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ON THE AXIOM OF PLANES AND THE AXIOM OF SPHERES IN THE ALMOST HERMITIAN GEOMETRY

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We prove analogues of Cartan's criterion for constancy of sectcional curvature to an arbitrary almost Hermitian manifold. As a consequence we establish for such a manifold analogues of a Cartan's theorem. Our results generalize some theorems in [2; 5; 7; 8].

1. Introduction. Les N be an n-dimensional submanifold of an m-dimensional Riemannian manifold M with Riemannian metric g and let $\tilde{\nabla}$ and ∇ be the Levi-Civita connections on M and N, respectively. It is well known, that the equation

$$\alpha(x,\ y)=\tilde{\nabla}_x y-\nabla_x y,$$

where $x,\ y \in \mathfrak{X}N$, defines a normal-bundle-valued symmetric tensor field, called the second fundamental form of the immertion. The submanifold N is said to be totally umbilical, if $\alpha(x,\ y)=g(x,\ y)H$ for all $x, y \in \mathfrak{X}N$, where $N=(1/n)$ trace α is the mean curvature vector of N in M. In particular, if α vanishes identically, N is called a totally geodesic submanifold of M.

For $x \in \mathfrak{X}N$, $\xi \in \mathfrak{X}N^1$ we write $\tilde{\nabla}_x\xi=-A_\xi x+D_x\xi$, where $-A_\xi x$ (respectively, $D_x\xi$) denotes the tangential (respectively, the normal) component of $\tilde{\nabla}_x\xi$. A normal vector field ξ is said to be parallel, if $D_x\xi=0$ for each $x \in \mathfrak{X}N$.

The manifold M is said to satisfy the axiom of n-planes (respectively, n-spheres), where n is a fixed integer $2\leq n<m$ if for each point $p \in M$ and for any n-dimensional subspace α of T_pM there exists an n-dimensional totally geodesic submanifold N (respectively an n-dimensional totally umbilical submanifold N with non-zero parallel mean curvature vector) containing p, such that $T_pN=\alpha$.

In his book on Riemannian geometry [1] E. Cartan proved the following theorem.

Theorem. *Let M be an m-dimensional Riemannian manifold, $m>2$, which satisfies the axiom of n-planes for some n, $2\leq n<m$. Then M has constant sectional curvature.*

In [4] Leung and Nomizu have substituted the axiom of n-planes with the axiom of n-spheres and have proved a generalization of the above mentioned Cartan's theorem.

Analogous results for Kaehler manifolds have been proved in [2; 5; 8] and in [7] it has been studied a similar problem for some almost Hermitian manifolds.

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2. Preliminaries. Let M be an m-dimensional Riemannian manifold with Riemannian metric g and let ∇ be its Levi-Civita connection. The curvature tensor R associated with ∇ has the following properties:

1) $R(X, Y) = -R(Y, X)$ for $X, Y \in T_pM$,

2) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$ for $X, Y, Z \in T_pM$,

3) $R(X, Y, Z, U) = -R(X, Y, U, Z)$ for $X, Y, Z, U \in T_pM$,

where $R(X, Y, Z, U) = g(R(X, Y)Z, U)$.

The curvature of a two dimensional plane in T_pM with an orthonormal basis X, Y is defined by $K(X, Y) = R(X, Y, Y, X)$.

It is easy to compute, that if N is a totally geodesic submanifold of M or a totally umbilical submanifold of M with parallel mean curvature vector, then $R(x, y, z, \xi) = 0$ for all vectors $x, y, z \in T_pN$, $\xi \perp T_pN$ and for each point $p \in N$.

Now, let M be a $2m$-dimensional almost Hermitian manifold with Riemannian metric g and almost complex structure J.

A subspace α in T_pM is said to be holomorphic (respectively, antiholomorphic or totally real) if $J\alpha = \alpha$ (respectively $J\alpha \perp \alpha$). For the dimension k of a holomorphic (respectively, antiholomorphic) subspace α of T_pM we have $k = 2n$, $1 \leq n \leq m$ (respectively, $1 \leq k \leq m$). If the holomorphic (respectively, antiholomorphic) sectional curvature in each point $p \in M$, i. e. the curvature of a holomorphic (respectively, antiholomorphic) subspace α of T_pM does not depend on α, then M is said to be of pointwise constant holomorphic (respectively, antiholomorphic) sectional curvature in p.

A connected Riemannian (respectively, Kaehler) manifold of global constant sectional curvature (respectively, of constant holomorphic sectional curvature) is called a real-space-form (respectively, a complex-space-fc m).

