} : x_n \in X_n,\ \|x\| := \|(\|x_n\|)\|_p < \infty\}.$$

The canonical projections $\mathrm{pr}_k : X \to X_k$ defined by $\mathrm{pr}_k((x_n)) = x_k$ and the canonical injections $\mathrm{inj}_k : X_k \to X$ defined by $\mathrm{inj}_k(x_k) = (\delta_{nk} x_k)_{n \in \mathbb{N}}$ are continuous.

2.4. Proposition. *Let $\mathcal{A}$ be a closed* BAN-*ideal. If* (T_n), (R_n) *are two sequences of operators* $T_n \in \mathcal{A}(X_n, Y)$ *and* $R_n \in \mathcal{A}(X, X_n)$ *satisfying* $\Sigma_{n=1}^{\infty} \|T_n\| < \infty$, $\Sigma_{n=1}^{\infty} \|R_n\| < \infty$, *then for all* $1 \leqq p \leqq \infty$ *the operators*

$$\Sigma : l_p((X_n)) \to Y \text{ defined by } \Sigma(x_n) := \Sigma_{n=1}^{\infty} T_n x_n,$$

$\Delta : X \to l_p((Y_n))$ *defined by* $\Delta(x) = (R_n x)$ *belong to* $\mathcal{A}$.

Proof. By assumption we have $\Phi_n := T_n \operatorname{pr}_n \in \mathcal{A}(l_p((X_n)), Y)$. The series $\Psi := \Sigma_{n=1}^{\infty} \Phi_n$ converges in the uniform operator topology because of $\Sigma_{n-1}^{\infty} \|\Phi_n\| = \Sigma_{1=n}^{\infty} \|T_n \operatorname{pr}_n\| \leqq \Sigma_{n=1}^{\infty} \|T_n\| < \infty$. Therefore, we have $\Psi \in \mathcal{A}(l_p(X_n), Y)$. Now, the statement follows from

$$\Psi(x_n) = \lim_{k \to \infty} \sum_{n=1}^{k} \Phi_n(x_n) = \lim_{k \to \infty} \sum_{n=1}^{k} T_n x_n = \sum_{n=1}^{\infty} T_n x_n = \Sigma(x_n).$$

By the analogous method one proves $\Delta \in \mathcal{A}$.

The following lemma is based on an idea of [8] and generalizes a method of construction of neighbourhoods in (DF)-spaces. Let M be a subset of a linear space E. By abs conv M we denote the *absolutely convex hull* of M.

2.5. Lemma. *Let E be a linear space. If* $(U_i)_{i \geqq 0}$ *is a decreasing sequence and* $(M_i)_{i \geqq 0}$ *is an increasing sequence of subsets of* E *with* $M_0 = \{0\}$, *then the sets* $V = \cap_{i=0}^{\infty} (U_i + M_i)$ *and* $W = \text{abs conv } \cup_{i=1}^{\infty} 2^{i+1} (U_{i-1} \cap M_i)$ *satisfy the following conditions:*

(i) *If* $E = \cup_{i=0}^{\infty} M_i$, *then* $V \subseteq W$.

(ii) *In general the inclusion is satisfied only approximately in the following sense:*
For each $x \in V$ *there is a sequence* (w_i) *such that*

a) $w_i \in 2^{i+1} (U_{i-1} \cap M_i)$ *for all* $i \geqq 1$,

b) $x - \Sigma_{i=1}^{n} 2^{-i} w_i \in U_n$ *for all* $n \geqq 1$.

Proof. For any $x \in V$ and every $i \geqq 0$ there exist partitions $x = u_i + m_i$ with $u_i \in U_i$ and $m_i \in M_i$. For $n \geqq 1$ we put $x_n := m_n - m_{n-1}$. Then we have

$$\sum_{i=1}^{n} x_i + u_n = m_n + u_n = x. \tag{1}$$

From $x_n \in 2M_n$ and $x_n = u_{n-1} - u_n \in 2U_{n-1}$ we obtain $x_n \in 2(U_{n-1} \cap M_n)$. Define $w_n = 2^n x_n \in 2^{n+1} (U_{n-1} \cap M_n)$. Equation (1) implies $x - \Sigma_{i=1}^{n} 2^{-i} w_i = u_n \in U_n$. This shows (ii). In the case (i) there is a $k \geqq 0$ such that $x \in M_k$. For this k we have $u_k = x - m_k \in 2M_k \cap U_k \subset 2(M_k \cap U_{k-1})$. Theorefore, $2^k u_k \in W$. From $w_i \in W$ and $\Sigma_{i=1}^{k} 2^{-i} + 2^{-k} = 1$ we get $x = \Sigma_{i=1}^{k} 2^{-i} w_i + 2^{-k} 2^k u_k \in W$.

