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MINIMAL TOPOLOGICAL RINGS

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A Hausdorff topological ring is called minimal if its topology is minimal in sense of Zorn among all Hausdorff ring topologies on the given ring. The minimal topological rings are closely related to the compact ones. The aim of this paper is to examine some permanence properties of the minimal rings and to give proofs of some results concerning the relationship between minimal rings and Krull dimension announced by the author in [19].

The minimal topologies on a given set X with a given property have been studied by various authors. Banaschewski [4] developed a general approach to minimal topologies on algebras. Doitchinov [15] and Stephenson [16] gave examples of non-compact minimal topological groups. Prodanov [11] studied the minimal precompact topologies on Abelian groups by means of the Pontrjagin's duality.

A Hausdorff topological ring (A, τ) is called minimal, if every ring topology on A, which is strictly coarser than τ, is not Hausdorff. Examples of non-compact minimal rings were given in [5] and [6]. Some permanence properties of minimal rings were established in [3]. Mutylin [2] studied minimal commutative rings without proper closed ideals. Minimal topological fields were studied much earlier (see for instance [8] and [9]).

In section 1 of this paper we introduce the notion of **E**-minimal topological ring as a minimal object in a category **E** of Hausdorff topological rings and continuous homomorphisms. Some permanence properties of **E**-minimal rings are established by passing to dense subrings, quotient rings, matrix rings, products and direct sums.

Section 2 is devoted to study **Lpc**-minimal rings, where **Lpc** is the category of topological rings whose completion is strictly linearly compact. In particular the minimal precompact topologies are included. The **Lpc**-minimal topologies on Noetherian rings are described in Theorem 2.2, on integral domains respectively in Lemma 2.7 and Theorem 2.8. The aim of the second half of the section is to establish a relationship between **Lpc**-minimality and Krull dimension. Theorem 2.12 gives a topological characterization of $\dim A = n$ for a complete local Noetherian integral domain A with a countable residue field. This is a partial generalization of the following theorem from [6].

Theorem A. *Let A be a compact Noetherian integral domain. Then* $\dim A = 1$ *iff every subring of A is minimal in the induced topology.*

The generalization was obtained as an answer to a question put by J. M. Smirnov. The source of theorem A was the following theorem due to Prodanov [11].

Theorem B. *The relative topology of every subgroup of an infinite compact Abelian group G is minimal iff G is algebraically and topologically isomorphic to some of the groups of p-adic numbers.*

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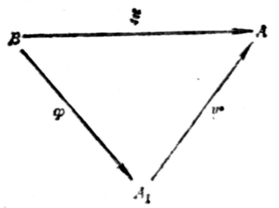
Theorem 2.15 gives a topological characterization of the Krull dimension of a countable Noetherian integral domain.

Throughout this paper the rings always possess a unit, ideal means two-sided ideal and the local rings are not necessarily Noetherian. The Krull dimension of a commutative ring A is denoted by $\dim A$, the completion of a topological ring A is denoted by $\widehat{A}$.

1. E-Minimal rings, definitions and permanence properties. The aim of this section is to introduce the notion of **E**-minimal ring and to examine the main properties of the **E**-minimal rings. Here **E** is a subcategory of the category **H** of all Hausdorff topological rings and continuous homomorphisms. The following abbreviation will be used: $A \in \mathbf{E}$ instead of $A \in \mathrm{Ob}(\mathbf{E})$ and $\varphi \in \mathbf{E}$ instead of $A, B \in \mathrm{Ob}(\mathbf{E})$ and $\varphi \in \mathrm{Mor}_{\mathbf{E}}(A, B)$.

Definition 1.1. *A subcategory* **E** *of* **H** *is called convenient, if the following conditions are fulfilled:*

(C1) *If*



is a commutative diagram in H, φ is an open epimorphism and ψ is a continuous isomorphism, then $B \in \mathbf{E}$ implies $\varphi \in \mathbf{E}$. Moreover $\xi \in \mathbf{E}$ implies $\psi \in \mathbf{E}$.

(C2) *If $B \in \mathbf{H}$, A is a topological subring of B and $i : A \to B$ is the canonical embedding, then $B \in \mathbf{E}$ implies $A \in \mathbf{E}$ and $i \in \mathbf{E}$. In the case A is dense in B, $A \in \mathbf{E}$ implies $B \in \mathbf{E}$ and $i \in \mathbf{E}$.*

(C3) *If $f : B \to C$ is a continuous epimorphism in* **H**, *and A is a dense subring of B such that the restriction $f|_A$ belongs to* **E**, *then $f \in \mathbf{E}$.*

Examples 1.2. Here we give examples of convenient categories.

(a) Let **P** be the full subcategory of **H** consisting of all precompact topological rings (a topological ring A is called precompact, if the completion $\widehat{A}$ is compact). Obviously **P** is a convenient category.

(b) Let **L** be the full subcategory of **H** consisting of all topological rings possessing a fundamental system of neighbourhoods of the zero which are open ideals (such topologies will be called linear, see [13]). It is a well known fact that **P** is a subcategory of **L** (remind that all rings in **H** possess unit). Clearly **L** is a convenient category.

(c) Let **B** be the full subcategory of all bounded rings in **H** (see [8]). Then **B** is convenient and **L** is a subcategory of **B**. Also the categories $\mathbf{B}_r$ and $\mathbf{B}_l$ of right bounded and respectively left bounded rings in **H** are convenient.

(d) Let **lB** be the full subcategory of all locally bounded rings in **H** (see [8]). Then **lB** is convenient and **B** is a subcategory of **lB**.

(e) The category **H** is convenient.

(f) Let $\mathfrak{m}$ be a cardinal number and $A \in \mathbf{H}$. The ring A is called $\mathfrak{m}$-topological ring (see [3]), if the intersection of every family Γ of neighbourhoods of the zero with card $\Gamma < \mathfrak{m}$ is a neighbourhood of the zero again. Denote by $\mathbf{H}_{\mathfrak{m}}$ the full subcategory of $\mathbf{H}$ consisting of all $\mathfrak{m}$-topological rings. It is easy to verify that $\mathbf{H}_{\mathfrak{m}}$ is convenient. In particular $\mathbf{H}_{a_0} = \mathbf{H}$.

(g) All examples given above are full subcategories of **H**, or briefly full convenient categories. Now we shall consider an example of convenient category which is not a full subcategory of **H**. A homomorphism of topological groups $f: A \to B$ is called almost open, if, for every neighbourhood of the zero in A, $f(U)$ is dense in some neighbourhood of the zero in $f(A)$. Let **E** be a full convenient category, denote by $\mathbf{E}'$ the category with the same objects and all almost open continuous homomorphisms as morphisms. Now $\mathbf{E}'$ is a convenient category and not a full subcategory of **H**.

Next we give an example of non-convenient category. Remind at first the definition of strictly linearly compact ring, we shall write briefly s. l. c.. A topological ring A is called s. l. c., if $A \in \mathbf{L}$, A is complete and for every open ideal $\mathfrak{L}$ of A the quotient ring $A/\mathfrak{L}$ satisfies the descending chains condition. More information about s. l. c. rings can be found in [14] and [17].

Definition 1.3. *A topological ring A is called strictly linearly precompact, if $A \in \mathbf{L}$ and the completion $\widehat{A}$ is s. l. c.*

We denote by **Lpc** the full subcategory of strictly linearly compact rings in **H**. The conditions (C1) and (C3) hold in **Lpc**, (C2) does not. For instance the formal power series ring $Q[[T]]$, where Q is the field of rational numbers, is in **Lpc** (endowed with the (T)-adic topology), but the subring Z of integers is discrete, hence $Z \notin \mathbf{Lpc}$. However **Lpc** satisfies the following (weaker) condition:

(C2*) If $B \in \mathbf{H}$ and $j: A \to B$ is a dense embedding $(A \in \mathbf{H})$ then $B \in \mathbf{E}$ iff $A \in \mathbf{E}$. In both cases $j \in \mathbf{E}$.

Definition 1.4. *A subcategory* **E** *of* **H** *is called quasi-convenient, if* (C1), (C2*) *and* (C3) *hold.*

The category **Lpc** is quasi-convenient.

Now we are able to introduce the notion of **E**-minimal ring, where **E** is a quasi-convenient category.

Definition 1.5. *Let* **E** *be a quasi-convenient category and* $(A, \tau) \in \mathbf{E}$. *We call the topological ring* (A, τ) **E**-*minimal ring and the topology* τ — **E**-*minimal ring topology, if every topology* τ' *on A with* $\tau' \leq \tau$, $(A, \tau') \in \mathbf{E}$ *and* $\mathrm{id}_A: (A, \tau) \to (A, \tau')$ *belonging to* **E**, *coincides with* τ.

Obviously (A, τ) is **E**-minimal iff every continuous isomorphism $f: (A, \tau) \to (A_1, \tau_1)$ in **E** is open. Every compact ring in **E** is **E**-minimal. **H**-minimal rings are exactly the minimal ones. Every **P**-minimal ring is minimal and every **Lpc**-minimal ring is **L**-minimal. The converse of the latter is true in Noetherian integral domains (see proposition 2.5.). $\mathbf{H}'$-minimal rings are similar to the B_r-complete topological groups (a Hausdorff topological group G is called B_r-complete, if every continuous almost open isomorphism $f: G \to H$, where H is a Hausdorff topological group, is open).