An almost Hermitian manifold is said an RK-manifold, if $R(X, Y, Z, U) = R(JX, JY, JZ, JU)$ for all $X, Y, Z, U \in T_pM$, $p \in M$.

For a two dimensional subspace α of T_pM with an orthonormal basis X, Y the angle $\theta \in [0, \pi/2]$ between α and $J\alpha$ is difined by $\cos\theta = |g(X, JY)|$.

We shall need the following theorems:

Theorem A [3]. *Let M be a $2m$-dimensional almost Hermitian manifold, $m \geq 2$, and let $T : (T_pM)^4 \to \mathbb{R}$ be a four-linear mapping, which satisfies the conditions:*

1) *for all* $X, Y, Z, U \in T_pM$

$$T(X, Y, Z, U) = -T(Y, X, Z, U),$$

$$T(X, Y, Z, U) + T(Y, Z, X, U) + T(Z, X, Y, U) = 0,$$

$$T(X, Y, Z, U) = -T(X, Y, U, Z);$$

2) $T(X, Y, Y, X) = 0$, *where X, Y is a basis of an arbitrary two dimensional subspace α in T_pM, for which the angle between α and $J\alpha$ is one of the numbers* $0, \pi/4, \pi/2$.

Then $T = 0$.

Let for all $X, Y, Z, U \in T_pM$

$$R_1(X, Y, Z, U) = g(X, U)g(Y, Z) - g(X, Z)g(Y, U),$$

$$R_2(X, Y, Z, U) = g(X, JU)g(Y, JZ) - g(X, JZ)g(Y, YU) - 2g(X, JY)g(Z, JU).$$

Theorem B [3]. *If M is a $2m$-dimensional RK-manifold, $m \geq 2$, with pointwise constant holomorphic sectional curvature c and with pointwise* *constant antiholomorphic sectional curvature* K, *then the curvature tensor* R *has the form*

$$R = KR_1 + (c-K)R_2/3. \tag{2.1}$$

As is proved in [6], if the curvature tensor of a $2m$-dimensional connected almost Hermitian manifold has the form (2.1) and if $m \geq 3$, then c and K are global constants. On the other hand, it is proved in [3], that if the curvature tensor of an almost Hermitian manifold M of dimension $2m \geq 4$ has the form (2.1) with global constants c and K, then either M is of constant sectional curvature $c=K$ or M is a Kaehler manifold of constant holomorphic sectional curvature. Hence we have:

Theorem C. *Let* M *be a connected RK-manifold of dimension* $2m \geqq 6$. *If* M *has pointwise constant holomorphic sectional curvature and pointwise constant antiholomorphic sectional curvature, then* M *is one of the following:*

1) *a real-space-form;*
2) *a compex-space-form.*

3. Criterions for constancy of the holomorphic and the antiholomorphic curvature at one point.

Lemma 1. *Let* M *be an almost Hermitian manifold with dimension* $2m$, $m \geq 2$ *and for a point* $p \in M$

$$R(X, JX, JX, Y) = 0 \tag{3.1}$$

holds for all $X, Y \in T_pM$, *with* $g(X, Y) = g(X, JY) = 0$. *Then* M *has constant holomorphic sectional curvature at* p *and*

$$R(X, Y, Y, X) = R(JX, JY, JX), \tag{3.2}$$

where X, Y *are as above.*

Proof. Taking two arbitrary unit vectors X, Y in T_pM with $g(X, Y) = g(X, JY) = 0$ and applying (3.1) for the vectors $X + \alpha Y$, $\alpha X - Y$, we obtain

$$H(X) - \alpha^2 H(Y) + (\alpha^2 - 1) R(X, JX, JY, Y) + (\alpha^2 - 1) R(X, JY, JX, Y) + \alpha^2 K(X, JY) - K(JX, Y) = 0, \tag{3.3}$$

where $H(X) = R(X, JX, JX, X)$ denotes the holomorphic sectional curvature, determined by X.

Let $\alpha = 1$:

$$H(X) - H(Y) + K(X, JY) - K(JX, Y) = 0. \tag{3.4}$$

From (3.3) and (3.4) it follows

$$H(Y) = R(X, JX, JY, Y) + R(X, JY, JX, Y) + K(X, JY). \tag{3.5}$$

Analogously $H(X) = R(X, JX, JY, Y) + R(X, JY, JX, Y) + K(JX, Y)$.

