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PROPERTIES OF CERTAIN TRANSFORMED CHARACTERISTIC FUNCTIONS

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1. Introduction. Characteristic functions are powerful tools for the solution of problems in Probability Theory and admit many important applications in Mathematical Statistics. During the last two decades there has been an increasing interest in transformations of characteristic functions, more precisely in operations which transform a given characteristic function into a new characteristic function. The investigation of the properties of the transformed characteristic function is of great importance in Probability Theory. The present paper deals with a transformation of characteristic functions belonging to distributions with finite second moment.

Let $\varphi(t)$ be the characteristic function of the distribution function $F(x)$ having finite second moment μ_2 and let $\varphi'(t)$ be the derivative of $\varphi(t)$. The operator T_1 defined by

$$T_1\varphi(t)=\varphi'(t)-\varphi'(0)/\varphi''(0)t \tag{1}$$

converts $\varphi(t)$ into a new characteristic function [2, Theorem 12.2.5]. We shall establish some properties of $T_1\varphi(t)$. We start by investigating a class of T_1-transformed characteristic functions satisfying a differential equation.

2. The differential equation. We shall investigate the differential equation

$$\varphi'(t)/\varphi''(0)t=[(e^{itb}-1)/itb]\,(\varphi(t))^a,$$

where $a\geq 0$, $b\geq 0$ real constants and $\varphi(t)$ the characteristic function of a distribution function with zero first moment and finite second moment. In other words, we shall look for characteristic functions $\varphi(t)$ such that the characteristic function $T_1\varphi(t)$ can be written as the product of the characteristic functions $\varphi_1(t)=e^{itb}-1/itb$, $\varphi_2(t)=[\varphi(t)]^a$. The characteristic function $\varphi_1(t)$ belongs to the uniform distribution and the characteristic function $\varphi_2(t)$ has "the same form" as $\varphi(t)$.

We shall consider several cases.

(1) $b=0$, $a=1$. Then $\varphi(t)=\exp(-\mu_2t^2/2)$. This is the characteristic function of the Normal distribution with zero mean value

(2) $v=0$, $a>1$. Then $\varphi(t)=(1+\frac{(a-1)}{2}\mu_2t^2)^{-1/(a-1)}$. This is the characteristic function of the Laplace distribution when $a=2$.

(3) $b=0$, $a<1$. Then $\varphi(t)=(1-(1-a)\mu_2t^2/2)^{1/(1-a)}$. This is not a characteristic function since for any sufficiently large t we have $|\varphi(t)|>1$.

(4) $b\neq 0$, $a=0$. Then $\varphi(t)=[\mu_2(e^{itb}-itb)+b^2-\mu_2]/b^2$. This is not a characteristic function since for $t_0=2\pi/b$ we have $|\varphi(t_0)|>1$.

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(5) $b \neq 0$, $a=1$. In this case $\varphi(t)=\exp(\mu_2(e^{itb}-itb-1)/b^2)$. This is the characteristic function of the random variable $Y=b(X-\mu_2/b^2)$, where X is the Poisson random variable with mean value μ_2/b^2.

(6) $b \neq 0$, $a>1$. Then $\varphi(t)=[1-\mu_2(a-1)(e^{itb}-itb-1)/b^2]^{-1/(a-1)}$. This characteristic function can be written in the form

$$\varphi(t)=\int_0^\infty \exp(x\mu_2(e^{itb}-itb-1)/b^2)dG(x)$$

where $G(x)$ is the distribution function having characteristic function $((1-(a-1)itb))^{-1/(a-1)}$. Hence $\varphi(t)$ is a power mixture of (5).

(7)$b \neq 0$, $a<1$. In this case $\varphi(t)=[1+\mu_2^{(1-a)}(e^{itb}-itb-1)/b^2]^{1/(1-a)}$. This is not a characteristic function since for $t_0=2\pi/b$ we have $|\varphi(t_0)|>1$.

3. Infinite divisibility of $T_1\varphi(t)$. A characteristic function $g(t)$ is called infinitely divisible if, and only if $g^p(t)$ is a characteristic function for all positive p. In this section we discuss certain cases of preservation of infinite divisibility under the transformation T_1.