Let B be a topological ring and A be a dense subring of B. The embedding $A \to B$ is called essential, if for every non-zero closed ideal $\mathfrak{J}$ of B $\mathfrak{J} \cap B \neq 0$ holds.

Proposition 1.6. *Let* **E** *be a quasi-convenient category, B be a Hausdorff topological ring and A be a dense subring of B. Then A is* **E**-*minimal iff B is* **E**-*minimal and the embedding* $A \subset B$ *is essential.*

The case $\mathbf{E}=\mathbf{P}$ is contained in [5], the case $\mathbf{E}=\mathbf{H}_{\mathfrak{m}}$ is partially contained in [3] and the case of topological algebras — in [4]. The proof is similar to that in [4] and will be omitted. There is an analogue to proposition 1.6 in [2] too. By proposition 1.6 every dense subring of a minimal topological division ring is minimal. Examples of minimal topological division rings can be found in [9] and [8].

Proposition 1.6 provides many examples of $\mathbf{H}'$-minimal rings too. From [7] every complete metrizable Abelian group is B_r-complete, hence every complete metrizable topological ring is $\mathbf{H}'$-minimal. By proposition 1.6 a metrizable ring A is $\mathbf{H}'$-minimal iff the embedding $A\subset\widehat{A}$ is essential. Similar result is true for locally precompact topological rings (i. e. topological rings with locally compact completion), since every locally compact Abelian group is B_r-complete [7].

Corollary 1.7. *A topological ring A in* **Lpc** *is* **Lpc**-*minimal iff the embedding $A\subset\widehat{A}$ is essential.*

Proof. It follows from 1.3 that the completion A is s. l. c. . Since every s. l. c. ring is **Lpc**-minimal [1], it remains only to apply proposition 1.6. Q. E. D.

The following lemma is used as a technical tool in section 2.

Lemma 1.8. *Let* **E** *be a quasi-convenient category and A be an* **E**-*minimal integral domain. Then $\widehat{A}$ is an* **E**-*minimal integral domain, too.*

The proof is similar to the proof of lemma 2 in [2] and will be omitted.

The next step is to establish productivity for **E**-minimal rings. It is natural to impose on **E** the following productivity condition:

(P) If $A_\alpha\in\mathbf{E}$ $(\alpha\in I)$, then $\Pi_{\alpha\in I}A_\alpha\in\mathbf{E}$.

Obviously (P) holds in **H**, **B**, $\mathbf{B}_r$, $\mathbf{B}_l$, **L**, **Lpc**, and **P**. In **lB** only finite products exist as well as in $\mathbf{H}_{\mathfrak{m}}$, when $\mathfrak{m}>a_0$.

Theorem 1.9. *Let* **E** *be a full convenient category satisfying* (P) *and $\{A_\alpha\}_{\alpha\in I}$ be a family of rings in* **E**. *Then the product $A=\Pi_{\alpha\in I}A_\alpha$ and the direct sum $A'=\Sigma_{\alpha\in I}A_\alpha$ endowed with the Tichonov topology are* **E**-*minimal* ($\mathbf{E}'$-*minimal*) *iff for every $\alpha\in I$ A_α is* **E**-*minimal* ($\mathbf{E}'$-*minimal*). *If τ is an* **E**-*minimal topology on A (A'), then there exist* **E**-*minimal topologies τ_α on each A_α such, that $\tau=\Pi_{\alpha\in I}\tau_\alpha$.*

Proof. We shall make use of the following property of the ring topologies on A. Let τ be a ring topology on A and for every $\alpha\in I$ τ_α be the induced topology on A_α by the canonical embedding $A_\alpha\to A$, then $\tau\geq\Pi_{\alpha\in I}\tau_\alpha$ In fact, suppose

$$(1)\qquad x_\mu\longrightarrow x\,(x,\ x_\mu\in A)$$

is a convergent net in (A,τ), it is enough to show that x_μ tends to x in $\Pi_{\alpha\in I}\tau_\alpha$ too. For every $\alpha\in I$ define $e_\alpha\in A$ by

$$e_\alpha(\beta)=\begin{cases}0, & \beta\in I\setminus\{\alpha\},\\ 1, & \beta=\alpha.\end{cases}$$

Multiplying (1) by e_α we note that the α-th coordinate of x_μ tends to the α-th coordinate of x in (A_α,τ_α). Hence x_μ tends to x in $(\Pi_{\alpha\in I}A_\alpha,\ \Pi_{\alpha\in I}\tau_\alpha)$. Thus $\Pi_{\alpha\in I}\tau_\alpha\leq\tau$ is proved.

Let for every $\alpha\in I$, τ_α be an **E**-minimal topology on A_α, we have to show that $\tau=\Pi_{\alpha\in I}\tau_\alpha$ is **E**-minimal. Clearly $(A,\tau)\in\mathbf{E}$ by (P). Suppose σ is a topology on A with $\sigma \leqq \tau$ and $(A, \sigma) \in \mathbf{E}$. Then for every $\alpha \in I$ the induced topology σ_α on A_α is coarser, than τ_α and $(A_\alpha, \sigma_\alpha) \in \mathbf{E}$ by (C2). The minimality of τ_α implies $\sigma_\alpha = \tau_\alpha$. On the other hand $\sigma \geqq \Pi_{\alpha \in I} \sigma_\alpha$ by the previous considerations. Hence $\sigma \geqq \Pi_{\alpha \in I} \sigma_\alpha = \Pi_{\alpha \in I} \tau_\alpha = \tau \geqq \sigma$, so $\sigma = \tau$ and the minimality of (A, τ) is established. Since A' is dense in A and every closed ideal $\mathfrak{L}$ of A has the form $\mathfrak{L} = \Pi_{\alpha \in I} \mathfrak{L}_\alpha$ ($\mathfrak{L}_\alpha = \mathfrak{L} \cap A_\alpha$ is a closed ideal of A_α for $\alpha \in I$), the embedding $A' \to A$ is essential. By proposition 1.6 the Tichonov topology on A' is $\mathbf{E}$-minimal.

Assume τ_α is an $\mathbf{E}'$-minimal topology on A_α for each $\alpha \in I$. We have to show that $\tau = \Pi_{\alpha \in I} \tau_\alpha$ is an $\mathbf{E}'$-minimal topology on A. Suppose σ is a topology on A with $(A, \sigma) \in \mathbf{E}$ and the identity

$$(A, \tau) \longrightarrow (A, \sigma) \tag{2}$$

is continuous and almost open. For every $\alpha \in I$ the induced topology σ_α on A_α is coarser, than τ_α. Moreover the identity

$$(A_\alpha, \tau_\alpha) \longrightarrow (A_\alpha, \sigma_\alpha) \tag{3}$$

is almost open. Indeed, let V_α be a τ_α-neighbourhood of the zero in A_α, we have to verify that $[V_\alpha]_{\sigma_\alpha}$ is a σ_α-neighbourhood of the zero in A_α (here $[\ ]_{\sigma_\alpha}$ denotes the closure with respect to σ_α). At first we prove that

$$[V_\alpha]_{\sigma_\alpha} = [V_\alpha]_\sigma \cap A_\alpha = [V_\alpha \times \prod_{\alpha' \neq \alpha} A_{\alpha'}]_\sigma \cap A_\alpha. \tag{4}$$

Only the second equality in (4) has to be verified. Obviously the left side is contained in the right one. Now take

$$x \in [V_\alpha \times \prod_{\alpha' \neq \alpha} A_{\alpha'}]_\sigma \cap A_\alpha,$$

then $x(\alpha') = 0$ for $\alpha' \in I \setminus \{\alpha\}$ and

$$x = \lim_\sigma x_\mu, \tag{5}$$

where $x_\mu \in V_\alpha \times \Pi_{\alpha' \neq \alpha} A_{\alpha'}$. Multiply (5) by e_α, then $x = xe_\alpha = \lim_\sigma x_\mu e_\alpha$, where $x_\mu e_\alpha \in V_\alpha$. Hence $x \in [V_\alpha]_\sigma \cap A_\alpha$ and (4) is proved. Since $V_\alpha \times \Pi_{\alpha' \neq \alpha} A_{\alpha'}$ is a τ-neighbourhood of the zero in A and (2) is almost open, the closure $[V_\alpha \times \Pi_{\alpha' \neq \alpha} A_{\alpha'}]_\sigma$ is a σ-neighbourhood of the zero in A. Hence $[V_\alpha]_{\sigma_\alpha}$ is a σ_α-neighbourhood of the zero in A_α by (4). Therefore (3) is almost open. By the minimality of τ_α we have $\sigma_\alpha = \tau_\alpha$. Hence

$$\tau = \Pi_{\alpha \in I} \tau_\alpha = \Pi_{\alpha \in I} \sigma_\alpha \leqq \sigma \leqq \tau,$$

so $\sigma = \tau$. Therefore (A, τ) is $\mathbf{E}'$-minimal. The minimality of the direct sum follows as well as above.

