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A CLASS OF INTEGRAL EQUATIONS INVOLVING H-FUNCTIONS

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Inversion integrals have been found for some integral transformations $\int_x^1 K(t/x)g(t)dt=f(x)$ $(0<a\leq x\leq 1)$. As Mellin transforms are the important tools to solve this class of integral equations one has, by necessity, to use such kernels which have Mellin-Barnes' integral representation. H-function falls in this category. We have found the solution to the above equation when K is the H-function or the Ψ-function.

1. Introduction. Sixteen years ago Johnson [5] considered the class of integral equations

$$\int_x^1 K(t/x)\,g(t)dt=f(x) \quad (0<a\leq x\leq 1), \tag{1.1}$$

with K as a solution of

$$\prod_{i=1}^{n}(\delta-c_i)y-z\prod_{i=1}^{n}(\delta+a_i)y=0. \tag{1.2}$$

These solutions have Mellin-Barnes integral representations (see Nørlund [8]) Johnson [5], in the second instance studied a slightly more general integra

$$R(z)=\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\frac{\prod_{j=1}^{n}\Gamma(b_j-B_js)}{\prod_{j=1}^{b}\Gamma(d_j-D_js)}(pz)^{-s}ds,$$

where the B_i's and D_i's are real.

We shall find a solution of (1.1) with K as the H-function of Fox [3]. As a simple transformation reduces (1.1) to the integral equation $\int_0^x K(x-t)g(t)dt$ $=f(x)$ $(0<x\leq a<\infty)$, our solution of (1.1) with H-function and Ψ-function as kernels indirectly involves (1.2).

Earlier, Ta Li [6] has solved (1.1) with Čebichev polynomial kernel. T. P. Higgins [4] and E. R. Love [7] solved (1.2) with Gauss hypergeometric function as kernels.

2. Preparation for main results. To find the form of the inversion integral of (1.1) we rewrite it, in the standard form of a convolution with respect to the Mellin transformation, viz.,

$$\int_0^\infty K(t/x)U(t/x-1)\,g(t)V(t-1)\,dt=f(x)V(x-1), \tag{2.1}$$

where $U(x)=1$ for $x>0$, $U(x)=0$ for $x<0$, and $V(x)=1-U(x)$. From Erdélyi [2], $M\{x^{\alpha}\int_0^{\infty} y^{\beta} f_1(xy) f_2(y)dy;\ s\}=g_1(s+\alpha)g_2(1-s-\alpha+\beta)$ and $M\{f(x^{-1});s\}=g(-s)$, we have

$$M\{K(x)U(x-1);\ -s\}\,M\{g(x)V(x-1);\ 1+s\}=M\{f(x)V(x-1);\ s\}.$$

The inverse Mellin transforms are assumed to converge for the individual transforms. Substituting $s-1$ for s in the last equation and rearranging, we have

(2.2) $M\{g(x)V(x-1);s\}=M\{f(x)V(x-1);\ s-1\}/M\{K(x)U(x-1);\ 1-s\}.$

A formal solution of (1.1) is then

$$g(x)=\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}[M\{f(x)V(x-1);\ s-1\}/M\{k(x)U(x-1);\ 1-s\}]\,x^{-s}ds.$$

However, we would prefer to express g in the form of a convolution similar to (2.1). We cannot use the general formulas, Erdelyi [2, 6.1 :(13), (14) and (4)] directly since the inverse Mellin transform of $M\{k(x)U(x-1);\ 1-s\}$ will not in general converge. There are two methods of modifying (2.2) which may allow us to form a convolution. In the first we must find a function h such that $M\{h(x)V(x-1);s\}=P(s)/M\{k(x)U(x-1);\ 1-s\}$, where P is a polynomial in s. In the second we insert

$$\Gamma(s)/\Gamma(s)=(-1+s)(-2+s)\ldots(-j+s)\Gamma(-1+s)/\Gamma(s)$$

in (2.2). We then let $P(s)=(-1+s)(-2+s)\ldots(-j+s)$ and define

$$h(x)=\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}[M\{K(x)V(x-1);\ 1-s\}\Gamma(s)]^{-1}(-j+s)x^{-s}ds,$$

provided the integral converges for some positive integer j and $M\{h(x)V(x-1)$ $s\}=\Gamma(-j+s)/[\Gamma(s)M\{K(x)U(x-1);\ 1-s\}]$. In either case (2.2) becomes

