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LAX-TYPE EQUIVALENCE THEOREMS WITH ORDERS FOR LINEAR EVOLUTION OPERATORS

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Lax-type equivalence theorems concerning the numerical solution of temporally inhomogeneous initial-value problems by difference schemes with arbitrary stepsizes are deduced in terms of the theory of discrete approximations. In this general setting it is shown that stability is not only sufficient but also necessary for convergence provided the difference scheme is consistent (analogous to the fundamental result of P. Lax 1954/56). Moreover it is shown that also consistency is necessary for convergence if some slight additional assumptions are made upon the differential equation. Both Lax-type theorems are given with orders, i. e., the concepts of stability, consistency, and convergence are equipped with orders.

1. Introduction. In this note, we are concerned with linear initial-value problems of the form

$$\frac{d}{dt}u(t)=A(t)u(t),\quad u(s)=f\in X,\quad 0\leq s\leq t, \tag{1.1}$$

where X denotes an arbitrary Banach space, $A(t)$, $t\geq 0$, are closed linear operators with domain and range in X, and the initial-value $f\in X$ is supposed to be given at time $s\geq 0$. Following up our previous investigations on Lax-type theorems with orders (cf. [4—6]), the main points of the present note are that not necessarily equidistant but arbitrary (time-) steps t_j are admitted in the construction of discretisations of (1.1), and that the entire reasoning is given in terms of the theory of discrete approximations (for this theory see e. g. [13, 14]). To this end, some preliminaries and notations are given in Sec. 2. The basic definitions of consistency, stability, and convergence with orders are then formulated in Sec. 3 in such a way that equivalence theorems of Lax-type hold true also in this more general situation. Thus, Thm. 1 states that stability is not only sufficient but also necessary for convergence, provided consistency is given. Moreover, under some stronger assumptions upon problem (1.1) (cf. (3.6)), consistency is not only sufficient but also necessary for convergence (cf. Thm. 2), all concepts taken with orders. Sec. 4 concludes with an elementary example in connection with a time-dependent heat conduction problem. Although in numerical analysis one is primarily interested in sufficient conditions for (certain rates of) convergence, their necessity indicates that they are adequate and in a certain sense minimal.

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2. Preliminaries and notations. For Banach spaces X and Y (with norms $\|\circ\|_X$ and $\|\circ\|_Y$, respectively) let $[X, Y]$ denote the space of bounded linear operators from X into Y. Concerning the initial-value problem (1.1), the Sobolewski — Tanabe theory states that under some regularity conditions upon the operators $A(t)$ the solution is given via a family of evolution operators (or propagators) $\{E(t, s);\ 0 \leq s \leq t\} \subset [X] := [X, X]$ by $u(t) = E(t, s)u(s) = E(t, t))f,\ t \geq s$ (cf. [7, p. 103 ff], [11, p. 55 ff] see also [1, p. 188 ff]). These evolution operators satisfy the principle of Hadamard, namely,

$$E(t, r) = E(t, s)E(s, r), \qquad 0 \leq r \leq s \leq t, \tag{2.1}$$

$$E(t, t) = I\ (:= \text{identity on } X), \qquad 0 \leq t, \tag{2.2}$$

and are uniformly bounded on $[0, T]$, i. e.,

$$\|E(t, s)f\|_X \leq L\|f\|_X, \qquad f \in X,\ 0 \leq s \leq t \leq T. \tag{2.3}$$

Here $[0, T]$ denotes an arbitrary but fixed, finite or infinite interval. Obviously conditions (2.1—3) are generalizations of those semigroup properties, that the solutions possess in case of a properly posed initial-value problem (1.1) for time-independent $A(t) \equiv A$.