Substituting X by JX and Y by JY we get

$$H(X) = R(X, JX, JY, Y) + R(X, JY, JX, Y) + K(X, JY). \tag{3.6}$$

From (3.5) and (3.6) we see that

$$H(X) = H(Y) \tag{3.7}$$

and combining this with (3.4) we find (3.2).

Let $m > 2$ and U, V be arbitrary unit vectors in T_pM. We choose $X \in \operatorname{span}\{U, JU\}^{\perp} \cap \operatorname{span}\{V, JV\}^{\perp}$. According to (3.7) we have $H(U) = H(X) = H(V)$ and the lemma is proved in the case $m > 2$. In the case $m = 2$ we put $c = H(X) = H(Y)$ and using (3.6) we see that $H(\alpha X + \beta Y) = c$, where $\alpha^2 + \beta^2 = 1$.

Hence it is not difficult to find that the holomorphic sectional curvature in p is a constant.

The following lemma is trivial.

Lemma 2. *Let α be a two-dimensional subspace in T_pM such that the angle between α and $J\alpha$ is $\pi/4$. Then α has an orthonormal basis X, $(JX+U)/\sqrt{2}$, where X, U are unit vectors in T_pM with $g(X, U)=g(X, JU)=0$.*

Lemma 3. *Let M be a $2m$-dimensional almost Hermitian manifold, $m\geq 2$ and for each point $p\in M$ (3.1) holds for all $X, Y\in T_pM$ with $g(X, Y)=g(X, JY)=0$. Then M is a RK-manifold.*

Proof. We put $T(X, Y, Z, U)=R(X, Y, Z, U)-R(JX, JY, JZ, JU)$ for all $X, Y, Z, U\in T_pM$. Obviously T has the property 1) of theorem A. Let α is a subspace in T_pM such that the angle between α and $J\alpha$ is θ. If $\theta=0$, α is a holomorphic plane and if $\alpha=\text{span}\{X, JX\}$ we have $T(X, JX, JX, X)=0$. If $\theta=\pi/2$, α is an antiholomorphic plane and we can choose two vectors X, $Y\in T_pM$ such that $\alpha=\text{span}\{X, Y\}$, $g\{X, Y)=g(X, JY)=0$. According to lemma 1 we have $T(X, Y, Y, X)=0$. Let $\theta=\pi/4$ and let X, $(JX+U)/\sqrt{2}$ be an orthonormal basis of α as in lemma 2. Then $T(X, JX+U, JX+U, X)=0$. According to theorem A we have $T=0$, which proves our assertion.

Lemma 4. *Let M be a $2m$-dimensional almost Hermitian manifold, $m\geq 2$ and for a point $p\in M$*

$$R(X, Y, Y, Z)=0 \tag{3.8}$$

holds for all $X, Y, Z\in T_pM$ with $g(X, Y)=g(X, JY)=g(X, Z)=g(Y, Z)=0$. Then M has constant antiholomorphic sectional curvature at p.

Proof. According to lemma 1 M has constant holomorphic sectional curvature c at p. We apply (3.8) for the vectors $X+JX$, Y, $JX-X$, where $X, Y\in T_pM$ are arbitrary unit vectors with $g(X, Y)=g(X, JY)=0$ and we get $K(JX, Y)=K(X, Y)$. As in the proof of lemma 1 we have

$$H(X)=R(X, JX, JY, Y)+R(X, JY, JX, Y)+K(X, JY). \tag{3.9}$$

Hence applying the first Bianchi's identity we obtain

$$H(X)=2R(X, JX, JY, Y)+R(JX, JY, X, Y)+K(X, JY). \tag{3.10}$$

The substitution of Y with JY in (3.9) gives

$$H(X)=R(X, JX, JY, Y)-R(X, Y, JX, JY)+K(X, Y).$$

Combining this with (3.10) we derive

$$2H(X)=3R(X, JX, JY, Y)+K(X, Y)+K(X, JY). \tag{3.11}$$

Let $m=2$. We put $K=K(X, Y)$ and from (3.9), (3.10), (3.11) we have $R(X, JX, JY, Y)=\frac{2}{3}\{c-K\}$; $R(X, JY, Y)=\frac{1}{3}\{c-K\}$; $R(JX, JY, X, Y)=\frac{1}{3}\{K-c\}$.

We put $R'=KR_1+\frac{c-K}{3}R_2$. A simple calculation shows that $R(X_1, X_2, X_3, X_4)=R'(X_1, X_2, X_3, X_4)$, whenever X_1, X_2, X_3, X_4 are chosen among the vectors X, JX, JY, Y. Consequenty $R=R'$ and the lemma is proved in the case $m=2$.