$$M\{g(x)V(x-1);\ s\}=M\{h(x)V(x-1);\ s\}P(s)M\{f(x)V(x-1);\ s-1\}.$$

With the aid of the general formulas in Erdelyi [2, § 6.1] we form from P a differential operator which operates on f and which is a polynomial P in the operators, d/dy, $y^{-1}d/dy$, $d/dy\ y^2$, and so forth. The coefficients of P are rational functions of y. The inversion integral, which is found from, Erdélyi [2, 6.1 :(14)], viz.,

$$M\{x^{\alpha}\int_0^{\infty} y^{\beta} f_1(x/y) f_2(y)dy;\ s\}=g_1(s+\alpha)\,g_2(s+\alpha+\beta+1),$$

is then

$$g(t)V(t-1)=\int_0^{\infty} h(t/y)V(t/y-1)P^{*}(D)f(y)V(y-1)dy,$$

where D is one of the operators; and this is equivalent to

(2.3) $$g(t)=\int_t^1 h(t/y)P^{*}(D)f(y)dy.$$

This gives us the form of the inversion integral. It is easier to show that (2.3) is the inversion of (2.1) by substitution rather than justifying the above steps. The outline of the proof runs as under:

Let I denote the formal integral obtained by substituting (2.3) into the integral (2.1), that is

$$I(x)=\int_x^1 K(t/x)dt\int_t^1 h(t/y)P^*(D)f(y)dy.$$

Using Dirichlet's formula, we have

$$I(x)=\int_x^1 P(D)f(y)\,dy\int_x^y K(t/x)\,h(t/y)\,dt.$$

If J denotes the inner integral, then J and the differential operator are related. Simplification, followed by a j-fold integration by parts leads to the function $f(x)$.

3. Definitions: *H*-function, *Ψ*-Function and Their Mellin-Transforms. The *Fox's H-function,* defined first by Fox [3] slightly differently is given by

$$R(z)=H_{p,q}^{m,n}\left[uz\left|\begin{matrix}(a_p,\alpha_p)\\(b_b,\beta_b)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Pi_{i=1}^m\Gamma(b_i-\beta_i s)\Pi_{i=1}^n\Gamma(l-a_i+\alpha_i s)}{\Pi_{i=m+1}^q\Gamma(1-b_i+\beta_i s)\,\Pi_{i=n+1}^p\Gamma(a_i-\alpha_i s)}(uz)^s ds$$

under sets of conditions which are too lengthy to be presented here. We shall write this function as $H(uz)$. The Mellin-Transform of the above function is given by

$$M\{H(uz);z\}=u^{-s}\frac{\Pi_{i=1}^m\Gamma(b_i+\beta_i s)\,\Pi_{i=1}^n\Gamma(l-a_i-\alpha_i s)}{\Pi_{i=m+1}^q\Gamma(1-b_i-\beta_i s)\,\Pi_{i=n+1}^p\Gamma(a_i+\alpha_i s)},\quad u>0,$$

$$-\min\operatorname{Re}(b_i/\beta_i)\ 1\leq i\leq m<\operatorname{Re}(s)<\alpha_i^{-1}-\max\operatorname{Re}(a_i/\alpha_i)\ 1\leq i\leq n$$

and $\theta>0$, $|\arg uz|<\theta\pi/2$ or $\theta\geq 0$, $|\arg uz|\leq\theta\pi/2$ and $\operatorname{Re}(\varphi+1)<0$, where

$$\theta=\sum_{i=1}^n\alpha_i-\sum_{i=n+1}^p\alpha_i+\sum_{i=1}^m\beta_i-\sum_{i=m+1}^q\beta_i,\quad \varphi=\frac{1}{2}(p-q)+\sum_{i=1}^q b_i-\sum_{i=1}^p a_i.$$