A numerical solution of the initial-value problem (1.1) then calls for difference schemes $\{D(t, s)\}$ which furnish an approximation to the corresponding evolution operators. Thus one approximates $E(t_n, t_0)$ by iterated difference operators $\Pi_{j=0}^{n-1} D(t_{j+1}, t_j)$, whereby the (time-) points t_j define a (finite) partition $0 \leq t_0 \leq t_1 \leq \cdots \leq t_{n-1} \leq t_n \leq T$ of the interval $[0, T]$. Here the iterated difference operators are only considered for those subclasses $\boldsymbol{Z}_0 \subset \boldsymbol{Z}$ ($:=$ set of all partitions of $[0, T]$) which satisfy the stability condition

$$Z \in \boldsymbol{Z}_0 \Rightarrow \boldsymbol{Z}_{a,b} := Z \cap [a, b] \in \boldsymbol{Z}_0, \qquad 0 \leq a \leq b \leq T. \tag{2.4}$$

Furthermore, since numerical calculations may only give and may only start off with a finite number of data (e. g. approximate values on a finite mesh), the numerical operators $D(t, s)$ are assumed to belong to $[X_s, X_t]$, where X_t is a Banach space for each $t \in [0, T]$ (with norm $\|\circ\|_t$). Thus one requires operators $p_t \in [X, X_t]$ in order to begin the numerical calculation $\Pi_{j=0}^{n-1} D(t_{j+1}, t_j)$ for any initial-value $f \in X$ as well as to compare the numerical result with the exact solution $E(t_n, t_0)f \in X$. These operators should satisfy (cf. [14, p. 8, 28])

$$\|p_t f\|_t \leq M\|f\|_{X}, \qquad f \in X,\ 0 \leq t \leq T, \tag{2.5}$$

$$\begin{cases} p_t(X) = X_t, \text{ i. e., } p_t \text{ is surjective, and} \\ \inf\limits_{f \in X, p_t f = h} \|f\|_X \leq M' \|h\|_t,\ h \in X_t,\ 0 \leq t \leq T. \end{cases} \tag{2.6}$$

Bounds for the approximation error

$$\Big\|\prod_{j=0}^{n-1} D(t_{j+1}, t_j) p_{t_0} f - p_{t_n} E(t_n, t_0) f\Big\|_{t_n}, \quad \{t_j\}_{j=0}^n \in \boldsymbol{Z}_0 \tag{2.7}$$

will be given in terms of the (modified) $\boldsymbol{K}$-functional, defined for subspaces $U \subset X$ (with seminorm $|\circ|_U$) by

$$\boldsymbol{K}(t, f) := \boldsymbol{K}(t, f; X, U) := \inf_{g \in U}\{\|f - g\|_X + t|g|_U\}$$

for $t \geq 0$, $f \in X$. It has the properties (cf. [2, p. 161 ff])

$$\text{(2.8)}\qquad K(s,f)\le\begin{cases}\|f\|_X, & f\in X\quad s\ge 0,\\ s|f|_U, & f\in U,\quad s\ge 0,\\ K(t,f), & f\in X,\quad 0\le s\le t,\end{cases}$$

and is an abstract analog for the classical modulus of continuity in concrete spaces. Indeed, in case $X=C_{UB}$, the space of uniformly continuous, bounded functions on the real axis $\mathbb{R}$ with usual sup-norm, one has for $0<\beta\le r$, $r\in\mathbb{N}$ (: = set of natural numbers)

$$\text{(2.9)}\qquad \begin{cases}K(t,f;C_{UB},C_{UB}^r)=O(t^{\beta/r}), \qquad t\to 0+,\\ \text{if and only if}\\ f\in \mathrm{Lip}_r\,\beta:=\{f\in C_{UB};\ \|[T_\delta-I]^r f\|_C=O(\delta^\beta),\ \delta\to 0+\}\end{cases}$$

(cf. [2, p. 191 ff]). Here $[T_\delta f](x):=f(x+\delta)$, $x,\delta\in\mathbb{R}$ is the translation operator and

$$C_{UB}^r:=\{f\in C_{UB};\ f^{(j)}\in C_{UB},\ 0\le j\le r\}\ \text{with}\ |f|_U:=\|f^{(r)}\|_C.$$

3. Main definitions and results. Stability and consistency with orders read as follows in the (foregoing) framework of discrete approximations with arbitrary stepsizes.

