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# THE CONGRUENCE LATTICE OF SIMPLE TERNARY ALGEBRAS

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It is shown that the congruence lattice of a simple ternary algebra is distributive and pseudo-complemented. Some specific structures are characterized, e. g. trees by congruences generated by prime ideals. It is shown that simple ternary algebras are join spaces and trees are characterized by prime ideals having a restricted linearity property.

**1. Introduction.** Let $V$ be a non-empty set and $Q$ a ternary operation defined on $V$. The pair $(V, Q)=\mathbf{A}$ is called a simple ternary algebra $\mathbf{A}$, provided that $Q$ satisfies the following demands:

(1) $$Q(a, a, b)=a, \quad a, b \in V;$$

(2) $Q(a, b, c)$ is invariant under all 6 permutations of $a, b, c \in V$;

(3) $$Q(Q(a, b, c), d, e)=(Q\ (Q(a, d, e), Q(b, d, e), c), \quad a, b, c, d, e \in V.$$

The purpose of this paper is to illuminate the properties of congruence relations on a simple ternary algebra $\mathbf{A}$. At first we characterize the congruence relations on $\mathbf{A}$ by means of ideals of $\mathbf{A}$, and show that the lattice $\mathbf{C}(\mathbf{A})$ of all congruence relations on $\mathbf{A}$ is a distributive lattice. Thereafter we consider some relations characterizing trees and similar structures, and finally we show that the operation $Q$ on $\mathbf{A}$ determines a join space over $V$. A few properties of this join space are characterized by means of ideals of $\mathbf{A}$.

The connection between simple ternary algebras $\mathbf{A}$ and partial lattices is considered e. g. by Avann [1]. The class of finite graphs having the same betweenness structure as simple ternary algebras are characterized in [6] and a few properties of the ideal structure of these algebras are given in [5]. Join spaces are defined in [7] and further considered by Varlet in [12]. We shall use here the basic notations of Nebeský given in [4]. An observation on congruences on a specific class of finite simple ternary algebras is given by Zelinka [13].

**2. The structure of the lattice C(A).** A binary relation $\theta$ on a simple ternary algebra $\mathbf{A}$ is a congruence relation on $\mathbf{A}$ if it is reflexive, symmetric, transitive and has the substitution property over the operation $Q$, i. e. if $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle, \langle a_3, b_3\rangle \in \theta$ then $\langle Q(a_1, a_2, a_3), Q(b_1, b_2, b_3)\rangle \in \theta$.

Let $U$ and $W$ be two non-empty subsets of $V$ and $s$ an element of $V$, then $Q(U, W, s)=\{Q(u, w, s) \mid u \in U$ and $w \in W\}$. A non-empty set $W \subset V$ is an ideal of $\mathbf{A}$ whenever $Q(W, W, s) \subset W$ for every $s \in V$. According to (1), $W$ is an ideal whenever $Q(W, W, s)=W$ for every $s \in V$. Let $\mathscr{W}$ be the family of all ideals of $\mathbf{A}$. As shown in [5], $\mathbf{W}(\mathbf{A})=(\mathscr{W}, Q)$ is a simple ternary algebra over the ideals of $\mathbf{A}$, where $Q(U, W, K)=\{Q(u, w, k) \mid u \in U, w \in W, k \in K\}$ for

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all $U, W, K \in \mathscr{W}$. Let $\mathscr{I} = \mathscr{W} \cup \{\emptyset\}$. Then $\mathbf{I}(\mathbf{A}) = (\mathscr{I}, \vee, \wedge)$ is the lattice of ideals of $\mathbf{A}$, where $U \wedge W = U \cap W$ and $U \vee W = W$ when $U = \emptyset$ and if $U, W \neq \emptyset$, $U \vee W = \{t \mid Q(u, w, t) = t$ for some $u \in U$ and some $w \in W, t \in V\}$. The concept of an ideal of simple ternary algebras is based on the definition of Nebeský given in [3].

Let $\mathbf{A} = (V, Q)$ be a simple ternary algebra and $x \in V$ an arbitrary element. As shown by Avann [1, Lemma 3], one can associate with $\mathbf{A}$ a partial lattice $\mathbf{L}(\mathbf{A}, x)$ having the following properties: (*i*) The order relation is given in $\mathbf{L}(\mathbf{A}, x)$ by $b \leq c \Leftrightarrow Q(x, b, c) = b$. (*ii*) The zero element of $\mathbf{L}(\mathbf{A}, x)$ is $x$. (*iii*) $\mathbf{L}(\mathbf{A}, x)$ is closed with respect to the meet given by $b \wedge c = Q(x, b, c)$. (*iv*) The existence of an element $m$, where $b, c \leq m$, implies the existence of the join $b \vee c = Q(m, b, c)$. (*v*) If $b \vee c$ exists, then $d \wedge (b \vee c) = (d \wedge b) \vee (d \wedge c)$. (*vi*) For all triples $b, c, d \in V$ there exists $(b \wedge c) \vee (b \wedge d) \vee (c \wedge d) = Q(b, c, d)$. Note that the partial lattices $\mathbf{L}(\mathbf{A}, x)$ and $\mathbf{L}(\mathbf{A}, y)$ need not be isomorphic when $x \neq y$.