Proof of Theorem A. Let $T \in \mathcal{A}^w(E, B)$ be given. By U_B we denote the unit ball in B. Its preimage $U := T^{-1}(U_B)$ is a neighbourhood in E. We choose a summable decreasing sequence (ν_i) of positive real numbers and define $a_i := 2^{-i} \nu_i$. Because E is quasinormable, there is an increasing sequence (M_i) of absolutely convex bounded subsets $M_i \subseteq E$ with $M_0 = \{0\}$ such that the set

$$V := \bigcap_{i=0}^{\infty} a_i U + M_i$$

is a neighbourhood of E. From $V \subset \alpha_0 U$ it follows $\|Tx\| \leq \alpha_0\, p_V(x)$. Therefore, by $T_V(C_V x) := Tx$ for $x \in E$ a linear continuous operator $T_V: E_V \longrightarrow B$ is defined. We will show $T_V \in \mathcal{A}(E_V, B)$. For $i \geq 1$ let us consider the bounded subsets $A_i = 2^{i-1}(\alpha_{i-1} U \cap M_i)$ of E. By assumption on T, the operators $T_i := TC_{A_i}: E_{A_i} \longrightarrow B$ belong to $\mathcal{A}$. The inclusion $A_i \subset 2^{i-1}\alpha_{i-1} U = \nu_{i-1} U$ implies $T(A_i) \subset \nu_{i-1} U_B$, therefore $\|T_i\| \leq \nu_{i-1}$. Let $Z = l_1((E_{A_i}))$. By proposition 2.4 the operator

$$\Sigma: Z \longrightarrow B, \ \Sigma(x_i) = \sum_{i=1}^{\infty} Tx_i$$

belongs to $\mathcal{A}(Z, B)$. Let U_Z be the unit ball of the Banach space Z. To complete the proof, it is sufficient by definition 2.3, to show the inclusion $T_V(C_V x) = T(V) \subset \Sigma(4U_Z)$. By the preceding lemma for each $x \in V$ there is a sequence of elements $w_i \in 2^{i+1}(\alpha_{i-1} U \cap M_i) = 4A_i$ $(i \geq 1)$ such that

$$x - \sum_{i=1}^{n} 2^{-i} w_i \in \alpha_n U \qquad \text{for all } n \geq 1.$$

Using this and $T(U) \subset U_B$ it follows $\|Tx - \sum_{i=1}^{n} T(2^{-i} w_i)\| \leq \alpha_n$, therefore $Tx = \sum_{i=1}^{\infty} T(2^{-i} w_i)$. On the other hand, the sequence $z = (2^{-i} w_i)_{i \geq 1}$ belongs to $4U_Z$ because of $\|z\| = \sum_{i=1}^{\infty} p_{A_i}(2^{-i} w_i) \leq \sup_{i \geq 1} p_{A_i}(w_i) \leq 4$. But $\Sigma z = \sum_{i=1}^{\infty} T(2^{-i} w_i) = Tx$. This shows $T(V) \subset \Sigma(4U_Z)$.

The Theorem B will be an easy consequence of the following much more stronger proposition, which yields further interesting corollaries.