Definition 1. *The difference scheme $\{D(t,s)\in[X_s,X_t];\ 0\le s\le t\le T\}$ is said to be stable of order $O(1/\psi)$ if for any $\{t_j\}_{j=0}^n:=Z\in Z_0\subset Z$, $h\in X_{t_0}$*

$$\text{(3.1)}\qquad \Big\|\prod_{j=0}^{n-1}D(t_{j+1},t_j)h\Big\|_{t_n}\le\frac{S}{\psi(Z)}\|h\|_{t_0},$$

where S is a positive constant and ψ a positive function on Z_0 with (cf. (2.4))

$$\text{(3.2)}\qquad S/\psi(\{t\})=1,\qquad \{t\}=Z\in Z_0,\ 0\le t\le T,$$

$$\text{(3.3)}\qquad S/\psi(Z_{a,b})\le S/\psi(Z),\qquad Z\in Z_0,\ 0\le a\le b\le T.$$

Since $\prod_{j=0}^{-1}D(t_{j+1},t_j):=I\in[X_{t_0}]$, $0\le t_0\le T$, condition (3.2) obviously is no restriction.

Definition 2. *The difference scheme $\{D(t,s)\in[X_s,X_t];\ 0\le s\le t\le T\}$ is said to be consistent with the evolution operators $\{E(t,s);\ 0\le s\le t\le T\}\subset[X]$ on $U\subset X$ of order $O(\varphi)$, if for any $g\in U$, $\{t_j\}_{j=0}^n:=Z\in Z_0$, $0\le j\le n-1$*

$$\text{(3.4)}\quad \|[D(t_{j+1},t_j)p_{t_j}-p_{t_{j+1}}E(t_{j+1},t_j)]E(t_j,t_0)g\|_{t_{j+1}}\le C[t_{j+1}-t_j]\varphi(t_j,t_{j+1})|g|_U,$$

where C is a positive constant and φ a non-negative function on $\{s,t\};\ 0\le s\le t\le T\}$.

Let us mention that (3.4) is equivalent to

$$\text{(3.5)}\qquad \|D(t,s)p_s g-p_t E(t,s)g\|_t\le C^*[t-s]\varphi(s,t)|g|_U$$

for any two-point partition $\{s,t\}=Z\in Z_0$, provided the evolution operators satisfy

$$\text{(3.6)}\qquad \begin{cases}E(t,s)g\in U \text{ for any } g\in U \text{ and}\\ |E(t,s)g|_U\le L^*|g|_U, \qquad 0\le s\le t\le T.\end{cases}$$

With this terminology one may state the following Lax-type equivalence theorems with orders.

Theorem 1. *Let the evolution operators* $\{E(t, s);\ 0 \leq s \leq t \leq T\} \subset [X]$ *satisfy* (2.3), *and let the difference scheme* $\{D(t, s) \in [X_s, X_t];\ 0 \leq s \leq t \leq T\}$ *be constistent of order* $O(\varphi)$. *Then the following assertions are equivalent*:

(i) *The difference scheme is stable of order* $O(1/\psi)$;

$$\text{(ii)} \qquad \left\| \prod_{j=0}^{n-1} D(t_{j+1}, t_j) p_{t_0} f - p_{t_n} E(t_n, t_0) f \right\|_{t_n} \leq \frac{SK}{\psi(Z)} \boldsymbol{K}(\Phi(Z), f)$$

for all $f \in X$, $\{t_j\}_{j=0}^n = Z \in \boldsymbol{Z}_0$, *where* K *is a positive constant and* $\Phi(Z) := \Sigma_{j=0}^{n-1}(t_{j+1} - t_j)\, \varphi(t_j, t_{j+1})$.