Lemma 1. *Let $\theta$ be a congruence relation on a simple ternary algebra $\mathbf{A} = (V, Q)$. Every congruence class $C \subset V$ of $\theta$ is an ideal of $\mathbf{A}$.*

Proof. $C$ is an ideal of $\mathbf{A}$ if for every two elements $a, b \in C$ and for an arbitrary $s \in V$ it holds: $Q(a, b, s) \in C$. Because $a, b \in C$, $\langle a, b \rangle \in \theta$. $\langle a, a \rangle, \langle s, s \rangle \in \theta$ follow from the reflexivity of $\theta$. By applying the substitution property, we obtain $\langle Q(a, b, c), Q(a, a, s) \rangle \in \theta$. But $Q(a, a, s) = a$, and the relation $\langle Q(a, b, s), a \rangle \in \theta$ implies that $Q(a, b, s) \in C$.

Let $\mathscr{C}_\theta$ be the family of all congruence classes of a congruence relation $\theta$ on $\mathbf{A}$. As well known, $C_1 \cap C_2 = \emptyset$ for every two classes $C_1, C_2 \in \mathscr{C}_\theta$ when $C_1 \neq C_2$, and $\cup \{C \mid C \in \mathscr{C}_\theta\} = V$.

Theorem 1. *A family $\mathscr{K}$ of non-empty ideals of a simple ternary algebra $\mathbf{A} = (V, Q)$ is the family of all congruence classes of a congruence relation $\theta$ on $\mathbf{A}$ if and only if $\mathscr{K}$ satisfies the conditions (i) — (iii):*
*(i) $K_1 \cap K_2 = \emptyset$ for every two ideals $K_1, K_2 \in \mathscr{K}$ when $K_1 \neq K_2$;*
*(ii) $\cup \{K \mid K \in \mathscr{K}\} = V$;*
*(iii) In $\mathbf{W}(\mathbf{A})$, $Q(Z, U, W) \subset K$ for every three ideals $Z, U, W \in \mathscr{K}$ and for some $K \in \mathscr{K}$.*

Proof. Let $\mathscr{K}$ be the family of all congruence classes of a congruence relation $\theta$ on $\mathbf{A}$. Then (*i*) and (*ii*) hold trivially. As shown in [5], $Q(Z, U, W)$ is an ideal of $\mathbf{A}$. Because $Z$, $U$ and $W$ are congruence classes of $\theta$, $\langle z_1, z_2 \rangle$, $\langle u_1, u_2 \rangle$, $\langle w_1, w_2 \rangle \in \theta$ for every elements $z_i \in Z$, $u_i \in U$ and $w_i \in W$, where $i = 1, 2$. According to the substitution property, $\langle Q(z_1, u_1, w_1), Q(z_2, u_2, w_2) \rangle \in \theta$, whence any two elements of the ideal $Q(Z, U, W)$ are in the relation $\theta$. Thus $Q(Z, U, W) \subset K$ for some $K \in \mathscr{K}$.

Conversely, let $\mathscr{K} \subset \mathscr{W}$ be a family satisifying the conditions (*i*) — (*iii*). We define a binary relation $S$ on $\mathbf{A}$ as follows: $\langle a, b \rangle \in S \Leftrightarrow a, b \in K$ for some $K \in \mathscr{K}$. According to (*ii*), any element $t \in V$ belongs to some ideal of $\mathscr{K}$, and thus $S$ is reflexive. The symmetry and the transitivity of $S$ follow from (*i*). Finally, (*iii*) implies the substitution property of $S$, and hence $S$ is a congruence on $\mathbf{A}$. This completes the proof.

As shown e. g. in the book [10, Section 56] of Szász, the congruence relations of an algebra $\mathbf{A}$ constitute a lattice $\mathbf{C}(\mathbf{A})$ with the operations $\vee$ and $\wedge$ defined as follows: Let $\theta, \varphi \in \mathbf{C}(\mathbf{A})$; then $\langle a, b \rangle \in \theta \wedge \varphi \Leftrightarrow \langle a, a \rangle \in \theta, \varphi$, and further, $\langle a, b \rangle \in \theta \vee \varphi \Leftrightarrow$ there is a finite sequence $x_0, x_1, \ldots, x_n, x_{n+1}$ elements of $\mathbf{A}$ such that $\langle x_j, x_{j+1} \rangle$ belongs either to $\theta$ or to $\varphi$ for every value of $j$, $j = 0, 1,$

$\ldots, n, a=x_0$ and $b=x_{n+1}$. Now we are able to prove a theorem on the structure of $\mathbf{C(A)}$.

