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RELATION BETWEEN WEYL'S AND BOCHN R'S CURVATURE TENSORS

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In the paper we study the following three linear mappings: a) $C: R \to C(R)$, where R in an LC-tensor over an n-dimensional real Euclidean vector space (V, g) and $C(R)$ is the Weyl's conformal curvature tensor; b) $B: R \to B(R)$, where R is a K-tensor over a $2n$-dimensional Hermitian vector space with complex structure (V, g, J) and $B(R)$ is the Bochner curvature tensor; c) $*: R \to R^*$, where R is an LC-tensor over (V, g, J) and R^* is the associated to R K-tensor defined by (3). The main results are: 1) C commutes with the product $B \circ *$; 2) If an almost Hermitian manifold (M, g, J) is conformally flat then it is a Bochner flat manifold as well, that is if $C(R)=0$, then $B(R^*)=0$.

Let (V, g) be an n-dimensional real Euclidean vector space. A tensor R over (V, g) of type (1,3) with

1. $R(x, y, z) = -R(y, x, z)$,
2. $R(x, y, z) + R(y, z, x) + R(z, x, y) = 0$,
3. $R(x, y, z, u) = -R(x, y, u, z)$,

where $R(x, y, z, u) = g(R(x, y, z), u)$, is called an LC (or Levi Civita) — tensor For every such tensor R by the decomposition theorem of Singer-Thorpe [1] and Nomizu [2] Weyl's conformal curvature tensor $C(R)$ is defined:

$$C(R) = R - \frac{1}{n-2} S \wedge g + \frac{2S(V)}{(n-1)(n-2)} g \wedge g, \tag{1}$$

where S is the Ricci tensor of R, $S(V)$ is the scalar curvature of V with respect to R and by definition

$$(S \wedge g)(x, y, z, u) = g(x, u)S(y, z) - g(x, z)S(y, u) + g(y, z)S(x, u) - g(y, u)S(x, z).$$

We consider the mapping $C: R \to C(R)$ defined by (1). It is easy to prove that C is a linear mapping and [3]: $C^2 = C$, $C(S \wedge g) = 0$.

Let (V, g, J) be a Hermitian vector space of real dimension $2n$ with complex structure J. An LC-tensor R over (V, g, J) with

4. $R(x, y, z, u) = R(x, y, Jz, Ju)$

is called a K (or Kähler) -tensor. For every such tensor R by the decomposition theorem of Mori [4] and Sitaramaya [5] Bochner's curvature tensor $B(R)$ is defined;

$$B(R) = R - \frac{1}{2n+4} S \underset{c}{\wedge} g + \frac{S(V)}{2n+2)(2n+4)} g \underset{c}{\wedge} g, \tag{2}$$

where by definition

$$(S \underset{c}{\wedge} g)(x, y, z, u) = g(x, u)S(y, z) - g(x, z)S(y, u) + g(y, z)S(x, u) - g(y, u)S(x, z)$$
$$+ g(x, Ju)S(y, Jz) - g(x, Jz)S(y, Ju) + g(y, Jz)S(x, Ju) - g(y, Ju)S(x, Jz)$$
$$-2g(x, Jy)S(z, Ju) - 2g(z, Ju)S(x, Jy).$$

We consider the mapping $B: R \to B(R)$, defined by (2) which is linear and [3]: $B^2 = B$, $B(S \underset{c}{\wedge} g) = 0$.

If R is an LC-tensor over (V, g, J) then the tensor R^*, defined by

(3) $$R^*(x, y, z, u)$$
$$= \frac{3}{16}[R(x, y, z, u) + R(x, y, Jz, Ju) + R(Jx, Jy, z, u) + R(Jx, Jy, Jz, Ju)]$$
$$+ \frac{1}{16}[R(Jx, Jz, y, z) - R(Jy, Jz, x, u) + R(x, z, Jy, Ju) - R(y, z, Jx, Ju)$$
$$+ R(y, Jz, Jx, u) - R(x, Jz, Jy, u) + R(Jy, z, x, Ju) - R(Jx, z, y, Ju)]$$

is a K-tensor [6; 3; 7]. R is a K-tensor iff $R^* = R$. It is very important that R^* is a single K-tensor defined by R with $R^*(x, Jx, Jx, x) = R(x, Jx, Jx, x)$.

Now we consider the mapping $*: R \to R^*$, defined by (3) which is linear and $*^2 = *$.

If S is a symmetric tensor of type (0,2) over (V, g, J) then [3]: $4(S \vee g)^* = S' \underset{c}{\wedge} g$, where $2S'(x, y) = S(x, y) + S(Jx, Jy)$.