Theorem 2. *Let the evolution operators* $\{E(t, s);\ 0 \leq s \leq t \leq T\} \subset [X]$ *satisfy* (2.3) *and* (3.6). *The following assertions are equivalent for a difference scheme* $\{D(t, s) \in [X_s, X_t];\ 0 \leq s \leq t \leq T\}$:

(i) *The difference scheme is stable of order* $O(1/\psi)$ *and consistent of order* $O(\varphi)$ *on* U;

$$\text{(ii)} \qquad \left\| \prod_{j=0}^{n-1} D(t_{j+1}, t_j) p_{t_0} f - p_{t_n} E(t_n, t_0) f \right\|_{t_n} \leq \frac{SK}{\psi(Z)} \boldsymbol{K}(\Phi^*(Z), f)$$

for all $f \in X$, $\{t_j\}_{j=0}^n = Z \in \boldsymbol{Z}_0$, *where* K *is a positive constant and* $\Phi^*(Z) := \psi(Z)$ $\Sigma_{j=0}^{n-1}(t_{j+1} - t_j)\varphi(t_j, t_{j+1})/\psi(Z_{t_{j+1}, t_n})$.

Proofs. For arbitrary $g \in U$, $\{t_j\}_{j=0}^n = Z \in \boldsymbol{Z}_0$, stability (2.4), (3.1) and consistency (3.4) deliver

$$\left\| \prod_{j=0}^{n-1} D(t_{j+1}, t_j)\, p_{t_0} g - p_{t_n} E(t_n, t_0) g \right\|_{t_n}$$

$$= \left\| \sum_{k=0}^{n-1} \prod_{j=k+1}^{n-1} D(t_{j+1}, t_j) [D(t_{k+1}, t_k) p_{t_k} - p_{t_{k+1}} E(t_{k+1}, t_k)]\, E(t_k, t_0) \right\|_{t_n}$$

$$\leq \sum_{k=0}^{n-1} (S/\psi(Z_{t_{k+1}, t_n})) C[t_{k+1} - t_k] \varphi(t_k, t_{k+1}) |g|_U \leq \frac{S}{\psi(Z)} C\Phi^*(Z) |g|_U.$$

Hence for arbitrary $f \in X$ conditions (2.3,5) yield

$$\left\| \prod_{j=0}^{n-1} D(t_{j+1}, t_j) p_{t_0} f - p_{t_n} E(t_n, t_0) f \right\|_{t_n}$$

$$\leq \inf_{g \in U} \Big\{ \left\| \prod_{j=0}^{n-1} D(t_{j+1}, t_j) p_{t_0} [f-g] - p_{t_n} E(t_n, t_0) [f-g] \right\|_{t_n}$$

$$+ \left\| \prod_{j=0}^{n-1} D(t_{j+1}, t_j) p_{t_0} g - p_{t_n} E(t_n, t_0) g \right\|_{t_n} \Big\}$$

$$\leq \inf \Big\{ \Big[\frac{SM}{\psi(Z)} + ML\Big] \|f-g\|_X + \frac{S}{\psi(Z)} C\Phi^*(Z) |g|_U \Big\} \leq \frac{SK}{\psi(Z)} K(\Phi^*(Z), f),$$

where $K := M + ML + C$ (cf. (3.2—3)). Thus the proof of Th. 2, (i)$\Rightarrow$(ii) is complete, likewise that of Th. 1, (i)$\Rightarrow$(ii) in view of (2.8) and (3.3).