Theorem 2. *Let* $\mathbf{A}=(V, Q)$ *be a simple ternary algebra. Then* $\mathbf{C(A)}$ *is a distributive lattice.*

Proof. Let $\theta, \varphi$ and $\psi$ be three congruence relations on $\mathbf{A}$. As well known, it is sufficient to show that $\theta\wedge(\varphi\vee\psi)\leq(\theta\wedge\varphi)\vee(\theta\wedge\psi)$, from which the distributivity of $\mathbf{C(A)}$ follows.

Let $\langle a, b\rangle\in\theta\wedge(\varphi\vee\psi)$, and so $\langle a, b\rangle\in\theta, \varphi\vee\psi$. The latter relation implies the existence of a sequence $x_0, \ldots, x_{n+1}$ with properties reported above. We assume now that $\langle x_n, x_{n+1}\rangle\in\varphi$ and denote $Q(x_0, x_n, x_{n+1})=x_n'$. Because $\langle x_0, x_0\rangle$, $\langle x_{n+1}, x_{n+1}\rangle$, $\langle x_n, x_{n+1}\rangle\in\varphi$, we obtain now $\langle Q(x_0, x_n, x_{n+1}), Q(x_0, x_{n+1}, x_{n+1})\rangle=\langle x_n', x_{n+1}\rangle\in\varphi$. Analogously we see that $\langle x_n', x_{n+1}\rangle\in\theta$. Without loosing the generality, we can assume that $\langle x_{n-1}, x_n\rangle\in\psi$. Because $\langle x_0, x_0\rangle, \langle x_{n+1}, x_{n+1}\rangle\in\psi$, we obtain $\langle Q(x_0, x_{n+1}, x_{n-1}), Q(x_0, x_{n+1}, x_n)\rangle=\langle x_{n-1}', x_n'\rangle\in\psi$. Further, the relations $\langle x_{n-1}, x_{n-1}\rangle, \langle x_0, x_{n+1}\rangle, \langle x_{n+1}, x_{n+1}\rangle\in\theta$ imply that $\langle x_{n-1}', x_{n+1}\rangle\in\theta$, and by transitivity of $\theta$, $\langle x_{n-1}', x_n'\rangle\in\theta$, too. By continuing this process we obtain a new sequence $x_0', x_1', \ldots, x_{n+1}'$, where $x_j'=Q(x_0, x_{n+1}, x_j)$. Moreover, $\langle x_j', x_{j+1}'\rangle\in\theta$ holds for every value of $j, j=0, \ldots, n$, as well as either $\langle x_j', x_{j+1}'\rangle\in\varphi$ or $\langle x_j', x_{j+1}'\rangle\in\psi$. By combining these results we obtain $\langle x_0', x_{n+1}'\rangle\in(\theta\wedge\varphi)\vee(\theta\wedge\psi)$, where $x_0'=Q(x_0, x_0, x_{n+1})=x_0$ and, analogously, $x_{n+1}'=x_{n+1}$. This completes the proof.

In the following we characterize the minimal congruence relation $\theta_{ab}$ collapsing two elements $a, b\in V$. Before we need a lemma.

Lemma 2. *An equivalence relation* $E$ *on a simple ternary algebra* $\mathbf{A}=(V, Q)$ *is a congruence relation on* $\mathbf{A}$ *if and only if* $\langle a, b\rangle\in E$ *implies* $\langle Q(a, t, r), Q(b, t, r)\rangle\in E$ *for any pair* $r, t\in V$.

Proof. If $E$ is a congruence on $\mathbf{A}$, the assertion follows from the reflexivity and the substitution property from $E$. Thus, let $E$ be an equivalence on $\mathbf{A}$ and $\langle a, b\rangle, \langle c, d\rangle, \langle f, g\rangle\in E$. We will show the substitution property of $E$. The property of the lemma gives the following sequence: $\langle Q(a, c, f), Q(b, c, f)\rangle$, $\langle Q(b, c, f), Q(b, d, f)\rangle$, $\langle Q(b, d, f), Q(b, d, g)\rangle\in E$, and from the transitivity of $E$ we obtain $\langle Q(a, c, f), Q(b, d, g)\rangle\in E$. This completes the proof.

Theorem 3. *Let* $a, b\in V$ *of a simple ternary algebra* $\mathbf{A}$ *and let* $\theta_{ab}$ *be a binary relation on* $\mathbf{A}$ *defined as follows:* $\langle c, d\rangle\in\theta_{ab}\Leftrightarrow$ *there are two elements* $z, y\in V$ *such that* $z=Q(a, b, z), y=Q(a, b, y), c=Q(z, c, d)$ *and* $d=Q(y, c, d)$. *Then* $\theta_{ab}$ *is a congruence relation on* $\mathbf{A}$ *and the least one collapsing* $a$ *and* $b$.