Let $m>2$. We choose a unit vector Z, normal to X, JX, Y, JY. Because of (3.8), from $R(X+Z, Y, Y, X-Z)=0$ we get

(3.12) $$K(X, Y)=K(Y, Z).$$

Let $m=3$. We shall show that

(3.13) $$R(X, JX, Y, Z)=R(X, Y, Z, JX)=0$$

and the case $m=3$ will follow as the case $m=2$. From $R(aX+JZ, aJX-Z, aJX-Z, Y)=0$, where a takes the values 1 and -1, we find

(3.14) $$R(X, JX, Z, Y)+R(X, Z, JX, Y)=0$$

and from $R(Y, X+JX, X+JX, Z)=0$ it follows

(3.15) $$R(X, Y, Z, JX)+R(X, Z, Y, JX)=0.$$

Using (3.14), (3.15) and the properties of the curvature tensor we get $R(X, Y, Z, JX)=0$ and together with (3.14) this gives (3.13).

Now let $m>3$. We take arbitrary antiholomorphic spaces α, β in T_pM with orthonormal bases X, Y and Z, U respectively such that $X\perp Y, JY$ and $Z\perp U, JU$. Let V, W be unit vectors in $\operatorname{span}\{X, JX\}^{\perp}\cap\operatorname{span}\{Z, JZ\}^{\perp}$ and $V\perp W, JW$. According to (3.12)

(3.16) $$K(X,V)=K(V,W)=K(V,Z).$$

Let $A\in\operatorname{span}\{V, JV\}^{\perp}\cap\operatorname{span}\{Z, JZ\}^{\perp}\cap\operatorname{span}\{U, JU\}^{\perp}$ be a unit vector. From (3.12)

(3.17) $$K(V, Z)=K(Z, A)=K(Z, U).$$

Analogously

(3.18) $$K(X, V)=K(X, Y).$$

From (3.16), (3.17), (3.18) it follows $K(X, Y)=K(Z, U)$ and the lemma is proved.

4. The main results. Let M be a $2m$-dimensional almost Hermitian manifold, $m\geq 2$.

Axiom of holomorphic $2n$-planes (respectively, $2n$-spheres). *For each point $p\in M$ and for any $2n$-dimensional holomorphic subspace α of T_pM there exists a totally geodesic submanifold N (respectively, a totally umbilical submanifold N with nonzero parallel mean curvature vector) containing p, such that $T_pN=\alpha$.*

Axiom of antiholomnrphic n-planes (respectively, n-spheres). *For each point $p\in M$ and for any n-dimensional antiholomorphic subspace α of T_pM there exists a totally geodesic submanifold N (respectively, a totally umbilical submanifold N with parallel mean curvature vector) containing p such that $T_pN=\alpha$.*

Theorem 1. *Let M be a $2m$-dimensional almost Hermitian manifold, $m\geq 2$. If M satisfies the axiom of holomorphic $2n$-planes or the axiom of holomorphic $2n$-spheres for some n, $1\leq n<m$, then M is an RK-manifold with pointwise constant holomorphic sectional curvature.*

Proof. The condition gives $R(X, JX, JX, Y)=0$ for all vectors $X, Y\in T_pM$ with $g(X, Y)=g(X, JY)=0$ and for each point $p\in M$ and the theorem follows from lemma 1 and lemma 3.

Theorem 2. *Let M be a $2m$-dimensional almost Hermitian manifold, $m\geq 2$. If M satisfies the axiom of antiholomorphic n-planes or the axiom of antiholomorphic n-spheres for some n, $2\leq n\leq m$, then M is an RK-manifold with pointwise constant holomorphic sectional curvature and with pointwise* *constant antiholomorphic sectional curvature and consequently the curvature tensor has the form* (2.1).

Proof. By the condition it follows $R(X, Y, Y, Z)=0$ for each point $p\in M$ and for all $X, Y, Z\in T_pM$ with $g(X, Z)=g(Y, Z)=g(X, Y)=g(X, JY)=0$. Now the theorem follows from lemmas 1, 3 and 4.

By theorem C and theorem 2 we derive

Theorem 3. *Let M be a 2m-dimensional connected almost Hermitian manifold,* $m\geq 3$. *If M satisfies the axiom of antiholomorphic n-planes or the axiom of antiholomorphic n-spheres for some n,* $2\leq n\leq m$, *then M is one of the following:*

1) *a real-space-form,*

2) *a complex-space-form.*

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