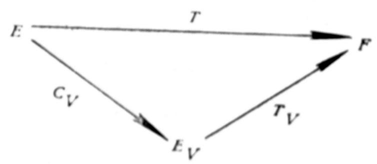
2.6. Proposition. *Let $\mathcal{A}_1$ be a closed injective and $\mathcal{A}_2$ a closed surjective* BAN-*ideal. If E is a complete (DF)-space and if F is an (F)-space, then for each operator $T \in \mathcal{A}_1^w(E, F) \cap \mathcal{A}_2^w(E, F)$ there is a neighbourhood V of E and a factorization $T = T^V C_V$ such that $C_V \in \mathcal{A}_1^w(E, E_V)$ and $T_V \in \mathcal{A}_2^w(E_V, F)$.*

Proof. Let T be given. By assumption, there are a countable increasing fundamental system (A_i) of the absolutely convex bounded subsets of E and a countable decreasing basis of neighbourhoods (U_i') of F. Now, we choose two decreasing summable sequences (ν_i) and (λ_i) of positive real numbers and define

(1) $$\alpha_i := 2^{-i}\nu_i, \ U_i := T^{-1}(U_1') \in \mathfrak{U}(E),$$

$$W_i := \alpha_i U_i + \lambda_i^{-1} A_i, \quad V := \bigcap_{i=0}^{\infty} W_i.$$

Obviously, the set V absorbs each bounded subset of E. Therefore, it must be a neighbourhood of the (DF)-space E. Because the A_i are bounded, there are numbers γ_i such that $V \subset \alpha_i U_i + \lambda_i^{-1} A_i \subset \gamma_i U_i$. From this it follows $T(V) \subset \gamma_i U_i'$. Therefore, by $T_V(C_V x) = Tx$ a linear continuous factorization



is defined. Now, we prove $T_V \in \mathcal{A}_2^w(E_V, F)$. Let j be fixed. Then we have

$$V \subset \gamma_j U_j \cap \bigcap_{i=j}^{\infty} (a_i U_j + \lambda_i^{-1} A_i) =: V_j.$$

As in the proof of Theorem A applied to the operator $T_j := C_{U'_j} T : E \longrightarrow F_{U'_j}$ one checks, that this operator factors through E_{V_j} in the form $T_j = T_{V_j} C_{V_j}$, where T_{V_j} belongs to $\mathcal{A}_2(E_{V_j}, F_{U'_j})$.

Because of $C_{U'_j} T_V C_V = C_{U'_j} T = T_j = T_{V_j} C_{V_j} = T_{V_j} C_{VV_j} C_V$ we have $C_{U'_j} T_V = T_{V_j} C_{VV_j} \in \mathcal{A}_2(E_V, F_{U'_j})$. This shows $T_V \in \mathcal{A}_2^w(E_V, F)$.

To prove $C_V \in \mathcal{A}_1^w(E, E_V)$, we consider the neighbourhoods $W'_i := U'_i + T(\lambda_i^{-1} A_i) \in \mathfrak{U}(F)$ and show

(2) $$p_{W_i}(x) = p_{W'_i}(Tx) \qquad \text{for all } x \in E \text{ and all } i \geq 0.$$

Because of $T(U_i) \subset U'_i$ we get $T(W_i) \subset W'_i$ and $p_{W'_i}(Tx) \leq p_{W_i}(x)$ for all $x \in E$. Conversely, let us suppose $Tx \in W'_i$. This means, that there are elements $u'_i \in U'_i$ and $a_i \in A_i$ such that $Tx = u'_i + T(\lambda_i^{-1} a_i)$. Therefore, $u'_i \in T(E)$ and there is a $u_i \in U_i$ such that $Tu_i = u'_i$. The element $w := u_i + \lambda_i^{-1} a_i$ belongs to W_i and we have $Tw = Tx$. This implies $T(w-x) = 0$, $p_{U_i}(w-x) = 0$, $p_{W_i}(w-x) = 0$, $p_{W_i}(x) = p_{W_i}(w) \leq 1$. This proves (2).

We define mappings $R_i := C_{W'_i} T : E \longrightarrow F_{W'_i}$ and fix j. By assumption, the mappings $R_i C_{A_j} : E_{A_j} \longrightarrow F_{W'_i}$ belong to $\mathcal{A}_1$. For $x \in A_j$ and all $i \geq j$ the following estimation holds:

$$\| R_i C_{A_j} x \|_{W'_i} = p_{W'_i}(Tx) = p_{W_i}(x) \leq \lambda_i \, p_{A_i}(x) \leq \lambda_i.$$