In [3] we have proved the following theorem:

(4) $$B \circ * \circ C = B \circ *.$$

By (1) for the tensor R^* we have

(5) $$C(R^*) = R^* - \frac{1}{2n-2} S^* \wedge g + \frac{2S^*(V)}{(2n-1)(2n-2)} g \wedge g$$

and

$$(C(R))^* = R^* - \frac{1}{8(n-2)} S' \underset{c}{\wedge} g + \frac{2S(V)}{8(n-1)(2n-1)} g \underset{c}{\wedge} g.$$

The last equality is a new decomposition for the K-tensor R^* which decomposition is different from (5) even in the Kählerian Geometry.

Now we shall give some new applications.

1. For a proper value λ of the linear mapping C we have $C(R) = \lambda R$, which gives $\lambda(\lambda - 1)R = 0$. In this way the existence of the following remarkable LC-tensors over (V, g) is estabilished:

a) $R = 0$; b) $R \neq 0$, $C(R) = 0$; c) $R = 0$, $C(R) = R$.

We note that in the case of a Riemannian manifold (M, g):

a) means that (M, g) is flat;

b) means that (M, g) is conformally flat but not flat;

c) means that (M, g) is an Einsteinan manifold but not flat.

2. For a proper value λ of the linear mapping B we have $B(R^*) = \lambda R^*$, which gives $\lambda(\lambda - 1)R^* = 0$.

Thus we establish the existence of the following remarkable K-tensors R over (V, g, J):

a_1) $R^*=0$; b_1) $R^*\neq 0$, $B(R^*)=0$; c_1) $R^*\neq 0$, $B(R^*)=R^*$.

We note that in the case of an almost Hermitian manifold (M, g, J):

a_1) means that (M, g, J) is holomorphically flat, that is $H(x)=R(x, Jx, Jx, x)=0$ for every $x\in M_p$ at every point $p\in M$. (3) implies that this is namely the case, when

$$3[R(x, y, z, u)+R(x, y, Jz, Ju)+R(Jx, Jy, z, u)+R(Jx, Jy, Jz, Ju)]$$
$$+R(Jx, Jz, y, u)-R(Jy, Jz, x, u)+R(x, z, Jy, Ju)-R(y, z, Jx, Ju)$$
$$+R(y, Jz, Jx, u)-R(x, Jz, Jy, u)+R(Jy, z, x, Ju)-R(Jx, z, y, Ju)=0;$$

b_1) means that (M, g, J) is Bochner-flat but not holomorphically flat;

c_1) means that (M, g, J) is not holomorphically flat but it is a $*$-Einsteinian manifold that is the Ricci tensor S^* of R^* is proportional to the metrical tensor g at every point $p\in M$ [3].

3. For a proper value λ of the linear mapping $*$ we have $R^*=\lambda R$, which gives $\lambda(\lambda-1)R=0$.

Thus we establish the existence of the following remarkable LC-tensors R over (V, g, J):

a_2) $R=0$; b_2) $R\neq 0$, $R^*=0$; c_2) $R\neq 0$, $R^*=R$.

We note now that in the case of an almost Hermitian manifold (M, g, J)

a_2) means that (M, g, J) is flat; b_2) means that (M, g, J) is holomorphically flat but not flat; c_2) means that (M, g, J) is a parakählerian manifold.

If we apply (2) for R^* we obtain:

$$B(R^*)=R^*-\frac{1}{2n+4}S^*\underset{c}{\wedge}g+\frac{S^*(V)}{(2n+2)(2n+4)}g\underset{c}{\wedge}g. \tag{6}$$

Then

$$C\circ B(R^*)=C(R^*)-\frac{1}{2n+4}C(S^*\underset{c}{\wedge}g)+\frac{S^*(V)}{(2n+2)(2n+4)}C(g\underset{c}{\wedge}g).$$

If we apply now (1) to the tensors R^*, $S^*\underset{c}{\wedge}g$, $g\underset{c}{\wedge}g$ and take into consideration (6) we obtain:

Theorem 1. *The linear mappings C, B, $*$ are connected by the relation*

$$C\circ B\circ * = B\circ *. \tag{7}$$

If we combine (7) with (4) we obtain:

Theorem 2. *The mapping C commutes with the product $B\circ *$ that is $C\circ B\circ * = B\circ *\circ C$.*

From (4) we have $B(C(R))^*=B(R^*)$. If $C(R)=0$ then $B(R^*)=0$. Thus we proved the following statement:

Theorem 3. *If an almost Hermitian manifold (M, g, J) is conformally flat then it is a Bochner flat manifold as well.*

In the case of the Kählerian Geometry we have: $C(R)=0\to B(R)=0$. This result gives us information about the geometrical meaning of the Bochner curvature tensor.

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