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SEQUENTIAL ESTIMATION FOR COMPOUND POISSON PROCESS I

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The problem of efficient (in the sense of a quadratic function of loss under Rao—Cramer regular conditions) sequential estimation is considered for the compound Poisson process. All efficient sequential plans for the whole domain of the parameters are determined.

1. Measures generated by stopping times and some sufficient statistics. Let Ω be the space of real functions $\omega(t)$, $t \in [0, +\infty)$, which are continuous from the right and let $\mathcal{F}$ be the least σ-algebra of subsets of Ω with respect to which the functions $\omega(t)$ are measurable. Let $\mathcal{F}_t$ denote the least σ-algebra of subsets of Ω with respect to which the functions $\omega(s)$ are measurable for $s \in [0, t]$. P_θ is a probability measure on $(\Omega, \mathcal{F})$ depending on the parameter $\theta \in \Theta$, where Θ is a subset of R^n.

Definition 1.1. *A Markov stopping time is a random variable* $\tau : \Omega \to [0, +\infty]$ *which satisfies the following condition:* $\forall_t \ \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$.

We shall denote by $\mathsf{P}_{\theta,t}$ the measure P_θ restricted to the σ-algebra $\mathcal{F}_t$ and shall assume that the measure $\mathsf{P}_{\theta,t}$ is absolutely continuous with respect to a measure $\mathsf{P}_{\theta_0,t}$ and the respective density is given by: $d\,\mathsf{P}_{\theta,t}/d\mathsf{P}_{\theta_0,t} = g(t, S_t(\omega), \theta, \theta_0)$, where $S_t : (\Omega, \mathcal{F}_t) \to (Y, \mathcal{B}_Y)$ is a random element for each t, Y is a locally compact metric separable space and $\mathcal{B}_Y$ denotes the Borel field of subsets of Y. We shall suppose that $d(S_{t_n}, S_t) \xrightarrow[t_n \searrow t]{} 0$ almost surely with respect to P_θ for each $\theta \in \Theta$, where d is the metric of Y. Moreover we shall suppose that $g(\circ, \circ, \theta, \theta_0)$ is a continuous function. It follows from Fisher-Neyman's factorisation theorem [1] that S_t is a sufficient statistics on the probability space $(\Omega, \mathcal{F}_t, \mathsf{P}_{\theta,t})$.

Lemma 1. *If* $\mathsf{P}_\theta\{\tau < +\infty\} = 1$, *then* S_τ *is measurable with respect to the* σ-*algebra* $\mathcal{F}$.

Proof. Let $\tau_n(\cdot) := -[-2^n\tau(\cdot)]/2^n$, where $[a]$ means the integer part of the number a. Of course $\tau_n \searrow \tau$ almost surely (P_θ). But S_{τ_n} is measurable with respect to $\mathcal{F}$, because $S^{-1}_{\tau_n}(A) = \cup_k(\{\tau_n = t^n_k\} \cap S^{-1}_{t^n_k}(A)) \in \mathcal{F}$, $A \in \mathcal{B}_Y$, where $\{t^n_k = k/2^n : k = 1, 2, 3, \dots\}$ means the set of values of τ_n. S_τ is also measurable with respect to $\mathcal{F}$ because $d(S_{\tau_n}, S_\tau) \xrightarrow[n\to\infty]{} 0$ almost surely (P_θ).

Let $U := [0, +\infty) \times Y$ and let $\mathcal{B}_U$ be the σ-algebra of Borel subsets of U. Let us denote by $t(u)$ the component of u which belongs to $[0, +\infty)$ and by $y(u)$ — the component of u which belongs to Y. The lemma 1 allows us to define the measure m_θ on $(U, \mathcal{B}_U)$ as follows: $m_\theta(A) := \mathsf{P}_\theta\{(\tau, S_\tau) \in A\}$, $A \in \mathcal{B}_U$.

Theorem 1. *Under the assumptions made above and if moreover,* $\mathsf{P}_\theta\{\tau < +\infty\} = 1$, $\mathsf{P}_{\theta_0}\{\tau < +\infty\} = 1$, *then* m_θ *is absolutely continuous with respect to* m_{θ_0} *and the respective density is given by:* $dm_\theta/dm_{\theta_0} = g(t, y, \theta, \theta_0)$, $t \in [0, +\infty)$, $y \in Y$.