Suppose now, that $(A, \Pi_{\alpha \in I} \tau_\alpha)$ is $\mathbf{E}$-minimal. Let for $\alpha_0 \in I$ σ_{α_0} be a topology on A_{α_0} with $\sigma_{\alpha_0} \leqq \tau_{\alpha_0}$ and $(A_{\alpha_0}, \sigma_{\alpha_0}) \in \mathbf{E}$. Set $\tau' = \sigma_{\alpha_0} \times \Pi_{\alpha \neq \alpha_0} \tau_\alpha$, then $(A, \tau') \in \mathbf{E}$ by (P) and $\tau' \leqq \tau$. Hence $\tau' = \tau$ and $\sigma_{\alpha_0} = \tau_{\alpha_0}$. Thus τ_{α_0} is $\mathbf{E}$-minimal.

Let $(A, \Pi_{\alpha \in I} \tau_\alpha)$ be $\mathbf{E}'$-minimal, we shall show that for every $\alpha \in I$ τ_α is $\mathbf{E}'$-minimal. Consider for $\alpha_0 \in I$ a topology σ_{α_0} on A_{α_0} with $(A_{\alpha_0}, \sigma_{\alpha_0}) \in \mathbf{E}$ and such that the identity

$$(A_{\alpha_0}, \tau_{\alpha_0}) \longrightarrow (A_{\alpha_0}, \sigma_{\alpha_0}) \tag{6}$$

is continuous and almost open. Set $\tau'=\sigma_{\alpha_0}\times\Pi_{\alpha\neq\alpha_0}\tau_\alpha$, then $(A,\tau')\in\mathbf{E}$ by (P). Let $V=V_{\alpha_0}\times U$ be a neighbourhood of the zero in (A,τ), where V_{α_0} is a neighbourhood of the zero in $(A_{\alpha_0},\tau_{\alpha_0})$ and U is a closed neighbourhood of the zero in $(\Pi_{\alpha\neq\alpha_0}A_\alpha, \Pi_{\alpha\neq\alpha_0}\tau_\alpha)$. Then $[V]_{\tau'}=[V_{\alpha_0}]_{\sigma_{\alpha_0}}\times U$, hence $[V]_{\tau'}$ is a τ'-neighbourhood of the zero in A, since (6) is almost open. Therefore the identity $(A,\tau)\rightarrow(A,\tau')$ is continuous and almost open. By the minimality of τ we have $\tau'=\tau$ and $\sigma_{\alpha_0}=\tau_{\alpha_0}$. Thus τ_{α_0} is $\mathbf{E}'$-minimal.

It remains to prove the last statement of the theorem. Let τ be an $\mathbf{E}$-minimal topology on A, then for every $\alpha\in I$ $(A_\alpha,\tau_\alpha)\in\mathbf{E}$, where τ_α is the induced topology on A_α by τ. By virtue of (P) $(A,\Pi\tau_\alpha)\in\mathbf{E}$ and $\tau\geqq\Pi_{\alpha\in I}\tau_\alpha$, hence $\tau=\Pi_{\alpha\in I}\tau_\alpha$. The minimality of each τ_α was established above. If τ is an $\mathbf{E}$-minimal topology on the direct sum A', then we form $\Pi_{\alpha\in I}\tau_\alpha$ as before. From $(A,\Pi_{\alpha\in I}\tau_\alpha)\in\mathbf{E}$ we have $(A',\Pi_{\alpha\in I}\tau_\alpha)\in\mathbf{E}$ by (C2). Now $\Pi_{\alpha\in I}\tau_\alpha\leqq\tau$ on A' and the minimality of τ implies $\tau=\Pi_{\alpha\in I}\tau_\alpha$. Q. E. D.

The cases $\mathbf{E}=\mathbf{L}$ and $\mathbf{E}=\mathbf{P}$ are obtained in [5]. The second case is partially obtained in [4] too. In [3] the $\mathfrak{m}$-product of topologies $\Pi^{\mathfrak{m}}_{\alpha\in I}\tau_\alpha$ is considered, where $\mathfrak{m}$ is a cardinal number (a fundamental system of neighbourhoods of the zero in $\Pi^{\mathfrak{m}}_{\alpha\in I}\tau_\alpha$ are all intersections of families Γ of neighbourhoods of the zero in $\Pi_{\alpha\in I}\tau_\alpha$ with card $\Gamma<\mathfrak{m}$). It is proved (theorem 5, [3]) that if $\{A_\alpha\}_{\alpha\in I}$ is a family of rings in $\mathbf{H}_{\mathfrak{m}}$, then $(\Pi_{\alpha\in I}A_\alpha, \Pi^{\mathfrak{m}}_{\alpha\in I}\tau_\alpha)$ is $\mathbf{H}_{\mathfrak{m}}$-minimal iff for every $\alpha\in I$ (A_α,τ_α) is $\mathbf{H}_{\mathfrak{m}}$-minimal. The reguirement of existence of a unit is weakened in [3]. However it is impossible to remove it completely, since Doitchinov [15] gives an example of a minimal Abelian group Z such, that $Z\times Z$ is not minimal. If we consider Z as a topological ring with zero multiplication, then Z is a minimal ring and $Z\times Z$ is not a minimal ring. For topological rings with a unit the product theorem from [3] can be obtained from the proof of theorem 1.9. It is enough only to prove in addition the following property of the ring $\mathbf{H}_{\mathfrak{m}}$-topologies τ on the product $\Pi_{\alpha\in I}A_\alpha : \tau\geqq\Pi^{\mathfrak{m}}_{\alpha\in I}\tau_\alpha$, where τ_α is the topology induced on A_α by τ.

Finally we remind in connection with theorem 1.9, that productivity does not hold for B_r-complete groups ([18]).

We begin now studying the minimal topologies on matrix rings. Theorem 1.10 below shows that they can be described in a very simple way. Let (A,τ) be a topological ring and n be a natural number. We denote by A_n the matrix ring of square matrices of order n and by τ^{n^2} the Tichonov topology on A_n. It is natural to require the convenient category $\mathbf{E}$ to be closed under forming the matrix ring endowed with the Tichonov topology. That's why we consider the following condition, where n is a natural number and $\mathbf{E}$ is a subcategory of $\mathbf{H}$:

(M_n) For every $(A,\tau)\in\mathbf{E}$ we have $(A_n,\tau^{n^2})\in\mathbf{E}$.

Clearly all categories considered in the examples above satisfy (M_n) for every natural number n.

Theorem 1.10. *Let* $\mathbf{E}$ *be a convenient category satisfying* (M_n) *and* (A,τ) *be a ring in* $\mathbf{E}$. *Then* (A,τ) *is* $\mathbf{E}$-*minimal* ($\mathbf{E}'$-*minimal*) *iff* (A_n,τ^{n^2}) *is* $\mathbf{E}$-*minimal* ($\mathbf{E}'$-*minimal*). *For every* $\mathbf{E}$-*minimal topology* Σ *on the matrix ring* A_n *there exists an* $\mathbf{E}$-*minimal topology* σ *on* A *such that* $\Sigma=\sigma^{n^2}$.

Proof. For a matrice M of A_n we denote by $(M)_{k,l}$ the element of M standing in the k-th row and in the l-th column ($k=1, 2, \ldots, n$; $l=1, 2, \ldots, n$). Now denote by $E_{k,l}$ the matrice of A_n with $(E_{k,l})_{ij}=\delta_{ik}\cdot\delta_{jl}$ ($i=1, 2, \ldots, n$; $j=1, 2, \ldots, n$), where δ_{ik} is the Kroneker symbol. We establish first the following property of ring topologies on A_n. Let Σ be a ring topology on A_n and σ be the induced topology on A by the embedding $a \to aE_1$; ($a \in A$), then $\Sigma \geqq \sigma^{n^2}$. To prove that take a net $\{M_\alpha\}$ in (A_n, Σ), which tends to $M \in A_n$. It is enough to prove that $\{M_\alpha\}$ tends to M in (A_n, σ^{n^2}). Since Σ is a ring topology, the net $\{E_{1k}M_\alpha E_{l1}\}$ tends to $E_{1k}ME_{l1}$ in Σ ($k=1, 2, \ldots, n$; $l=1, \ldots, n$). In other words the net $(M_\alpha)_{k,l}E_{11}$ tends to $(M)_{k,l}E_{11}$ in Σ. By the definition of σ this means, that $(M_\alpha)_{k,l}$ tends to $(M)_{k,l}$ in (A, σ). Hence $\{M_\alpha\}$ tends to M in (A_n, σ^{n^2}) and $\Sigma \geqq \sigma^{n^2}$ is verified.

Assume τ is an **E**-minimal topology on A, we shall show that τ^{n^2} is **E**-minimal. Let Σ be a topology on A_n with $(A_n, \Sigma) \in$ **E** and $\Sigma \leqq \tau^{n^2}$. Then the induced topology σ on A is coarser than τ and $(A, \sigma) \in$ **E**. By the **E**-minimality of τ we get $\sigma=\tau$. By the above property $\tau^{n^2}=\sigma^{n^2} \leqq \Sigma \leqq \tau^{n^2}$, hence $\Sigma=\tau^{n^2}$ and the **E**-minimality of τ^{n^2} is established. In the case τ is **E**′-minimal we use the same argument. It remains only to prove that the identity $(A, \tau) \to (A, \sigma)$ is continuous and almost open, if the identity $(A_n, \tau^{n^2}) \to (A_n, \sigma^{n^2})$ has the corresponding properties. This is easily established as in the proof of theorem 1.9.