Proof. It holds for any $f\in V$ and for any $z=Q(a, b, z)$ that $Q(z, f, f)=f$, whence $\theta_{ab}$ is reflexive. Obviously $\theta_{ab}$ is symmetric. Next we show that $\langle Q(c, p, s), Q(d, p, s)\rangle\in\theta_{ab}$ for any two $p, s\in V$. Let $z$ and $y$ have the property of the theorem, and because $\langle c, d\rangle\in\theta_{ab}$, $c=Q(z, c, d)$ and $d=Q(y, c, d)$. Then $Q(z, Q(c, p, s), Q(d, p, s))=Q(Q(z, c, d), p, s)=Q(c, p, s)$ and similarly $Q(y, Q(c, p, s), Q(d, p, s))=Q(d, p, s)$. Thus $\langle Q(c, p, s), Q(d, p, s)\rangle\in\theta_{ab}$, and the desired property follows.

Finally, we show the transitivity of $\theta_{ab}$. Let $\langle t, r\rangle, \langle r, m\rangle\in\theta_{ab}$. According to the definition of $\theta_{ab}$, $t=Q(t, r, u)$, $r=Q(t, r, z)$, $r=Q(r, m, w)$ and $m=Q(r, m, x)$, where $u, z, w$ and $x$ have the property of the definition with respect to $a$ and $b$. We form a new element $q=Q(r, z, w)$. This element has also the property of the definition with respect to $a$ and $b$, i. e. $Q(q, a, b)=Q(a, b, Q(r, z, w))=Q(r, Q(z, a, b), Q(w, a, b))=G(r, z, w)=q$. Moreover, $Q(t, r, q)=$

$=Q(t, r, Q(r, z, w))=Q(w, Q(t, r, t), Q(t, r, z))=Q(w, r, t)=r$, and similarly, $Q(r, m, q)=r$. We consider now the situation in the partial lattice $\mathbf{L}(\mathbf{A}, r)$ because the manipulations are simple to perform in the lattice formulation. In $\mathbf{L}(\mathbf{A}, t)$, $Q(r, t, u)=t=t\wedge u$, $Q(m, r, x)=m\wedge x=m$ and $Q(t, r, q)=r=0=t \wedge q=m\wedge q=Q(m, r, q)$. As above we see that $Q(Q(u, q, t), a, b)=Q(u, q, t)=h$ and $Q(Q(x, q, m), a, b)=Q(x, q, m)=k$. Moreover, $h=Q(u, q, t)=(u\wedge q)\vee(q\wedge t) \vee(t\wedge u)=t\vee(u\wedge q)$, and $Q(h,m,t)=(h\wedge t)\vee(m\wedge t)\vee(h\wedge m)=t\vee(m\wedge t)\vee(h\wedge m) =t\vee(m\wedge t)\vee(m\wedge u\wedge q)=t$ because $m\wedge q=0$. Similarly, $Q(k, m, t)=m$, and the transitivity of $\theta_{ab}$ follows. According to Lemma 2, $\theta_{ab}$ is a congruence on $\mathbf{A}$.

Let $\theta$ be a congruence on $\mathbf{A}$ collapsing $a$ and $b$. If $\langle c, d\rangle\in\theta_{ab}$, then $\langle z, a\rangle =\langle Q(z, a, b), Q(z, a, a)\rangle\in\theta$, and similarly, $\langle y, a\rangle\in\theta$, whence $\langle y, z\rangle\in\theta$ as well. Further, $\langle c, d\rangle=\langle Q(z, c, d), Q(y, c, d)\rangle\in\theta$, and thus $\theta\geqq\theta_{ab}$, which shows the minimality of $\theta_{ab}$, and the theorem follows.

Obviously the set $\{u\,|\, u=Q(u, x, y)u\in V\}$, where $x$ and $y$ are fixed elements of $V$, is an ideal of $\mathbf{A}=(V, Q)$; we denote it by $I[x, y]$. The following theorem shows that $\mathbf{C}(\mathbf{A})$ is pseudo-complemented.

Theorem 4. *Let $\mathbf{A}=(V, Q)$ be a simple ternary algebra and $\theta$ a congruence relation on $\mathbf{A}$. We define a binary relation $\theta^*$ on $\mathbf{A}$ as follows: $\langle x, y\rangle\in\theta^*\Leftrightarrow$ in the ideal $I[x, y]$ of $\mathbf{A}$ every congruence class of $\theta$ consists of a single element. Then $\theta^*$ is a congruence on $\mathbf{A}$ and it is the pseudo-complement of $\theta$ in $\mathbf{C}(\mathbf{A})$.*