Conversely, to prove (ii)$\Rightarrow$(i) one has for arbitrary $f \in X$ using (2.3, 5, 8),

$$\left\| \prod_{j=0}^{n-1} D(t_{j+1}, t_j) p_{t_0} f \right\|_{t_n} \leq \left\| \prod_{j=0}^{n-1} D(t_{j+1}, t_j) p_{t_0} f - p_{t_n} E(t_n, t_0) f \right\|_{t_n} + \| p_{t_n} E(t_n, t_0) f \|_{t_n}$$

$$\leq \frac{SK}{\psi(Z)} \|f\|_X + ML \|f\|_X \leq \frac{S(K+ML)}{\psi(Z)} \|f\|_X$$

(cf. (3.2, 3)). Therefore condition (2.6) implies for any $h \in X_{t_0}$

$$\Big\|\prod_{j=0}^{n-1} D(t_{j+1}, t_j)h\Big\|_{t_n} = \inf_{p_{t_0}f=h} \Big\|\prod_{j=0}^{n-1} D(t_{j+1}, t_j)p_{t_0}\Big\|_{t_n} \leq \frac{S(K+ML)}{\psi(Z)} \inf_{p_{t_0}f=h} \|f\|_X \leq \frac{S'}{\psi(Z)} \|h\|_{t_0},$$

where $S' := SM'(K+ML)$. Thus stability of order $O(1/\psi)$ is necessary for the convergence assertions (ii) in both theorems.

To complete the proof of Th. 2 one has to establish the necessity of the consistency of order $O(\varphi)$. But applying the convergence assertion (ii) to partitions $\{s, t\} = Z \in \mathbf{Z}_0$ and elements $f \in U$, condition (3.6) yields (cf. (2.8))

$$\|D(t, s)p_s f - p_t E(t, s) f\|_t \leq \frac{SK}{\psi(Z)} \Phi^*(Z) \|f\|_U = \{SK[t-s]\varphi(s, t)/\psi(\{t\})\} \|f\|_U,$$

thus consistency (3.5) with $C^* = K$ in view of (3.2).

Remark. To prove Th. 1 it would be sufficient to have

$$(2.4)' \qquad Z \in \mathbf{Z}_0 \Rightarrow Z_a := Z \cap [a, T] \in \mathbf{Z}_0, \qquad 0 \leq a \leq T.$$

The stronger condition (2.4) was only used in the proof of Th. 2 to have that for any $\{t_j\}_{j=0}^n = Z \in \mathbf{Z}_0$ also any two-point partition $\{t_j, t_{j+1}\}$ belongs to $\mathbf{Z}_0$ (cf. (3.5) $\Rightarrow$ (3.4)). In [10] it was shown by a counterexample that it is not possible to drop (2.4)' completely for the special case of equidistant stepsizes.

4. An example. Let us consider the initial-value problem

$$(4.1) \qquad \begin{cases} \frac{d}{dt} u(x, t) = a(t) \frac{d^2}{dx^2} u(x, t), & x \in \mathbb{R},\ 0 \leq s \leq t, \\ u(x, s) = f(x) \in C_{UB} =: X, \end{cases}$$

where $a(t)$ is a positive continuous function on $[0, \infty)$. The corresponding evolution operators are given by

$$(4.2) \qquad E(t, s) := W\Big(\int_s^t a(u)du\Big), \qquad 0 \leq s \leq t \leq T := \infty,$$

with Gauss-Weierstraß integral

$$[W(t)f](x) := (4\pi t)^{-1/2} \int_{-\infty}^{+\infty} f(x+u) \exp\{-u^2/4t\} du.$$

The operators (4.2) satisfy conditions (2.1—3) and (3.6) with $L = L^* = 1$ on $U = C_{UB}^r$. Let us consider the explicit difference scheme

$$(4.3) \qquad D(t, s) := [1 - 2\lambda(t, s)]I + \lambda(t, s)[T_\delta + T_{-\delta}],$$

where $\lambda(t, s) := \delta^{-2}\int_s^t a(u)du$, $0 \leq s \leq t$. This scheme is defined on ($\delta > 0$)

$$C_\delta := \{f_\delta : \delta Z \to \mathbb{R};\ \|f_\delta\|_\delta := \sup_{j \in Z} |f(\delta j)| < \infty\},$$

$$\delta Z := \{x \in \mathbb{R};\ x = \delta j,\ j \in Z (:= \text{set of integers})\}.$$