One proves that τ is **E**-minimal (**E**′minimal) if τ^{n^2} is **E**-minimal (**E**′minimal) in the same way as in theorem 1.9.

It remains only to prove that all **E**-minimal topologies on A_n are of the type considered above. Suppose that Σ is an **E**-minimal topology on A_n and σ is the induced topology on A. Then $(A, \sigma) \in$ **E** by (C2), hence $(A_n, \sigma^{n^2}) \in$ **E** by (M_n). Since $\Sigma \geqq \sigma^{n^2}$, the minimality of Σ implies $\Sigma=\sigma^{n^2}$. Q. E. D.

An $\mathbf{H}_{\mathrm{m}}$-version of this theorem is proved in [3].

Theorems 1.9 and 1.10 show that the class of **E**-minimal rings is closed under forming products, direct sums and matrix rings. Now we consider a subclass which is closed under forming quotient rings with respect to closed ideals.

Definition 1.11. *Let* **E** *be a quasi-convenient category. An* **E**-*minimal ring* (A, τ) *is called* **E**-*totally minimal ring and the topology* τ *is called* **E**-*totally minimal ring topology, if for every closed ideal* $\mathfrak{J}$ *of* A *the quotient ring* $A/\mathfrak{J}$ *is* **E**-*minimal.*

Obviously every compact ring in **E** is **E**-totally minimal. It is easy to prove that the ring A is **E**-totally minimal iff $A \in$ **E** and every epimorphism $f: A \to A_1$ in **E** is open. Clearly **E**-minimal rings without proper closed ideals are **E**-totally minimal. In the commutative case for **E**$=$**H** they are exactly the dense subrings of complete minimal topological fields [2]. **H**-totally minimal rings will be called for brevity totally minimal.

A subring A of a topological ring B is called totally dense, if for every closed ideal $\mathfrak{J}$ of B $[A \cap \mathfrak{J}]=\mathfrak{J}$ holds.

Proposition 1.12. *Let* **E** *be a quasi-convenient category and* A *be a dense subring of the topological ring* B. *Then* A *is* **E**-*totally minimal iff* B *is* **E**-*totally minimal and* A *is totally dense in* B.

Proof. Let A be **E**-totally minimal and $\mathfrak{J}$ be a closed ideal of B. Then for $\mathfrak{J}_1=[\mathfrak{J} \cap A]$ we have $A \cap \mathfrak{J}_1=\mathfrak{J}$ and the quotient ring $A/_{\mathfrak{J} \cap A}$ is **E**-minimal by the **E**-total minimality of A. Hence the embedding $A/_{\mathfrak{J} \cap A} \to B/\mathfrak{J}_1$ is essential and $B/_{\mathfrak{J}_1}$ is **E**-minimal by proposition 1.6. Now $\mathfrak{J}/\mathfrak{J}_1$ is a closed ideal of $B/\mathfrak{J}_1$ which intersects $A/_{\mathfrak{J}\cap A}$ in a trivial way, hence $\mathfrak{J}_1=\mathfrak{J}$. This proves the total density of A in B and the **E**-total minimality of B at the same time.

To prove the sufficiency we have to show that for every closed ideal $\mathfrak{L}$ of A the quotient ring $A/\mathfrak{L}$ is **E**-minimal. Set $\mathfrak{J}=[\mathfrak{L}]_B$, then $B/\mathfrak{J}$ is **E**-minimal and $A/\mathfrak{L}$ is a dense subring of $B/\mathfrak{J}$. By proposition 1.6 it is enough to show that this embedding is essential. Let $\bar{\mathfrak{L}}_1$ be a closed ideal of $B/\mathfrak{J}$ which intersects $A/\mathfrak{L}$ in a trivial way. If $\varphi:B\to B/\mathfrak{J}$ is the canonical homomorphism, then $\mathfrak{J}_1=\varphi^{-1}(\bar{\mathfrak{L}}_1)$ is a closed ideal of B with $\mathfrak{J}_1\cap A=\mathfrak{L}$ by the above property of $\bar{\mathfrak{L}}_1$. Since A is totally dense in B, we have $\mathfrak{J}_1=[\mathfrak{J}_1\cap A]=[\mathfrak{L}]=\mathfrak{J}$, hence $\bar{\mathfrak{L}}_1$ is the zero ideal of $B/\mathfrak{J}$. Then theorem is proved. Q. E. D.

From this proposition we obtain the following total minimality criterion: a precompact topological ring A is totally minimal iff A is totally dense in the (compact) completion $\widehat{A}$.

The notion of **E**-total minimality can be introduced also in a convenient category of topological algebras (satisfying (C1), (C2) and (C3)) and an analogue of proposition 1.12 can be proved.

Our next goal is to establish productivity for **E**-totally minimal rings.

Theorem 1.13. *Let* **E** *be a full convenient category satisfying* (P) *and* $\{A_\alpha\}_{\alpha\in\mathfrak{A}}$ *be a family of rings in* **E**. *Then the product* $A=\Pi_{\alpha\in\mathfrak{A}}A_\alpha$ *and the direct sum* $A'=\Sigma_{\alpha\in\mathfrak{A}}A_\alpha$ *are* **E**-*totally minimal* (**E**′-*totally minimal*) *iff for every* $\alpha\in\mathfrak{A}$ *the ring* A_α *is* **E**-*totally minimal* (**E**′-*totally minimal*).

Proof. We have only to apply theorem 1.9 and definition 1.11., and to remark that every closed ideal of A has the form $\mathfrak{J}=\Pi_{\alpha\in\mathfrak{A}}\mathfrak{J}_\alpha$, where each $\mathfrak{J}_\alpha$ is a closed ideal of A_α. Q. E. D.

Theorem 1.14. *Let* n *be a natural number,* **E** *be a full convenient category satisfying* (M_n) *and* A *be a ring in* **E**. *Then* A_n *is* **E**-*totally minimal* (**E**′-*totally minimal*) *iff* A *has the same property.*

Proof. We have only to apply theorem 1.10 and definition 1.11, and to remark that every closed ideal of A_n has the form $\mathfrak{J}_n$, where $\mathfrak{J}$ is a closed ideal of A and $\mathfrak{J}_n$ is the ideal of all matrices in A_n with coefficients from $\mathfrak{J}$. Q. E. D.

The last two theorems show that the class of **E**-totally minimal rings is closed under forming products, direct sums, completions, matrix rings and quotient rings with respect to closed ideals. A closed subgroup of a minimal Abelian group is minimal, too [11]. The following example shows that an open (and consequently closed) subring of a totally minimal ring may be non-minimal. Let k be a field and $L=k\langle\langle T\rangle\rangle$ be the field of formal power series of one variable over k. Then the topology of L generated by the usual valuation of L is minimal [8]. The ring $A=k[[T]]$ of formal power series of one variable over k is an open subring of L and the induced topology on A is not minimal, if k is infinite [4].

2. Lpc-minimal topologies on commutative rings. This section has two general lines. The first one is to describe by means of $\mathfrak{M}$-adic topologies and $\mathfrak{M}$-adic completions all **Lpc**-minimal topologies on two classes of commutative rings — the class of all Noetherian rings and the class of all locally Noetherian integral domains. The results in this line generalize similar results about minimal precompact topologies on locally Noetherian integral domains in [6]. On the other hand we obtain a description of all **L**-totally minimal topologies in the classes mentioned above. In the second line some relations are established between Krull dimension and **Lpc**-minimality.

Throughout this section all rings are commutative. If A is a ring and $\mathfrak{M}$ is an ideal of A, then the $\mathfrak{M}$-adic topology on A will be denoted by $\tau_{\mathfrak{M}}$ and the $\mathfrak{M}$-adic completion of A will be denoted by $\widehat{A}_{\mathfrak{M}}$.

We shall often make use of the following lemma, which combines results from [1] and [17].