Proof. Clearly $\theta^*$ is reflexive and symmetric. Let $\langle x, y\rangle\in\theta^*$ and $r, p$ be two arbitrary elements of $\mathbf{A}$, and assume that the ideal $I[Q(x, r, p), Q(y, r, p)]$ contains two elements $a$ and $b$ such that $\langle a, b\rangle\in\theta$. Let us consider the elements $Q(a, x, y)$ and $Q(b, x, y)$. Because $\theta$ is a congruence relation, $\langle Q(a, x, y), Q(b, x, y)\rangle\in\theta$. Thus $Q(a, x, y)=Q(b, x, y)$, since $\langle x, y\rangle\in\theta^*$. Now $Q(Q(a, x, y), r, p)=Q(Q(x, r, p), Q(y, r, p), a)=a$ because $a\in I[Q(x, r, p), Q(y, r, p)]$. Similarly, $Q(Q(b, x, y), r, p)=b$. But as $Q(a, x, y)=Q(b, x, y)$, we obtain $a=Q(Q(a, x, y), r, p)=Q(Q(b, x, y), r, p)=b$. Therefore $\langle a, b\rangle\in\theta$ implies $a=b$ for every two $a, b\in I[Q(x, r, p), Q(y, r, p)]$, and so $\langle Q(x, r, p), Q(y, r, p)\rangle\in\theta^*$. After proving the transitivity of $\theta^*$, Lemma 2 implies now the congruence property of $\theta^*$.

Let $\langle x, y\rangle, \langle y, z\rangle\in\theta^*$ and assume that $a, b\in I[x, z]$ such that $\langle a, b\rangle\in\theta$. As above, we can conclude that $Q(a, x, y)=Q(b, x, y)$ and $Q(a, z, y)=Q(b, z, y)$. In the partial lattice $\mathbf{L}(\mathbf{A}, y)$ these results mean that $a\wedge x=b\wedge x$ and $a\wedge z =b\wedge z$. Now, $Q(a, x, z)=(a\wedge x)\vee(x\wedge z)\vee(a\wedge z)=(b\wedge x)\vee(x\wedge z)\vee(b\wedge z) =Q(b, x, z)$. Because $a, b\in I[x, z]$, $a=Q(a, x, z)$ and $b=Q(b, x, z)$ and so the defining property of $\theta^*$ holds for $I[x, z]$, whence $\langle x, z\rangle\in\theta^*$. Accordingly, $\theta^*$ is a congruence relation on $\mathbf{A}$.

Clearly $\langle x, y\rangle\in\theta\wedge\theta^*\Leftrightarrow x=y$, whence $\theta\wedge\theta^*$ is equal to the least element 0 of $\mathbf{C}(\mathbf{A})$. On the other hand, if $\theta\wedge\varphi=0$, then $\langle x, y\rangle\in\varphi$ only if $\langle a, b\rangle\in\theta$ and $a, b\in I[x, y]$ together imply $a=b$. Therefore $\theta^*\geqq\varphi$ and so $\theta^*$ is the pseudo-complement of $\theta$ in $\mathbf{C}(\mathbf{A})$.

Now we are ready to prove a theorem about the Boolean property of $\mathbf{C}(\mathbf{A})$. Following Grätzer and Schmidt [2, Def. 3] we say that a congruence relation $\theta$ on a simple ternary algebra $\mathbf{A}$ is separable if for all $a, b\in V$ there exists a sequence $x_0, x_1, \ldots, x_n, x_{n+1}$ of elements of the ideal $I[a, b]$, $x_0=a$, $x_{n+1}=b$, such that for every $i$ either $\langle x_i, x_{i+1}\rangle\in\theta$ or $\langle x_i, x_{i+1}\rangle\notin\theta$ and $x, y\in I[x_i, x_{i+1}]$, $\langle x, y\rangle\in\theta$ imply $x=y$, $i=0, 1, \ldots, n$.

Theorem 5. *The congruence lattice $\mathbf{C}(\mathbf{A})$ of a simple ternary algebra $\mathbf{A}=(V, Q)$ is a Boolean lattice if and only if all congruences on $\mathbf{A}$ are separable.*

Proof. Let all congruence relations on $\mathbf{A}$ be separable. The definitions of separability and of $\theta^*$ imply that $\langle x, y\rangle \in \theta \vee \theta^*$ for every pair $x, y \in V$, i. e. $\theta \vee \theta^* = 1$ in $\mathbf{C}(\mathbf{A})$. Because $\theta \wedge \theta^* = 0$ and $\mathbf{C}(\mathbf{A})$ is distributive, it is Boolean, too. The converse part follows from the definition of $\theta^*$.

The observations given in Theorems 3, 4 and 5 generalize the corresponding results proved by Grätzer and Schmidt in [2] for distributive lattices

**3. Some Special Cases.** In this section we will consider some congruence relations which are characteristic for specific classes of simple ternary algebras.

An ideal $P \in \mathscr{W}$ is called prime, if $Q(x, y, z) \in P$ implies $|\{x, y, z\} \setminus P| \leq 1$, where $|B|$ denotes the cardinality of the set $B$. At first we prove two simple lemmas.

Lemma 3. *Let $P$ be a prime ideal of the simple ternary algebra $\mathbf{A} = (V, Q)$. Then $V \setminus P$ is a prime ideal of $\mathbf{A}$, too.*

Proof. Let $Q(x, y, z) \in V \setminus P$. If $|\{x, y, z\} \cap P| \geq 2$ and, let us say $x, y \in P$. then $Q(x, y, z) \in P$ because $P$ is an ideal of $\mathbf{A}$, which is a contradiction. Hence, the lemma.