The *Ψ-function,* defined by Varma [9], is given by

$$(3.1)\quad \Psi(x)=\Psi_{\nu_1,\ldots,\nu_m,\lambda_1,\ldots,\lambda_n}^{\xi_1,\ldots,\xi_p}(x)=\frac{1}{\eta_1\eta_2\ldots\eta_p}\int_0^\infty\int_0^\infty\Psi_{\nu_1,\ldots,\nu_m,\lambda_1,\ldots,\lambda_n}(xT_1\ldots T_f)$$

$$\times T^{1/\eta_1-1/2}(T_p)^{1/\eta_p-1/2}J_\xi(T^{1/\eta_1})\ldots J_{\xi_p}(T_p^{1/\eta_p})dT_1\ldots dT_p.$$

$$(0<\eta_r\leq 1,\ \xi_r+1/2>0,\ r=1,\ldots,p;\ m+n>p).$$

$$(3.2)\qquad M\{\Psi(x);s\}=\Phi(s)={}_2(s-1/2)(m+\mu_1+\cdots+\mu_n-\eta_1-\cdots-\eta_n)$$

$$\times\prod_{r=1}^m\frac{\Gamma(s/2+\nu_r/2+1/4)}{\Gamma(\nu_r/2-s/2+3/4)}\times\prod_{k=1}^n\frac{\Gamma(\lambda_k/2+(\mu_k/2)s-\mu_k/4+1/2)}{\Gamma(\lambda_k/2-(\mu_k/2)s+\mu_k/4+1/2)}\times\prod_{i=1}^p\frac{\Gamma(\xi_i/2-(\eta_i/2)s+\eta_i/4+1/2)}{\Gamma(\xi_i/2+\eta_i s/2-\eta_i/4+1/2)}$$

valid for

$$\min(-\nu_r,-1/2,\ 1\leq r\leq m-(\lambda_k+1)/\mu_k 1\leq k\leq n+1/2)<R(s)((\xi_1+1)/\eta_i+\eta_i/2)1\leq i\leq p$$

From (3.2) we obtain, on using Mellin's inversion formula and Erdelyi [2, 6.1 (2), p. 307], the following Mellin-Barnes integral for the Ψ-function:

$$(3.3)\quad \Psi(ax)=\Psi^{\xi_1,\ldots,\xi_p}_{\nu_1,\ldots,\nu_m,\lambda_1,\ldots,\lambda_n}(ax)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}2^{(s-1/2)(m+\mu_1+\cdots+\mu_n-\eta_1-\cdots-\eta_p)}$$

$$\times\prod_{r=1}^{m}\frac{\Gamma(\nu_r/2+s/2+1/4)}{\Gamma(\nu_r/2-s/2+3/4)}\prod_{k=1}^{n}\frac{\Gamma(\lambda_k/2+\mu_k s/2-\mu_k/4+1/2)}{\Gamma(\lambda_k/2-\mu_k s/2+\mu_k/4+1/2)}\prod_{i=1}^{p}\frac{\Gamma(\xi_i/2-\eta_i s/2+\mu_i/4+1/2)}{\Gamma(\xi_i/2+\eta_i s/2-\eta_i/4+1/2)}$$

If we use the same symbols as in Erdélyi [1, pp. 49—50], we have for (3.3) the following notations

$$(3.4)\qquad a=0,\ \beta=m+\sum_1^n\mu_k-\sum_1^p\eta_i,\ =\frac{1}{2}(\sum_n^p\eta_i-\sum_1^n\mu_k-m)=-\frac{\beta}{2},$$

$$=2^{-\beta}\prod_{k=1}^{n}(\mu_k)^{\mu_k}\prod_{i=1}^{p}(\eta_i)^{-\eta_i}.$$

So, the integral (3.3) does not converge for complex z, where $z=2^\beta a^{-1}x^{-1}$.

For $z>0$ it converges absolutely if γ is so chosen that $-\beta\gamma<1+\lambda$; and there éxists an analytic function of z, defined over $|\arg z|<\pi$, whose values for positive z are given by (3.3).

We are set now for the main results which we write as theorems.

4. Main results. Theorem 4.1. *The solution of*

$$\int_x^1 R(t/x)\,g(t)dt=f(x)\quad(0<a\leqq x\leqq 1),$$

where R is the H-function defined by (3.1), *is given by*

$$g(t)=\int_t^1 T(t/y)y^{-1}(-D)^j[y^{j-1}f(y)]dy,$$

where $D=d/dy$, $j+\varphi>1$ and

$$T(t/y)=u^{-1}H^{q-m+1,\,p-n}_{p+1,\,q+1}$$

$$\times\left[pty^{-1}\,\middle|\,\begin{matrix}\{(1-a_{n+1,p}-\alpha_{n+1,p},\ \beta_{n+1,p})\},\ \{(1-a_n-\alpha_n,\ \alpha_n)\},\ (0,1)\\ \{(1-b_{m+1,q}-\beta_{m+1,q},\ \beta_{m+1,q})\},\ \{(1-b_m-\beta_m,\ \beta_m)\},\ (-j,1)\end{matrix}\right],$$