Lemma 2.1. *Every s. l. c. ring is a product of local s. l. c. rings endowed with the Tichonov topology. In particular every s. l. c. integral domain is local. The topology of a s. l. c. local ring with maximal ideal $\mathfrak{M}$ is coarser than $\tau_{\mathfrak{M}}$.*

Theorem 2.2. *Let A be a Noetherian ring,*

$$\bigcap_{i=1}^{m} \mathfrak{Q}_i=(0) \tag{1}$$

be an irreducible primary decomposition of the zero ideal in A and τ be an **Lpc**-*minimal topology on A. Then there exist a finite number uniquely determined open maximal ideals $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$ of A, such that*

$$\bigcup_{k=1}^{n} \mathfrak{M}_k \supset \bigcup_{i=1}^{m} \mathfrak{Q}_i \tag{2}$$

and $\tau \leq \tau_{\mathfrak{M}}$, where $\mathfrak{M}=\cap_{k=1}^{n}\mathfrak{M}_k$. There exists a unique ideal $\mathfrak{P}$ of $\widehat{A}_M$ satisfying
(3) $A \cap \mathfrak{P}=(0)$
and maximal with this property, such that the canonical embedding

$$A \longrightarrow \widehat{A}_{\mathfrak{M}}/\mathfrak{P} \tag{4}$$

induces on A the original topology τ. Moreover $\mathfrak{P} \subset \operatorname{rad}(\widehat{A}_{\mathfrak{M}})$. Conversely, if $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$ are maximal ideals of A satisfying (2) *and for $\mathfrak{M}=\cap_{k=1}^{n}\mathfrak{M}_k$ the ideal $\mathfrak{P}$ of $\widehat{A}_{\mathfrak{M}}$ satisfies* (3), *then the topology induced on A by the embedding* (4) *is* **Lpc**-*minimal. If in addition $\mathfrak{P} \subset \operatorname{rad}(\widehat{A}_{\mathfrak{M}})$ holds, then the ideals $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$ are open*

Proof. By the definition of **Lpc** the completion $\widehat{A}$ with respect to τ is s. l. c. By lemma 2.1 there exist a family $\{A_\alpha\}_{\alpha\in I}$ of s. l. c. local rings such that $\widehat{A}=\Pi_{\alpha\in I}A_\alpha$. If the maximal ideal of A_α is $\mathfrak{M}_\alpha$ $(\alpha\in I)$, then the topology of A_α is coarser than $\tau_{\mathfrak{M}_\alpha}$ by the same lemma. The canonical image of A_α in $\widehat{A}$ is a closed ideal, hence $A_\alpha \cap A \neq (0)$ for every $\alpha \in I$, since the embedding $A \subset \widehat{A}$ is essential by proposition 1.6. The ring A satisfies the ascending chains condition, that is why the family of non-zero ideals $\{A_\alpha \cap A\}_{\alpha\in I}$ is finite, hence I is finite. Let $\widehat{A}=\Pi_{k=1}^{n} A_k$, where A_k is a local s. l. c. ring with maximal ideal $\mathfrak{M}_k$ $(k=1, \ldots, n)$. Set

$$\overline{\mathfrak{M}}_k = A_1 \times \ldots \times A_{k-1} \times \mathfrak{M}_k \times A_{k+1} \times \ldots \times A_n \quad (k=1, 2, \ldots, n),$$

clearly $\overline{\mathfrak{M}}_k$ is an open maximal ideal of $\widehat{A}$. By lemma 2.1 the topology of $\widehat{A}$ is coarser than $\Pi_{k=1}^{n} \tau_{\mathfrak{M}_k} = \tau_{\overline{\mathfrak{M}}}$, where $\overline{\mathfrak{M}} = \cap_{k=1}^{n} \overline{\mathfrak{M}}_k$. Therefore $\tau \leq \tau_{\mathfrak{M}}$, where $\mathfrak{M} = A \cap \overline{\mathfrak{M}}$. It is a well known fact that $\tau_{\mathfrak{M}}$ is Hausdorff iff (2) holds. The identity $i:(A, \tau_{\mathfrak{M}}) \to (A, \tau)$ is continuous, hence we can extend it to a continuous homomorphism $\widehat{i} : \widehat{A}_{\mathfrak{M}} \to \widehat{A}$ of the corresponding completions. Now $\widehat{A}_{\mathfrak{M}}$ is s. l. c. [14] and $\widehat{i}(\widehat{A}_{\mathfrak{M}})$ is dense in $\widehat{A}$. Since $\widehat{i}(\widehat{A}_{\mathfrak{M}})$ is also s. l. c., it follows that $\widehat{i}(\widehat{A}_{\mathfrak{M}})$ is complete, hence closed and $\widehat{i}$ is an epimorphism. If $\mathfrak{P} = \ker \widehat{i}$, then $\widehat{A} \simeq \widehat{A}_{\mathfrak{M}}/\mathfrak{P}$ algebraically and $A \cap \mathfrak{P} = (0)$. Since the quotient topology of $\widehat{A}_{\mathfrak{M}}/\mathfrak{P}$ is s. l. c. (hence **L**-minimal), the isomorphism $\widehat{A} \cong \widehat{A}_{\mathfrak{M}}/\mathfrak{P}$ is topological too. Thus the embedding (4) induces on A the original topology τ. By proposition 1.6 the embedding (4) is essential, hence the ideal $\mathfrak{P}$ is maximal with the property $A \cap \mathfrak{P} = (0)$ ($\widehat{A}_{\mathfrak{M}}$ is a Zariski ring, hence every ideal of $\widehat{A}_{\mathfrak{M}}$ is closed). Suppose $\mathfrak{Q}$ is an ideal of $\widehat{A}_{\mathfrak{M}}$ with $A \cap \mathfrak{Q} = (0)$ and the embedding $A \to \widehat{A}_{\mathfrak{M}}/\mathfrak{Q}$ induces on A the original topology τ. Then $\widehat{A}_{\mathfrak{M}}/\mathfrak{Q}$ is a completion of A. Since the completion is unique, there exists an isomorphism $g : \widehat{A}_{\mathfrak{M}}/\mathfrak{P} \to \widehat{A}_{\mathfrak{M}}/\mathfrak{Q}$ such that $g/_A = id_A$. Hence $\mathfrak{P} = \mathfrak{Q}$ and the uniqueness of $\mathfrak{P}$ is established.

The maximal ideals $\widehat{A}_{\mathfrak{M}}$ are $\mathfrak{M}_k \widehat{A}_{\mathfrak{M}}$ $(\mathfrak{M}_k = A \cap \overline{\mathfrak{M}}_k)$. Since $\mathfrak{M}_k$ is open in (A, τ) we have $\mathfrak{P} \subset \mathfrak{M}_k \widehat{A}_{\mathfrak{M}}$ $(k=1, 2, \ldots, n)$, hence $\mathfrak{P} \subset \operatorname{rad}(\widehat{A}_{\mathfrak{M}})$. It remains to prove that the ideals $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$ are uniquely determined. In fact, they are the sole open maximal ideals of (A, τ). Assume $\mathfrak{M}'$ is an open maximal ideal of (A, τ), then $\mathfrak{M}'$ is also $\tau_{\mathfrak{M}}$-open, hence $\mathfrak{M} \subset \mathfrak{M}'$ and $\mathfrak{M}_k \subset \mathfrak{M}'$ for some $k=1, 2, \ldots, n$. The first part of the theorem is proved.

Let $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$ be maximal ideals of A satisfying (2) and for $\mathfrak{M} = \Pi_{k=1}^{n} \mathfrak{M}_k$ the ideal $\mathfrak{P}$ of $\widehat{A}_{\mathfrak{M}}$ satisfy (3). Then $\widehat{A}_{\mathfrak{M}}/\mathfrak{P}$ is a complete local Noetherian ring, hence s. l. c. Thus $\widehat{A}_{\mathfrak{M}}/\mathfrak{P}$ is **Lpc**-minimal. By (3) the embedding (4) is essential, hence by proposition 1.6 the induced on A topology is **Lpc**-minimal. The last assertion can be established as above. Q. E. D.

The existence of ideals with (2) and (3) is provided by the Zorn's lemma. Therefore on every Noetherian ring there exist **Lpc**-minimal topologies.

Corollary 2.3. *The minimal precompact topologies on a Noetherian ring can be described as in theorem 2.2., all maximal ideals being necessarily of finite index.*

If A is a Noetherian ring with irreducible primary decomposition (1) of the zero ideal, then there exist minimal precompact topologies on A, iff there exist maximal ideals $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$ of A of finite index satisfying (2).

Corollary 2.4. *The* **L**-*totally minimal topologies on a Noetherian ring A are described as in Theorem 2.2 the condition* (3) *replaced by*
(3*) $\mathfrak{P} \cap A = 0$ *and for every ideal* $\mathfrak{A}$ *of* $\widehat{A}_{\mathfrak{M}}$ *with* $\mathfrak{A} \supset \mathfrak{P}$ $\mathfrak{P} + (\mathfrak{A} \cap A)\widehat{A}_{\mathfrak{M}} = \mathfrak{A}$ *holds.*

Proof. First we prove that every **L**-totally minimal topology (on an arbitrary ring) is **Lpc**-minimal. Let $\mathfrak{L}$ be an open ideal of A. Suppose $\mathfrak{L}_1 \supset \mathfrak{L}_2 \supset \ldots \supset \mathfrak{L}_n \supset \ldots$ is a descending chain of ideals in $A/\bar{\mathfrak{L}}$. Set $\bar{\mathfrak{L}} = \bigcap_{n=1}^{\infty} \mathfrak{L}_n$, then $\mathfrak{L}_1/\bar{\mathfrak{L}} \supset \mathfrak{L}_2/\bar{\mathfrak{L}} \supset \ldots \supset \mathfrak{L}_n/\bar{\mathfrak{L}} \supset \ldots$ is a descending chain of ideals in $A/\bar{\mathfrak{L}}$ with zero intersection. Hence they generate an **L**-topology on $A/\bar{\mathfrak{L}}$ which is necessarily discrete, since $A/\bar{\mathfrak{L}}$ is **L**-minimal and discrete at the same time. Hence both chains are stable.