Therefore, we have $\Sigma_{i=1}^{\infty} \| R_i C_{A_j} \| < \infty$. Using proposition 2.4 there is an operator $\Delta \in \mathcal{A}_1(E_{A_j}, l_\infty((F_{W'_i})))$ defined by $\Delta x = (R_i C_{A_j} x)_{i \in \mathbb{N}}$ for all $x \in E_{A_j}$. Furthermore, for all $x \in E_{A_j}$ the equation (2) implies

$$\| C_{A_j V} x \|_V = p_V(x) = \sup_i p_{W_i}(x) = \sup_i p_{W'_i}(Tx) = \| \Delta x \|.$$

Because $\mathcal{A}_1$ is injective, it follows $C_{A_j V} \in \mathcal{A}_1(E_{A_j}, E_V)$. But j was arbitrary, therefore $C_V \in \mathcal{A}_1^w(E, E_V)$. This completes the proof of the proposition.

P r o o f o f T h e o r e m B. By the above proposition the operator $T \in \mathcal{A}^w(E, F)$ has a factorization $T = T_V C_V$ such that $C_V \in \mathcal{A}^w(E, E_V)$. Applying Theorem A we obtain $C_V \in \mathcal{A}^s(E, E_V)$. Therefore,

$$T = T_V C_V \in \mathcal{A}^s(E, F).$$

From proposition 2.6 we get an interesting corollary. The part on BAN-ideals was proved indepedently by H e i n r i c h [4] using the method of interpolation.

2.7. Corollary. *Let $\mathcal{A}_1$ be an injective and $\mathcal{A}_2$ be a surjective closed* BAN-*ideal. If E is a complete (DF)-space and F is an (F)-space, then the following equation holds:* $(\mathcal{A}_1^w \cap \mathcal{A}_2^w)(E,F) = (\mathcal{A}_2\ \mathcal{A}_1)^w(E, F)$. *Especially,* $\mathcal{A}_1 \cap \mathcal{A}_2 = \mathcal{A}_2\ \mathcal{A}_1$.

Proof. Let $T \in (\mathcal{A}_1^w \cap \mathcal{A}_2^w)(E, F)$ and $A \in \mathfrak{B}(E)$, $U \in \mathfrak{U}(F)$ be given. By proposition 2.6 there is a factorization $T = T_V C_V$ with $V \in \mathfrak{U}(E)$, $C_V C_A \in \mathcal{A}_1(E_A, E_V)$, $C_U T_V \in \mathcal{A}_2(E_V, F_U)$. Therefore, $C_U T C_A = C_U T_V\ C_V C_A \in \mathcal{A}_2\ \mathcal{A}_1$. This shows $T \in (\mathcal{A}_2\ \mathcal{A}_1)^w$. The other inclusion is obvious. Banach spaces are at the same time (DF)- and (F)-spaces. This implies the equation $\mathcal{A}_1 \cap \mathcal{A}_2 = \mathcal{A}_2\ \mathcal{A}_1$.

Immediately from 2.7 we obtain:

2.8. Corollary. *Each injective and surjective closed BAN-ideal $\mathcal{A}$ is idempotent, i. e. $\mathcal{A}\ \mathcal{A} = \mathcal{A}$.*

3. Applications and counterexamples. We consider some applications. Of course, Theorem A yields the mentioned above Grothendieck's Theorem and the results of [2], because it is applicable to the BAN-ideals $\mathfrak{K}$ and $\mathfrak{W}$ of the compact and weakly compact operators, respectively. The idempotency of these ideals, which follows from corollary 2.8, was proved only in 1972 [7] and 1974 [1], respectively. There is another corollary, implicitly contained in [3, Theorem 2]. A linear operator T is called *strongly bounded,* if there is a neighbourhood which is mapped by T into a bounded set.

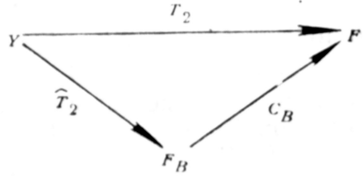
3.1. Corollary. *Any linear continuous operator from a (DF)-space into an (F)-space is strongly bounded.*

Proof. Without loss of generality we may suppose, that the (DF)-space is complete. The assertion follows now from Theorem B with $\mathcal{A} = \mathfrak{L}$.

3.2. Now we apply the results to another question. In general it is only a little known about the structure of the system of the bounded subsets in an (F)-space.