Lemma 4. *Let $P$ be a prime ideal of a simple ternary algebra $\mathbf{A} = (V, Q)$ and $\theta(P)$ a binary relation on $\mathbf{A}$ defined as follows: $\langle x, y\rangle \in \theta(P) \Leftrightarrow x, y \in P$ or $x, y \in V \setminus P$. Then $\theta(P)$ is a congruence on $\mathbf{A}$.*

Proof. The relation $\theta(P)$ is obviously an equivalence on $\mathbf{A}$. Let $\langle a, b\rangle \in \theta(P)$ and $p, r \in V$. Because $a$ and $b$ belong to one class of $\theta(P)$, there are in each of the sets $\{a, r, p\}$ and $\{b, r, p\}$ always two elements, say $\{a, p\}$ and $\{b, p\}$, belonging to the same class, say $P$, whence $Q(a, p, r)$ and $Q(b, p, r)$ belong to the same class (in this case to $P$). Hence $\langle Q(a, p, r), Q(b, p, r)\rangle \in \theta(P)$, and lemma follows from Lemma 2.

A simple ternary algebra $\mathbf{A} = (V, Q)$ is a tree if for any $x \in V$ the partial lattice $\mathbf{L}(\mathbf{A}, x)$ is a tree, i. e. no two non-comparable elements $a$ and $b$ of $\mathbf{L}(\mathbf{A}, x)$ have a common upper bound in $\mathbf{L}(\mathbf{A}, x)$.

Theorem 6. *A simple ternary algebra $\mathbf{A} = (V, Q)$ is a tree if and only, if $\theta[P]$ is a congruence relation on $\mathbf{A}$ for every prime ideal $P$ of $\mathbf{A}$, where, $\langle x, y\rangle \in \theta[P] \Leftrightarrow x = y$ or $x, y \in P$.*

Proof. Assume that $\mathbf{A}$ is a tree. The definition of $\theta[P]$ implies that $\theta[P]$ is reflexive, symmetric and transitive. The substitution property is valid for $\theta[P]$ if we can show that $Q(a, b, x) = Q(a, b, y)$ for any pair $a, b \in V \setminus P$ and for every two elements $x, y \in P$; the other cases concerning the substitution property follow from the ideal property of $P$.

We consider the situation in the partial lattice $\mathbf{L}(\mathbf{A}, a)$. Then $Q(a, b, x) = b \wedge x$ and $Q(a, b, y) = b \wedge y$. Because $\mathbf{A}$ and $\mathbf{L}(\mathbf{A}, a)$ are trees and $b$ is a common upper bound of $b \wedge x$ and $b \wedge y$, these two elements are comparable. We suppose that $b \wedge x > b \wedge y$. The element $x \wedge y$ exists in $\mathbf{L}(\mathbf{A}, a)$ and $b \wedge x > b \wedge y \geq b \wedge x \wedge y$. If $x \wedge y \geq b \wedge x$, then $b \wedge x = b \wedge y = b \wedge x \wedge y$, which contradicts the assumption $b \wedge x > b \wedge y$. If $x \wedge y > b \wedge y$, then $x \wedge y$ and $b \wedge y$ are comparable and $y$ is their common upper bound in $\mathbf{L}(\mathbf{A}, a)$. Thus $b \wedge y = b \wedge x \wedge y$ and because $b \wedge x > b \wedge y$, then also $b > x \wedge y$, whence $x \wedge y = b \wedge x \wedge y = b \wedge y$. On the other hand, $x \wedge y = Q(a, x, y) \in P$ as $P$ is an ideal of $A$. Since now $b > x \wedge y$, $Q(a, b, Q(a, x, y)) = x \wedge y \in P$ and as $P$ is prime, $a$ or $b$ belongs to $P$, which is a contradiction. Hence $x \wedge b = y \wedge b$, and the first part of the proof follows.

Let us consider a partial lattice $\mathbf{L}(\mathbf{A}, z)$ and assume that there are two non-comparable elements $a$ and $b$ having a common upper bound $q$ in $\mathbf{L}(\mathbf{A}, z)$

Because $q$ exists, there is also an element $a \vee b$ in $\mathbf{L}(\mathbf{A}, z)$. The intervals $[a \wedge b, a]$ and $[b, a \vee b]$ of $\mathbf{L}(\mathbf{A}, z)$ are ideals $I[a \wedge b, a]$ and $I[b, a \vee b]$ of $\mathbf{A}$, respectively. As $a$ and $b$ are non-comparable, $[a \wedge b, a] \cap [b, a \vee b] = \emptyset$. Now we can apply [5, Thm. 1]: there is in $\mathbf{A}$ a prime ideal $P$ such that $I[a \wedge b, a] \subset P$ and $P \cap I[b, a \vee b] = \emptyset$. According to Theorem 3, $\langle b, a \vee b \rangle \in \theta_{a \wedge b, a}$, because $Q(a, a \vee b, b) = Q(a, b, Q(q, a, b)) = Q(q, a, b) = a \vee b$ and $Q(a \wedge b, a \wedge b, b) = Q(Q(z, a, b), Q(q, a, b), b) = Q(Q(z, q, b), a, b) = Q(b, a, b) = b$. Since now $I[a \wedge b, a] \subset P$, every congruence relation having $P$ as a congruence class must also collaps the elements $a \vee b$ and $b$, whence the relation $\theta[P]$ does not hold in $\mathbf{A}$. Thus the assumption is false, and $\mathbf{A}$ is a tree. This completes the proof.