Now to **L**-totally minimal topologies on A we can apply theorem 2.2. The notations being as there it remains to prove that A is totally dense in $\widehat{A}_{\mathfrak{M}}/\mathfrak{P}$ iff (3*) holds, which is straight-forward. Q. E. D.

Remark. In the case A is a Noetherian integral domain in theorem 2.2. there exists a unique open open maximal ideal $\mathfrak{M}$. Indeed, by (3) the ideal $\mathfrak{P}$ is prime ($A \setminus \{0\}$ is multiplicatively closed), hence $\mathfrak{P} \subset \operatorname{rad} \widehat{A}_{\mathfrak{M}} = \mathfrak{M}_1 \widehat{A}_{\mathfrak{M}_1} \times \ldots \times \mathfrak{M}_n \widehat{A}_{\mathfrak{M}_n}$ implies $n=1$. Therefore theorem 3 from [5] as well as corollary 1 from [6] describing the minimal precompact topologies on Noetherian integral domains can be obtained from corollary 2.3. Moreover, theorem 2.2 describes the **L**-minimal topologies on Noetherian integral domains. It suffices to prove the following proposition (we have mentioned before that every **Lpc**-minimal topology is **L**-minimal).

Proposition 2.5 *Every* **L**-*minimal topology on a Noetherian integral domain is* **Lpc**-*minimal.*

Proof. Let A be a Noetherian integral domain and τ be an **L**-minimal topology on A. It is enough to show that $(A, \tau) \in$ **Lpc**, since for every linear topology σ on A $\sigma \leqq \tau$ implies $(A, \sigma) \in$ **Lpc**. The following definition is given in [1] and [12]. An ideal $\mathfrak{J}$ of A is called sheltered, if the intersection of all non-zero ideals of the quotient ring $A/\mathfrak{J}$ is a non-zero ideal. Denote by τ^* the linear topology on A having as a fundamental system of neighbourhoods of zero all τ-open sheltered ideals of A. Obviously $\tau^* \leqq \tau$ and τ^* is Hausdorff [17]. By the **L**-minimality of τ we have $\tau = \tau^*$. Now take an open ideal $\mathfrak{J}$ of (A, τ). From [17] it follows that there exists an (open) sheltered ideal $\mathfrak{L}$ contained in $\mathfrak{J}$. The ideal $\mathfrak{L}$ is irreducible, hence $\mathfrak{L}$ is primary. It is easy to prove, that the radical of $\mathfrak{L}$ is a maximal ideal (see for instance [12]). Thus the quotient ring $A/\mathfrak{L}$ satisfies the descending chains condition, still more $A/\mathfrak{J}$ does. This proves $(A, \tau) \in$ **Lpc**. Q. E. D.

Before passing to integral domains recall a definition [10]. A local Noetherian integral domain $\mathfrak{A}$ with a maximal ideal $\mathfrak{M}$ is called analytically irreducible, if the completion $\widehat{A}_{\mathfrak{M}}$ has no zero divisors. We introduce the following notion.

Definition 2.6. *Let* (A, τ) *be a non-discrete topological ring in* **L**. *We call the ring* A *analytically irreducible and the topology* τ*-analytically irreducible, if the completion* $\widehat{A}$ *has no zero divisors.*

By lemma 1.8 every commutative **L**-minimal ring is analytically irreducible. From now on we write a. i. instead of analytically irreducible.

The following lemma enables us to study the a. i. topologies on integral domains by means of localization with respect to maximal ideals $\mathfrak{M}$ such that the $\mathfrak{M}$-adic topology is Hausdorff.

Lemma 2.7. *Let* A *be an integral domain and* τ *be an a. i.* **Lpc**-*topology on* A. *Then there exists a unique open maximal ideal* $\mathfrak{M}$ *of* A *with* $\tau \leqq \tau_{\mathfrak{M}}$. *There exists a unique a. i.* **Lpc**-*topology* τ' *on the ring of quotients* $A_{\mathfrak{M}}$ *which induces on* A *the original topology* τ. *Moreover,* τ *is* **Lpc**-*minimal iff* τ' *is.*

Proof. The completion $\widehat{A}$ is s. l. c. and has no zero divisors. By lemma 2. 1 $\widehat{A}$ is local. Let $\overline{\mathfrak{M}}$ be the maximal ideal of $\widehat{A}$, by lemma 2.1 the topology of $\widehat{A}$ is coarser than $\tau_{\overline{\mathfrak{M}}}$. Since A is dense in $\widehat{A}$, the ideal $\mathfrak{M}=A \cap \overline{\mathfrak{M}}$ is open and maximal in A. Obviously the topology induced by $\tau_{\overline{\mathfrak{M}}}$ on A is coarser than $\tau_{\mathfrak{M}}$, hence $\tau \leqq \tau_{\mathfrak{M}}$. The uniqueness of $\mathfrak{M}$ follows as in theorem 2.2. .

As in theorem 2.2 we have a continuous epimorphism $\widehat{i}\colon \widehat{A}_{\mathfrak{M}} \to \widehat{A}$ with $\widehat{i}/_A = \mathrm{id}_A$. We may assume $A \subset A_{\mathfrak{M}} \subset \widehat{A}_{\mathfrak{M}}$ since $\tau_{\mathfrak{M}}$ is Hausdorff by $\tau \leqq \tau_{\mathfrak{M}}$. By $A \cap \ker \widehat{i} = 0$ it follows $A_{\mathfrak{M}} \cap \ker \widehat{i} = 0$, hence $A_{\mathfrak{M}}$ is embedded into $\widehat{A}$ as a dense subring. Denote by τ' the induced topology on $A_{\mathfrak{M}}$ by this embedding. Since $\widehat{A}$ is s. l. c. integral domain (by lemma 1.8) τ' is a. i. and **Lpc**. The uniqueness of τ' follows by the uniqueness of the completion.

Now suppose σ is an arbitrary a. i. **Lpc**-topology of $A_{\mathfrak{M}}$, we show that the induced topology τ on A is a. i. and **Lpc**. Obviously τ is a. i. By the first part of the proof $\sigma \leqq \tau_{\mathfrak{M}'}$, where $\mathfrak{M}'$ is the maximal ideal of the local ring $A_{\mathfrak{M}}$. It is easy to prove that A is dense in $(A_{\mathfrak{M}}, \tau_{\mathfrak{M}'})$, hence A is dense in $(A_{\mathfrak{M}}, \sigma)$, thus $(A, \tau) \in$ **Lpc**.

Assume τ is **Lpc**-minimal, then the embedding $A \subset \widehat{A}$ is essential, still more the embedding $A_{\mathfrak{M}} \subset \widehat{A}$ is essential. Therefore τ' is **Lpc**-minimal, too. Conversely, suppose $(A_{\mathfrak{M}}, \tau')$ is **Lpc**-minimal. Then the embedding $A_{\mathfrak{M}} \subset \widehat{A}$ is essential. Since the embedding $A \subset A_{\mathfrak{M}}$ is essential even in the case $A_{\mathfrak{M}}$ is provided with the discrete topology, the embedding $A \subset \widehat{A}$ is essential, too. Hence τ is **Lpc**-minimal. Q. E. D.

Theorem 2.8. *Let* A *be a local integral domain with finitely generated maximal ideal* $\mathfrak{M}$ *and* τ_M *be Hausdorff. Then there is a decreasing bijection* σ *between the set of all a. i.* **Lpc**-*topologies on* A *and the set* W *of all prime ideals* $\mathfrak{P}$ *of the* $\mathfrak{M}$-*adic completion* $\widehat{A}_{\mathfrak{M}}$ *satisfying*

(5) $$A \cap \mathfrak{P} = 0,$$

such that for every $\mathfrak{P} \in W$ *the embedding* (4) *induces on* A *topology* $\sigma(\mathfrak{P})$. *The topology* $\sigma(\mathfrak{P})$ *is* **Lpc**-*minimal iff* $\mathfrak{P}$ *is maximal with the property* (5).

Proof The completion $\widehat{A}_{\mathfrak{M}}$ is a complete local Noetherian ring [10], hence s. l. c. Clearly for every $\mathfrak{P} \in W$ the topology $\sigma(\mathfrak{P})$ induced on A by the embedding (4) is a. i. and **Lpc**. On the other hand for every a. i. **Lpc**-topology τ on A we have $\tau \leq \tau_{\mathfrak{M}}$ by the previous lemma. Hence there exists an open and continuous epimorphism $\widehat{i}: \widehat{A}_{\mathfrak{M}} \to \widehat{A}$ such that $\widehat{i}|_A = \mathrm{id}_A$, where $\widehat{A}$ is the completion of (A, τ). Now for $\mathfrak{P} = \ker \widehat{i}$ clearly $\mathfrak{P} \in W$ and $\tau = \sigma(\mathfrak{P})$. The uniqueness of $\mathfrak{P}$ can be established as in theorem 2.2. Obviously for $\mathfrak{P}, \mathfrak{Q} \in W$ with $\mathfrak{P} \supseteq \mathfrak{Q}$ we have $\sigma(\mathfrak{P}) \leq \sigma(\mathfrak{Q})$. It remains to prove the last assertion. By proposition 1.6 for $\mathfrak{P} \in W$ the topology $\sigma(\mathfrak{P})$ is **Lpc**-minimal iff the embedding (4) is essential which is equivalent to the maximality of $\mathfrak{P}$ with respect to (5). Q. E. D.