where $\{(1-a_{n+1,p}-\alpha_{n+1,p},\ \alpha_{n+1,p})\}$ *denotes the* $p-n$ *parameter pairs* $(1-a_{n+1}-\alpha_{n+1},\ \alpha_{n+1})$, $(1-a_{n+2}-\alpha_{n+2},\ \alpha_{n+2}),\ldots,(1-a_p-\alpha_p,\ \alpha_p)$, *and* $\{(1-a_n-\alpha_n,\ \alpha_n)\}$ *denotes the* n *parameter pairs* $(1-a_1-\alpha_1,\ \alpha_1),(1-a_2-\alpha_2,\ \alpha_2),\ldots,(1-a_n-\alpha_n,\ \alpha_n)$.

In the solution it has been supposed that $f^{(j+1)}$ is sectionally continuous for $0<a\leqq x\leqq 1$ and $f^{k}(1)=0$ for $0\leqq k\leqq j$.

Note 1. We can weaken the restriction of $f^{(j+1)}$ to be just integrable instead of sectionably continuous. In order that the inversion integral be convergent, it is needed in addition to being integrable that the resolvent kernel be bounded almost everywhere.

Note 2. We were interested in such convergence conditions which made $\theta=0$. Integral involved is still convergent, for the exponential function in the asymptotic expansion of the quotient of gamma function in the definition integral disappears in this case.

Putting $\alpha_j=1(j=1,\ldots,n,\ n+1,\ldots,p)$; $\beta_j=1\,(j=1,\ldots,m,\ m+1,\ldots,q)$, we have the following

Corollary 4.2. *The solution of*

$$\int_x^1 G_{p,q}^{m,n}\left(\frac{ut}{x} \,\middle|\, \begin{matrix}(a_p)\\(b_q)\end{matrix}\right) g(t)dt = f(x) \quad (0 < a \leq x \leq 1)$$

is given by

$$g(t) = u^{-1}\int_t^1 G_{p+1,q+1}^{q-m+1,p-n}\left(uty^{-1} \,\middle|\, \begin{matrix}(-a_{n+1,p}),(-a_n),(0,1)\\(-b_{m+1,q}),(-b_m),(-j,1)\end{matrix}\right)(-D)^j\,[\,y^{j-}f(y)]dy,$$

where $D = d/dy$, $j+\varphi > 1$. *In the solution it being supposed that* $f^{(j+1)}$ *is sectionally continuous for* $0 < a \leq x \leq 1$ *and* $f^k(1) = 0$ *for* $0 \leq k \leq j$.

If we particularize the parameters further we get a number of pairs involving lesser transcendents.

5. An Integral Equation Involving Ψ-Function. We apply the methods of the section 2 to obtain

Theorem 5.1. *The solution of* $\int_x^1 \Psi(t/x)g(t)dt = f(x)$, $(0 < a \leq x \leq 1)$, *where* $\Psi(z)$ *is the function defined by* (3.1), *is given by*

$$g(t) = \int_t^1 T(t/y)y^{-1}(-D)^j\,[\,y^{j-1}f(y)]dy,$$

where $D = d/dy$, $\beta\gamma < j - \lambda - 1$ *and*

$$T(z) = 2^{-\beta/2}\,a\,H_{2n+2n+1,\,2p+1}^{p,m+n+1}$$

$$\times\left[\frac{2a\beta}{z} \,\middle|\, \begin{matrix}\{(3/4 \mp \nu_m/2,\,1/2)\},\ \{(1/2 \mp \lambda_n/2 + \mu_n/4,\,\mu_n/2)\},\ (-j+1,\,1)\\ \{(1/2 \pm \xi_p/2 + \mu_p/2,\,\eta_p/2)\},\end{matrix}\right](0,1),$$

where α, β, λ *and parameters are given by* (3.3) *and* (3.4) *and* $z = 2^\beta a^{-1}x^{-1}$ *and* γ *occurs in the limits of the contour integral definition for* $\Psi(z)$. *In the solution it being supposed that* f^{j+1} *is sectionally continuous for* $0 < a \leq x \leq 1$ *and* $f^k(1) = 0$ *for* $0 \leq kk \leq j$.

Theorem 5.1 covers a class of integral equations which have the kernel of the type (3.1) and whose inversion formula involves Fox's H-function.

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