Theorem 7. *Every congruence on a simple ternary algebra* $\mathbf{A}$ *is the meet of maximal congruences* $\theta(F)$ *if and only if for every collection* $k$ *of ideals satisfying the conditions (i)—(iii) of Theorem 1 it holds: for any two disjoint ideals* $k, m \in k$ *there exists a prime ideal* $P$ *of* $\mathbf{A}$ *such that* $k \subset P$, $P \cap m = \emptyset$ *and if* $u \in k$ *then* $u \subset p$ *or* $u \cap P = \emptyset$.

The proof is obvious. Trees are a class of simple ternary algebras satisfying the demands of Theorem 7.

**4. Simple Ternary Algebras and Join Spaces.** The concept of a join space was introduced in [7]; Tagamlitzki presented an analogous concept considerably earlier in [1]. Varlet showed in [12] that every distributive lattice is a join space with respect to the most general betweenness relation on distributive lattices. This section generalizes Varlet's results.

A join operation $\circ$ in a set $J$ is a mapping of $J \times J$ into the family of subsets of $J$. The image of $(a, b) \in J \times J$ under $\circ$ is denoted by $a \circ b$ and called the join of $a$ and $b$. By definition, if $A \subset J$ and $B \subset J$, $A \circ B = \cup \{a \circ b \mid a \in A$ and $b \in B\}$. An "inverse" operation is defined as follows: $a/b = \{x \mid a \in b \circ x\}$ and $A/B = \cup \{a/b \mid a \in A$ and $b \in B\}$.

The system $(J, \circ)$, where $J$ is an arbitrary set, is a join space if the join operation satisfies the following postulates: $JS\,1$: $a \circ b \neq \emptyset$; $JS\,2$: $a \circ b = b \circ a$; $JS\,3$: $(a \circ b) \circ c = a \circ (b \circ c)$; $JS\,4$: $a/b \cap c/d \neq \emptyset$ implies $a \circ d \cap b \circ c \neq \emptyset$; $JS\,5$: $a/b \neq \emptyset$.

At first we prove a lemma.

Lemma 5. *Let* $\mathbf{A} = (V, Q)$ *be a simple ternary algebra* and let $a$ and $b$ be some fixed elements of V. *Then the set* $D_{ab} = \{x \mid x \in V, Q(x, a, b) = a\}$ *is an ideal of* $\mathbf{A}$.

Proof. We consider the situation in the partial lattice $\mathbf{L}(\mathbf{A}, b)$. Then $Q(a, b, x) = a \wedge x = a$ for every $x \in D_{ab} = [a)$, hence $Dab = [a)$. If now $z, y \in D_{ab}$, then $Q(s, z, y) = (s \wedge y) \vee (s \wedge z) \vee (z \wedge y) \geq a$ because $z \wedge y \geq a$, and thus $Q(s, z, y) \in D_{ab}$ for any $s \in V$. Hence $D_{ab}$ is an ideal of $\mathbf{A}$.

Theorem 8. *Let* $\mathbf{A} = (V, Q)$ *be a simple ternary aglebra. Then* $(V \circ)$ *is a join space with respect to the* $\circ$*—operation defined as follows*: $a \circ b = I[a, b]$.

Proof. The postulates $JS\,1$, $JS\,2$ and $JS\,3$ hold obviously for $a \circ b = I[a, b]$. $a/b = \{x \mid, x \in VQ(x, a, b) = a\} = D_{ab}$, and at least $a \in D_{ab}$, whence $JS\,5$ holds.

Let $D_{ab} \cap D_{cd} \neq \emptyset$, and let $y \in D_{ab} \cap D_{cd}$. In the partial lattice $\mathbf{L}(\mathbf{A}, y)$, $Q(y, a, b) = a \wedge b = a$ and $Q(y, c, d) = c \wedge d = c$. We will show that the element $b \wedge d = Q(y, b, d)$ belongs to $I[a, d] \cap I[b, c]$. Now $Q(a, b \wedge d, d) = (a \wedge b \wedge d) \vee (a \wedge d) \vee (b \wedge d) = (a \wedge d) \vee (b \wedge d) = d \wedge (a \wedge b) = d \wedge b$; $a \vee b$ exists because $a \vee b = a$. Similarly, $Q(c, b \wedge d, d) = b \wedge d$, and thus $b \wedge d \in I[a, d] \cap I[b, c]$. Thus also $JS\,4$ holds and the theorem follows.