We suppose that $F(x)$ is a distribution function on $[0, \infty]$ having finite first moment μ_1 and $\varphi(t)$ is its characteristic function. Then $1-F(x)/\mu_1$ is the renewal density and $T_0\varphi(t)=\varphi(t)-1/\varphi'(0)t$ is its characteristic function, Steutel [4]. Since $T_1\varphi(t)=T_0\varphi'(t)/\varphi'(0)$ [4, theorem 1] implies that $T_1\varphi(t)$ is infinitely divisible if, and only if

$$\log x-\sum_{k=1}^\infty \frac{1}{k}\Big(\frac{1}{\mu_1}\int_0^x u\,dF(u)\Big)^{*k}$$

is non-increasing for $x>0$ [*denotes convolution].

Below we give examples of characteristic functions which preserve their infinite divisiblity under the transformation T_1. The infinite divisibility of these T_1-transformed characteristic functions cannot be established by [4, theorem 1].

a) Let $\varphi(t)=\int_0^\infty (1+xt^2)^{-1}dH(x)$ with $b=\int_0^\infty x\,dH(x)<\infty$. $\varphi(t)$ is a scale mixture of Laplace distribution. It follows from [5] that $T_1\varphi(t)=\int_0^\infty(1+xt^2)^{-2}dH_1(x)$ is infinitely divisible when the distribution $H_1(x)=\int_0^x u\,dH(u)/b$ is unimodal.

b) Let $\varphi(t)=\int_0^\infty \exp(-xt^2)dH(x)$, which means that $\varphi(t)$ is a variance mixture of Normal distribution having zero mean value. It follows from [5, Theorem 4.1] that $\varphi(t)$ is infinitely divisible if $H(x)$ is infinitely divisible. Then $T_1\varphi(t)=\int_0^\infty \exp(-xt^2)dH_1(x)$ is infinitely dlvisible if $H_1(x)$ is infinitely divisible.

4. The transformation $\varphi^{(n)}(t)-\varphi^{(n)}(0)/\varphi^{(n+1)}(0)t$ and the uniform distribution. When $\varphi(t)$ is the characteristic function of the distribution function $F(x)$ having left extremity at zero and $\mu_n=\int_0^\infty x^n dF(x)<\infty$ then $\varphi^{(n)}(t)/\varphi^{(n)}(0)$ has distribution function $F_n(x)=\int_0^x u^n dF(u)/\mu_n$, $(n=1, 2, \dots)$. Furthermore

$$\gamma_n(t)=\varphi^{(n-1)}(t)-\varphi^{(n-1)}(0)/\varphi^{(n)}(0)t=\frac{1}{t}\int_0^t \frac{\varphi^{(n)}(u)}{\varphi^{(n)}(0)}du$$

is also a characteristic function (see [2, p. 321]). Suppose that the support of $F(x)$ is of the form $[0, b]$ with $b<\infty$. Then lemma 3 of [1] implies that $\lim_{n\to\infty} \mu_{k+n}/\mu_n=b^k, k=1, 2, \dots$ Since $\mu_{k+n}/\mu_n=\int_0^\infty u^k dF_n(u)$ and b^k is the kth moment of the degenetate distribution with characteristic function e^{itb} it follows that

$$\lim_{n\to\infty} \gamma_n(t) = e^{itb} - 1/itb.$$

5. $T_1\varphi(t)$ as superposition of two characteristic functions. When $\varphi(t)$ is the characteristic function of a distribution function $F(x)$ having finite second moment, then $T_2\varphi(t) = 2[\varphi(t) - \varphi'(0)t - 1]/\varphi''(0)t^2$ is a characteristic function. Theorem 1 of [3] implies that $T_2\varphi(t)$ belongs to a distribution having a convex density. Furthermore

$$T_3\varphi(t) = 2[t\varphi'(t) - \varphi(t) + 1]/\varphi''(0)t^2 = \frac{2}{t^2}\int_0^t \frac{\varphi''(u)}{\varphi''(0)}\, u du$$

is a characteristic function, (see [2, pp. 321, 323]). Since $T_1\varphi(t) = T_2\varphi(t)/2 + T_3\varphi(t)/2$ is follows that $T_1\varphi(t)$ is the superposition of $T_2\varphi(t)$ and $T_3\varphi(t)$ with $T_2\varphi(t)$ having a convex density.

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