Remarks 1. If in theorem 2.8 $\tau_{\mathfrak{M}}$ is a. i., then $\tau_{\mathfrak{M}}$ is the maxmal a. i. **Lpc**-topology on A. In the general case there are only a finite number of maximal a. i. **Lpc**-topologies on A, they correspond to the minimal elements of W.

2. By Zorn's lemma every a. i. **Lpc**-topology on A majorizes some **Lpc**-minimal topology on A.

3. It was proved in corollary 2.4 that every **L**-totally minimal topology is **Lpc**-minimal. Hence the **L**-totally minimal topologies on integral domains can be dealt with lemma 2.7 and theorem 2.8. By lemma 2.7 the **L**-totally minimal topologies on integral domains can be studied locally, by theorem 2.8 they can be obtained by embeddings (4), only (3) must be replaced by (3*).

If in theorem 2.8 the quotient ring $A/\mathfrak{M}$ is finite, then $\tau_{\mathfrak{M}}$ is precompact and all a. i. **Lpc**-topologies are precompact. That's why all **Lpc**-minimal topologies in this case are minimal. In this way we obtain theorem 1 from [6], which characterizes the minimal precompact topologies on integral domains.

Corollary 2.9. *Let A be an integral domain and $\mathfrak{M}$ be a finitely generated maximal ideal of A with* height $\mathfrak{M} = 1$ *and* $\tau_{\mathfrak{M}}$ *be Hausdorff. Then* $\tau_{\mathfrak{M}}$ *is* **Lpc**-*minimal iff* $\tau_{\mathfrak{M}}$ *is a. i.*.

Proof. Suppose $\tau_{\mathfrak{M}}$ is a. i., then $\widehat{A}_{\mathfrak{M}}$ is complete local Noetherian integral domain with $\dim \widehat{A}_{\mathfrak{M}} = 1$. Hence every non-zero ideal of $\widehat{A}_{\mathfrak{M}}$ is open and intersects A in a non-trivial way. The minimality of $\tau_{\mathfrak{M}}$ follows from theorem 2.8. The necessity follows from lemma 1.8. Q. E. D.

This corollary is a correction to corollary 2 from [6], where a. i. was not suggested. Corollary 3 obtained there is wrong too. It can be replaced by the following statement: on a one-dimensional Noetherian integral domain A having a. i. rings of quotients $A_{\mathfrak{M}}$ with respect to all maximal ideals $\mathfrak{M}$, the **Lpc**-minimal topologies are exactly the $\mathfrak{M}$-adic. On the other hand, the **Lpc**-minimal topologies on almost Dedekind integral domains are exactly the $\mathfrak{M}$-adic, where $\mathfrak{M}$ is a maximal ideal.

In what follows the relationship between Krull dimension and minimality is examined.

Proposition 2.10. *Let A be a s. l. c. integral domain with* $\dim A=1$. *Then every dense subring of A is* **Lpc**-*minimal.*

Proof. By proposition 1.6 it is enough to show that every dense subring of A is essentially embedded. Suppose B is a dense subring of A. By the Zorn's lemma there exists an ideal $\mathfrak{L}$ of A with $B\cap\mathfrak{L}=0$ and maximal with this property. Since $B\setminus\{0\}$ is multiplicatively closed, $\mathfrak{L}$ is a prime ideal. By lemma 2.1 A is local hence from $\dim A=1$ it follows $\mathfrak{L}=0$ (the maximal ideal of A is open, hence intersects A in a non-trivial way). This proves the essentiality of the embedding $B\subset A$. Q. E. D.

The following proposition is a partial converse to the previous one.

Proposition 2.11. *Let A be a complete local Noetherian integral domain and A_1 be a dense subring of A with* $\operatorname{card} A_1<\mathfrak{c}$. *If the induced topology on A_1 is* **Lpc**-*minimal, then* $\dim A=1$.

Proof. Suppose $\dim A>1$, then we shall establish that the embedding $A_1\subset A$ is not essential which contradicts the **Lpc**-minimality of A_1 by virtue of proposition 1.6.

According to Cohen's structure theory of complete local Noetherian integral domains A contains a complete regular ring S such that:

i) A is finite S-module;

ii) S is a power series ring $S=I[[x_1, x_2, \dots, x_n]]$, where I is a coefficient ring of A and $x_1, x_2, \dots, x_n$ are analytically independent over I [10].

The ring I is a complete local Noetherian integral domain with maximal ideal pI, where p is the characteristic of the residue field of A. Now i) implies $\dim S>1$, whence $n\geqq 1$. If $n=1$, then

$$\dim I+1=\dim S>1,$$

hence $\dim I=1$ and I is not a field. By the completeness of I $\operatorname{card} I\geqq\mathfrak{c}$. Set $S_1=I$ if $n=1$ and $S_1=I[[x_1, \dots, x_{n-1}]]$ if $n>1$, then $\operatorname{card} S_1\geqq\mathfrak{c}$. Let $\mathfrak{N}$ be the maximal ideal of S_1, then $\operatorname{card}\mathfrak{N}\geqq\mathfrak{c}$. For every $\nu\in\mathfrak{N}$ denote by $\mathfrak{P}_\nu$ the principal ideal of $S=S_1[[x_n]]$ generated by $x_n-\nu$. Since S is a unique factorization domain, $\mathfrak{P}_\nu$ is a minimal prime ideal and for every ν, $\mu\in\mathfrak{N}$, $\nu\neq\mu$ we have $\mathfrak{P}_\nu\neq\mathfrak{P}_\mu$. By virtue of the lying over theorem of Cohen-Seidenberg we can find a family $\{\mathfrak{Q}_\nu\}_{\nu\in N}$ of distinct minimal prime ideals of A. Every non-zero element of A is contained in at most a finite number ideals $\mathfrak{Q}_\nu$ by the primary decomposition theorem. Now $\operatorname{card}\mathfrak{N}>\operatorname{card} A_1$ provides an ideal $\mathfrak{Q}_\nu$ with $A_1\cap\mathfrak{Q}_\nu=0$. Since A is a Zariski ring, $\mathfrak{Q}_\nu$ is closed, therefore the embedding $A_1\subset A$ is not essential. Q. E. D.

It follows from the preceding proposition that the $(x_1, \dots, x_n)$-adic topology on the ring of polynomials $k[x_1, \dots, x_n]$ over a finite field is not minimal, if $n>1$.

The following theorem is a partial generalization of theorem A (see the introduction).

Theorem 2.12. *Let A be a complete local Noetherian integral domain with maximal ideal $\mathfrak{M}$ and* $\operatorname{card}(A/\mathfrak{M})<\mathfrak{c}$. *Then for every natural number n the following conditions are equivalent:*

a) $\dim A\leqq n$;

b) *for every dense subring A_1 of A and for every decreasing chain of a. l.* **Lpc**-*topologies $\tau_1>\tau_2>\dots>\tau_m$ on A_1, where τ_1 is the induced on A_1 topology, $m\leqq n$ holds;*

c) *there exists a subring A_1 of A with the above properties and* card $A_1 < \mathfrak{c}$.

Proof a) $\Rightarrow$ b). Let A_1 be a dense subring of A and $\tau_1, \ldots, \tau_m$ be as in b). Denote by $\widehat{A}_k$ the completion of (A_1, τ_k) $(k=1, 2, \ldots, m)$, in particular $\widehat{A}_1 = A$. For $k=1, 2, \ldots, m$ the identity $(A_1, \tau_1) \xrightarrow{i_k} (A_1, \tau_k)$ is continuous, after passing to the corresponding completions we have an open and continuous epimorphism $\widehat{i}_k : A \rightarrow \widehat{A}_k$. Now $\mathfrak{P}_k = \ker \widehat{i}_k$ is a prime ideal of A with $A_1 \cap \mathfrak{P}_k = 0$ and

$$0 = \mathfrak{P}_1 \subset \mathfrak{P}_2 \subset \ldots \subset \mathfrak{P}_m. \tag{6}$$

By the density of A_1 in A $\mathfrak{M} \cap A \neq 0$ holds, hence $P_m \neq \mathfrak{M}$ in (6). Thus $m \leqq \dim A \leqq n$. Observe that this implication was proved without the restriction card $(A/\mathfrak{M}) < \mathfrak{c}$.

b) $\Rightarrow$ c). First we clarify the topological meaning of card$(A/\mathfrak{M}) < \mathfrak{c}$. Denote for brevity card$(A/\mathfrak{M}) = \tau$. Let $d(A)$ be the least cardinal number such, that there exists a dense subset of A with cardinality $d(A)$. We shall prove that $d(A) = \max\{\aleph_0, \tau\}$ and this is true without the restriction $\tau < \mathfrak{c}$.