The following three theorems illuminate the properties of the join space $(V, \circ)$.

A subset $B$ of a join space is convex if $a, b \in B$ implies $a \circ b \subset B$ [7]; see also [11], where an equivalent concept is introduced.

Theorem 9. *Let* $\mathbf{A}=(V, Q)$ *be a simple ternary algebra. A subset* $B \subset V$ *is a convex set of the join space* $(V, \circ)$ *if and only if* $B$ *is an ideal of* $\mathbf{A}$.

Proof. Let $B$ be an ideal of $\mathbf{A}$. Then for any two elements $a, b \in B$ all the elements $t$ for which $Q(a, b, t)=t$ belong to $B$, whence $B$ is convex.

Conversely, let $B$ be a convex set of the join space $(V, \circ)$ and $a, b \in B$. Now, $Q(a, b, s)=q$ and, as well known, $Q(a, b, q)=q$, whence $q \in a \circ a \subset B$. Thus $Q(a, b, s) \in B$ for any $s \in V$, whence $B$ is an ideal of $\mathbf{A}$.

A subset $B$ of a join space $J$ is a linear set of $J$ if $a, b \in B$ implies $a \circ b \subset B$ and $a/b \subset B$ [7]; this concept was found and studied earlier by Prodanov in [8] and [9].

Theorem 10. *Let* $\mathbf{A}=(V, Q)$ *be a simple ternary algebra. A subset* $B \subset V$ *is a linear set of the join space* $(V, \circ)$ *if and only if* $B=V$.

Proof. Trivially $V$ is a linear set of the join space $(V, \circ)$. Conversely, let $B$ be a linear set of $(V, \circ)$. If $b \in B$, then $b/b \subset B$, and $b/b=\{x \mid Q(x, b, b)=b\}$. According to (1), every $x$ of $V$ belongs to $b/b$ and hence $V \subset B$.

If the simple ternary algebra $\mathbf{A}$ is a tree, the prime ideals of $\mathbf{A}$ have a restricted linearity property as shown in the following theorem.

Theorem 11. *Let* $\mathbf{A}$ *be a simple ternary algebra.* $A$ *is a tree if and only if for every prime ideal* $P$ *of* $\mathbf{A}$ *and for any two distinct elements* $a, b \in P$ *it holds:* $a/b \subset P$ *or* $b/a \subset P$.

Proof. Let $\mathbf{A}=(V, Q)$ be a tree, $P$ a prime ideal of $\mathbf{A}$, $a, b \in P$ and $a \neq b$. We assume that $x$ is an element such that $a=Q(x, a, b)$ but $x \notin P$, i. e. $a/b \not\subset P$. Let $y \in b/a$, i. e. $Q(a, b, y)=b$. We consider the situation in the partial lattice $\mathbf{L}(\mathbf{A}, x)$. $Q(a, b, x)=a \wedge b=a$, whence $a \leq b$. According to the assumption $a \neq b$, and thus $a<b$. On the other hand, $Q(a, b, y)=(a \wedge b) \vee(a \wedge y) \vee(b \wedge y)=a \vee(y \wedge(a \vee a))=a \vee(y \wedge b)=b$. Because $\mathbf{L}(\mathbf{A}, x)$ is a tree and $b \geq a, y \wedge b, a$ and $y \wedge b$ are comparable. Moreover, as $b>a$ and $a \vee(y \wedge b)=b$, $y \wedge b=b$, whence $y \geq b$. If $b/a=\{b\}$, $b/a \subset P$ without any proof, and so we can assume that $y>b$. Because $x<b<y$ in $\mathbf{L}(\mathbf{A}, x)$, $Q(y, x, b)=b \in P$. Because $P$ is prime, at least one of the elements $x$ and $y$ belongs to $P$. According to the assumption, $x \notin P$, and thus $y \in P$. Hence $a/b \not\subset P$ implies $b/a \subset P$ and the first part of the proof follows.

Let $(V, \circ)$ be a join space having the property given in the theorem. If $\mathbf{A}$ is not a tree, there exists an element $x \in V$ such that two non-comparable elements $a$ and $b$ have a common upper bound in $\mathbf{L}(\mathbf{A}, x)$. Because $y>a, b$, then also $a \vee b$ exists and $a \vee b \leq y$. Accordingly, $I[b, a \vee b]$ and $I[b, a \vee B]$ are two ideals of $A$ such that $I[a \wedge b, a] \cap I[b, a \vee b]=\varnothing$, whence there is a prime ideal $P$ containing $I[b, a \vee b]$ and $I[a \wedge b, a] \cap P=\varnothing$ [5, Thm. 1]. Now $Q(a \wedge b, b, a \vee b)=b$ and $Q(a, b, a \vee b)=a \vee b$, which imply that $a \wedge b \in b/a \vee b$ and $a \in a \vee b/b$, respectively. But $a \vee b, a \notin P$ whence $b/a \vee b$, $a \vee b/b \not\subset P$. This is a contradiction, and so $A$ is a tree.

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