Proposition. *Let $\mathcal{A}$ be an injective, surjective and closed ideal, let F be an (F)-space. If $A \in \mathfrak{B}(F)$ and $C_A \in \mathcal{A}^w(F_A, F)$, then there exists a set $B \in \mathfrak{B}(F)$ such that $A \subset B$ and $C_{AB} \in \mathcal{A}$, $C_B \in \mathcal{A}^w(F_B, F)$.*

Proof. We apply proposition 2.6 to the mapping $C_A : F_A \longrightarrow F$. Therefore, there are a Banach space Y and operators $T_1 \in \mathcal{A}(F_A, Y)$, $T_2 \in \mathcal{A}^w(Y, F)$ such that $C_A = T_2 T_1$. Without loss of generality we may assume, that $T_1(A)$ is contained in the unit ball S_Y of Y. The set $B := \overline{T_2(S_Y)}$ is an absolutely convex bounded subset of F and we have $A \subset \overline{C_A(A)} = \overline{T_2 T_1(A)} \subset \overline{T_2(S_Y)} = B$. By definition of B, there is a factorization



and because $\mathcal{A}$ is surjective, from $T_2 \in \mathcal{A}^w(Y, F)$ it follows $C_B \in \mathcal{A}^w(F_B, F)$. Finally, we have $C_{AB} = \widehat{T}_2 T_1 \in \mathcal{A}$.

3.3. Let us turn to the theory of $\mathcal{A}$-spaces. Let $\mathcal{A}$ be an operator ideal. A locally convex space E is called an *$\mathcal{A}$-space,* if for any $U \in \mathfrak{U}(E)$ there is a

$V \in \mathfrak{V}(E)$ such that the mapping C_{VU} belongs to $\mathcal{A}$. E is called a co *$\mathcal{A}$-space,* if for any $A \in \mathfrak{B}(E)$ there is a $B \in \mathfrak{B}(E)$ such that C_{AB} belongs to $\mathcal{A}$. From Theorem A we conclude a result, which is dual in some sence to [5, Theo rem 4.1]:

Proposition. *If $\mathcal{A}$ is a closed surjective ideal, then any quasinormable co$\mathcal{A}$-space in an $\mathcal{A}$-space.*

A BAN-ideal $\mathcal{A}$ is called *symmetric,* if $T \in \mathcal{A}(B_1, B_2)$ implies $T' \in \mathcal{A}(B_2', B_1')$. The ideals $\mathfrak{K}$ and $\mathfrak{W}$ are symmetric.

Corollary. *If $\mathcal{A}$ is a closed surjective symmetric ideal, then the dual space of each (F)-$\mathcal{A}$-space F is an $\mathcal{A}$-space.*

Proof. By the above proposition it is sufficient to show, that F_b' is a co$\mathcal{A}$-cpace. Because the (F)-space F is barrelled, the bounded subsets of F' are equicontinuous. By assumption on F for any $U \in \mathfrak{V}(F)$ there is a $V \in \mathfrak{V}(F)$ that $V \subset U$ and $C_{VU}: F_V \longrightarrow F_U \in \mathcal{A}$. This implies $C'_{VU}: (F_U)' \longrightarrow (F_V)' \in \mathcal{A}$. On the other hand, the spaces $(F_U)'$ and $(F_V)'$ are isomorphic in a natural way to F'_{U^0} and F'_{U^0}, where U^0 and V^0 denotes the polar sets of U and V. Therefore, we have $C_{U^0V^0}: F'_{U^0} \longrightarrow F'_{V^0} \in \mathcal{A}$. Thus, F' is a co$\mathcal{A}$-space.

3.4. The Theorem A fails in general, if the assumption on E is weakened to be an (F)-space. Indeed, the (F)-space $E = R^{\mathbb{N}}$ with the pointwise topology is quasinormable and, of course, $1_E \in \mathfrak{L}^w(E, E)$. But the canonical spaces E_U are isomorphic to some R^n. Therefore, $\mathfrak{L}^s(E, E)$ contains only finite dimensional operators. This shows $1_E \notin \mathfrak{L}^s(E, E)$.

3.5. The Theorem A fails in general, if the assumption of quasinormability of E is replaced by the metrizability. This follows from [5, 5.3 Cor. 2].

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