Suppose τ is finite, then A is compact and metrizable, therefore $d(A) = \aleph_0$. It remains to consider the case when τ is infinite. Then we have to prove that $d(A) = \tau$.

By the Cohen's structure theory there is a subring S of A with the properties i) and ii) from the proof of proposition 2.11. In the case of equal characteristics the coefficient ring I is a field isomorphic to $A/\mathfrak{M}$. By i) there exist $a_1, a_2, \ldots, a_s \in A$ such that $A = a_1 S + \ldots + a_s S$. Then the polynomial ring $I[x_1, \ldots, x_n]$ is dense in S, hence the ring $B = I[x_1, \ldots, x_n, a_1, \ldots, a_s]$ is dense in A and card $B \leqq \tau$. Thus $d(A) \leqq \tau$. In the case char $A \neq$ char $A/\mathfrak{M}$ the coefficient ring I is a complete discrete valuation ring. Let πI be the maximal ideal of I, then $I/\pi I$ is isomorphic to $A/\mathfrak{M}$. Choose a full system L of representatives of $I/\pi I$ in I. Then card $L = \tau$ and the elements of I can be (uniquely) expressed as convergent series $\Sigma_{\nu=0}^{\infty} l_\nu \pi^\nu$, where $l_\nu \in L$ $(\nu = 0, 1, \ldots)$. Hence the set H of all polynomials of π with coefficients from L is dense in I and card $H = \tau$. Now the set H_1 of polynomials of $x_1, \ldots, x_n, a_1, \ldots, a_s$ with coefficients from L is dense in A and card $H_1 = \tau$. Thus $d(A) \leqq \tau$ is established in this case too.

On the other hand, $\{l + \mathfrak{M}\}_{l \in L}$ is a family of disjoint open sets in A with cardinality τ. Hence, $d(A) \geqq \tau$ and the equality $d(A) = \tau$ is established.

To finish the proof of the second implication let's remark that $d(A) < \mathfrak{c}$ implies also the existence of a dense subring of A with cardinality less than $\mathfrak{c}$. Now we apply directly b).

c) $\Rightarrow$ a). Let A_1 be a dense subring of A with the property mentioned in b) and card $A_1 < \mathfrak{c}$ and assume $\dim A = d$. By the Zorn's lemma there exists an ideal $\mathfrak{P}$ of A maximal with the property $A_1 \cap \mathfrak{P} = 0$. Then the embedding $A_1 \rightarrow A/\mathfrak{P}$ is essential, hence the topology induced on A_1 by this embedding is **Lpc**-minimal. By proposition 2.11 depth $\mathfrak{P} = 1$, hence there exists a chain of prime ideals of A

$$0 = \mathfrak{P}_1 \subset \mathfrak{P}_2 \subset \ldots \subset \mathfrak{P}_d = \mathfrak{P}.$$

Obviously $P_j \cap A_1 = 0$ $(j=1, 2, \dots)$, denote by τ_j the topology induced on A_1 by the embedding $A_1 \to A/\mathfrak{P}_j$ $(j=1, 2, \dots, d)$, in particular τ_1 is the induced topology on A_1 by the embedding $A_1 \subset A$. Clearly τ_j is a. i. **Lpc**-topology on A_1 and $\tau_1 > \tau_2 > \dots > \tau_d$. By c) $d \leqq n$. Therefore $\dim A \geqq n$. Q. E. D.

Remark. Suppose in the notations of theorem 2.12 $\dim A = n$, then in every chain $\tau_1 > \dots > \tau_n$ of a. i. **Lpc**-topologies the last one $-\tau_n-$ is **Lpc**-minimal. In the case $n=1$ we have the following analogue of theorem A:

Let A be a complete local Noetherian integral domain with maximal ideal $\mathfrak{M}$ and $\operatorname{card} A/\mathfrak{M} < \mathfrak{c}$. Then $\dim A = 1$ iff every dense subring of A is **Lpc**-minimal.

In the non-compact case there exist non-minimal subrings of A, even if $\dim A = 1$ (see the example after definition 1.3.).

Now we prove the necessity in theorem A.

Proposition 2.13. *Let A be a compact Noetherian integral domain with* $\dim A = 1$. *Then every subring of A is totally minimal in the induced topology.*

Proof. Let B be a subring of A, then the completion $\widehat{B}$ is compact, hence to prove the total minimality of B it suffices to show that B is totally dense in $\widehat{B}$. In what follows we prove that $\widehat{B}$ is a Noetherain integral domain with $\dim \widehat{B} = 1$. Hence by the primary decomposition theorem every non-zero ideal of $\widehat{B}$ is open and this implies the total density of B in $\widehat{B}$. Denote by D the ring of integers in the case $\operatorname{char} \widehat{B} = 0$, otherwise if k is the prime subfield of $\widehat{B}$ there exists an element t of $\widehat{B}$ which is transcendental over k, in this case denote by D the ring of polynomials $k[t]$. In both cases D is a Dedekind subring of $\widehat{B}$. Let $\mathfrak{M}$ be the maximal ideal of A, then $\mathfrak{N} = D \cap \mathfrak{M}$ is a maximal ideal of D. Obviously the induced topology on D is coarser than $\tau_{\mathfrak{N}}$, but the latter is minimal [5], hence they coincide. Now the completion $\widehat{D}_{\mathfrak{N}}$ is a complete discrete valuation ring with maximal ideal $\bar{\mathfrak{N}} = \widehat{D}_{\mathfrak{N}} \cap \mathfrak{M}$. The ideal $\bar{\mathfrak{N}}A$ is non-zero, hence it is open in A and the guotient ring $A/\bar{\mathfrak{N}}A$ is finite. By (30.6) from [10] A is a finitely generated $\widehat{D}_{\mathfrak{N}}$-module. From $\widehat{D}_{\mathfrak{N}} \subset \widehat{B}$ it follows that $\widehat{B}$ is finitely generated $\widehat{D}_{\mathfrak{N}}$- module, hence a Noetherian ring. On the other hand, A is a finitely generated $\widehat{B}$-module, hence $\dim \widehat{B} = \dim A = 1$. Q. E. D.

Corollary 2.14. *Let A be a minimal precompact Noetherian integral domain with* $\operatorname{card} A < \mathfrak{c}$. *Then every subring of A is totally minimal in the induced topology.*

Proof. Let $\widehat{A}$ be the completion of A. By theorem 2.8 $\widehat{A}$ is a compact Noetherian integral domain. From proposition 2.11 $\dim \widehat{A} = 1$. Now apply proposition 2.13. Q. E. D.

The last theorem characterizes the dimension of Noetherian integral domains with cardinality less than $\mathfrak{c}$.

Theorem 2.15. *Let A be a Noetherian integral domain with* $\operatorname{card} A < \mathfrak{c}$. *Then for every natural number n the following three conditions are equivalent:*

a) $\dim A \geqq n$;

b) *for every decreasing chain of a. i.* **Lpc**-*topologies*

(7) $$\tau_1 > \tau_2 > \ldots > \tau_m$$
on A $m \leq n$ *holds*;

c) *For every* **Lpc**-*minimal topology* τ *on* A *and for every decreasing chain of a. i.* **Lpc**-*topologies* (7) *on* A *with* $\tau_m = \tau$ $m \leq n$ *holds.*

Proof. a) $\Rightarrow$ b). Let (7) be a decreasing chain of a. i. **Lpc**-topologies on A. By lemma 2.7 there exists a maximal ideal M of A with $\tau_{\mathfrak{M}} \geqq \tau_1$. Applying lemma 2.7 and theorem 2.8 we obtain an increasing chain

$$\mathfrak{P}_1 \subset \mathfrak{P}_2 \subset \ldots \subset \mathfrak{P}_m$$

of prime ideals of the completion $\widehat{A}_{\mathfrak{M}}$ with $\mathfrak{P}_j \cap A = 0$ $(j = 1, 2, \ldots, m)$. Obviously they are not maximal. From a) we have $\dim \widehat{A}_{\mathfrak{M}} = \text{height } \mathfrak{M} \leq n$, hence $m \leq n$.

b) $\Rightarrow$ c). Trivial.

c) $\Rightarrow$ a). Let M be a maximal ideal of A, we have to prove that $h = \text{height } \mathfrak{M} = \dim \widehat{A}_{\mathfrak{M}} \leq n$.

Choose a prime ideal $\mathfrak{P}$ of $\widehat{A}_{\mathfrak{M}}$ with depth $\mathfrak{P} = h$. Since A has no zero divisors $A \cap \mathfrak{P} = 0$, hence A is embedded in $B = \widehat{A}_{\mathfrak{M}}/\mathfrak{P}$ and the induced topology σ on A by this embedding is a. i. and **Lpc**. Now we can apply to the complete local Noetherian integral domain B theorem 2.12. Q. E. D.

In the case $n = 1$ we have the following corollary.

Corollary 2.16. *Let* A *be a Noetherian integral domain with* card $A < \mathfrak{c}$. *Then* $\dim A = 1$ *iff every a. i.* **Lpc**-*topology on* A *is* **Lpc**-*minimal.*

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