Thus taking $p_t f := f|_{\delta Z}$ ($:=$ restriction of $f \in C_{UB}$ on δZ), one has (2.5, 6) with $M = M' = 1$. Since the difference operators (4.3) are contractions whenever $0 \leqq \lambda(t, s) \leqq 1/2$, one has stability (3.1) of order $O(1)$ (with $S=1$) on the classes

$$\boldsymbol{Z}_0 := \{\{t_j\}_{j=0}^n \in \boldsymbol{Z};\ \int_{t_j}^{t_{j+1}} a(u)du \leqq \delta^2/2,\ 0 \leqq j \leqq n-1\} \tag{4.4}$$

and also on each (sub-) class $(0 < \sigma \leqq 1/2)$

$$\boldsymbol{Z}_\sigma := \{\{t_j\}_{j=0}^n \in \boldsymbol{Z};\ \int_{t_j}^{t_{j+1}} a(u)du = \sigma\delta^2,\ 0 \leqq j \leqq n-1\}. \tag{4.5}$$

Consistency (3.5) (and thus (3.4)) may be shown by means of Taylor's formula with $C^*(=C)=1$ and

$$(t-s)\varphi(s, t) = \frac{1}{2}\left(\frac{1}{6\lambda(t, s)} + 1\right)\left[\int_s^t a(u)du\right]^2$$

on $U = C_{UB}^4$ for $0 < \lambda(t, s) \leqq 1/2$, and even with $(t-s)\varphi(s, t) = \frac{4}{15}\left[\int_s^t a(u)du\right]^3$ on $U = C_{UB}^6$ for $\lambda(t, s) = 1/6$.

Thus Ths. 1, 2, (i) $\Rightarrow$ (ii), show that the approximation error is bounded by $3\boldsymbol{K}(\Phi(Z), f; X, U)$ with (cf. (4.4—5))

$$\Phi(Z) = \frac{1}{2}\sum_{j=0}^{n-1}\left(\frac{1}{6\lambda(t_{j+1}, t_j)} + 1\right)\left[\int_{t_j}^{t_{j+1}} a(u)du\right]^2, \quad \{t_j\}_{j=0}^n \in \boldsymbol{Z}_0,\ U = C_{UB}^4,$$

$$\Phi(Z) = \frac{\sigma}{2}\left(\sigma + \frac{1}{6}\right)n\delta^4, \qquad \{t_j\}_{j=0}^n \in \boldsymbol{Z}_\sigma,\ U = C_{UB}^4,$$

$$\Phi(Z) = \frac{4}{15}n(\delta^2/6)^3 = \frac{1}{810}n\delta^6, \qquad \{t_j\}_{j=0}^n \in \boldsymbol{Z}_{1/6},\ U = C_{UB}^6.$$

In view of (2.9) one therefrom abtains rates of convergence for the approximation error depending on the smoothness of the initial-value $f \in C_{UB}$, measured in terms of the classical Lipschitz spaces $\mathrm{Lip}_r\,\beta$.

Let us mention that the orders of convergence for this particular example are not best possible. Since problem (4.1) is of parabolic type, one may use methods similiar to those in [12] (see also [9]) for equidistant stepsizes, and therefore improve the orders of convergence (see [3]).

Apart from numerical applications Lax-type theorems in Banach space terminology may also be applied to certain probality — theoretic "difference methods". This leads to the weak law of large numbers, the central limit theorem, and the stable limit laws (with orders) see [3; 9] for the case of independent, identically distributed random variables). The results presensed here may furthermore be applied to the case of not necessarily identically distributed, but independent random variables (see [8]).

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