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Proof. Let us denote by m_θ^n the measure generated by (τ_n, S_{τ_n}) on $(U, \mathcal{B}_U)$, where τ is as in the Lemma 1. Moreover let

$A_{n,k} := \{(t, y) \in U : t = t_k^n\}$, $C_{n,k} := \{\omega \in \Omega : \tau_n(\omega) = t_k^n\}$, $m_{\theta,k}^n(A) := m_\theta^n(A \cap A_{n,k})$,

$A \in \mathcal{B}_U$. It is easy to see that:

$$m_{\theta,k}^n(A) = \int_{(\tau_n S_{\tau_n})^{-1}(A \cap A_{n,k})} d\,\mathsf{P}_\theta = \int_{C_{n,k} \cap S_{t_k^n}^{-1}(A \cap A_{n,k})\; t_k^n} d\,\mathsf{P}_\theta$$

$$= \int_{C_{n,k} \cap S_{t_k^n}^{-1}(A \cap A_{n,k})\; t_k^n} d\,\mathsf{P}_{\theta, t_k^n} = \int_{C_{n,k} \cap S_{t_k^n}^{-1}(A \cap A_{n,k})\; t_k^n} g(t_k^n, S_{t_k^n}, \theta, \theta_0)\, d\,\mathsf{P}_{\theta_0, t_k^n}$$

$$= \int_{C_{n,k} \cap S_{t_k^n}^{-1}(A \cap A_{n,k})\; t_k^n} g(t_k^n, S_{t_k^n}, \theta, \theta_0)\, d\,\mathsf{P}_{\theta_0} = \int_{(\tau_n, S_{\tau_n})^{-1}(A \cap A_{n,k})} g(t_k^n, S_{t_k^n}, \theta, \theta_0)\, d\,\mathsf{P}_{\theta_0}$$

$$= \int_{(\tau_n, S_{\tau_n})^{-1}(A \cap A_{n,k})} g(\tau_n, S_{\tau_n}, \theta, \theta_0)\, d\,\mathsf{P}_{\theta_0} = \int_{A \cap A_{n,k}} g(t, y, \theta, \theta_0)\, dm_{\theta_0}^n .$$

But $m_\theta^n(A) = \sum_{k=1}^\infty m_{\theta,k}^n(A)$, i. e. $m_\theta^n(A) = \int_A g(t, y, \theta, \theta_0)\, dm_{\theta_0}^n$. It follows from $d(S_{\tau_n}, S_\tau) \underset{n\to\infty}{\longrightarrow} 0$ a. s. (P_θ) that for every continuous and bounded function $f: U \to R$ we have:

$$\int f(t, y)\, dm_\theta^n = \int f(\tau_n, S_{\tau_n})\, d\,\mathsf{P}_\theta \underset{n\to\infty}{\longrightarrow} \int f(\tau, S_\tau)\, d\,\mathsf{P}_\theta = \int f(t, y)\, dm_\theta$$

i. e. $m_\theta^n \Rightarrow m_\theta$. The following lemma completes the proof of the Theorem 1.

Lemma 2. *Let X be a locally compact metric separable space and $\mathcal{B}_X$ — the σ-algebra of Borel subsets of X. Let $\{\mu_n\}_{n \geq 1}$, μ be probability measures on $(X, \mathcal{B}_X)$ and let K be the class of all continuous functions $f: X \to R$, with compact support. Then: $\mu_n \Rightarrow \mu \Leftrightarrow \int f d\mu_n \to \int f d\mu \;\forall f \in K$.*

Theorem 1 is shown by R. Różański in the case $X = R^n$. V. Sudakov has proved a similar theorem (see [4; 6]).

2. Sequential plans for the compound Poisson process. The compound Poisson process is the process given by the formula (see [2])

$$X_t := \sum_{i=1}^{N_t} \xi_i, \; X_0 = 0, \; t \in (0, +\infty), \tag{2.1}$$

where $\{\xi_i\}_{i \geq 1}$ are independent identically distributed random variables, $\{N_t\}_{t \geq 0}$ is the classical Poisson process with parameter $\lambda \in (0, +\infty)$, $\{N_t\}_{t \geq 0}$ and $\{\xi_i\}_{i \geq 1}$ are independent. In this paper we shall assume that ξ_i is exponentially distributed with parameter $\mu \in (0, +\infty)$. Of course the sample functions of this process are continuous from the right and we shall assume that it is defined on $(\Omega, \mathcal{F})$. It is easy to show that

$$d\,\mathsf{P}_{\lambda,\mu,t} / d\,\mathsf{P}_{\lambda_0, \mu_0, t} = \lambda^{N_t} \mu^{N_t} e^{-\lambda t - \mu X_t} e^{\lambda_0 t + \mu_0 X_t} \lambda_0^{-N_t} \mu_0^{-N_t}, \tag{2.2}$$

where $\{\mathsf{P}_{\lambda,\mu,t}\}_{t>0}$ are the restrictions of $\mathsf{P}_{\lambda,\mu}$ on $\{\sigma(X_s: s\leq t\}_{t>0}$. We are unde the conditions of the Theorem 1, where the sufficient statistic is (N_t, X_t). There exists a σ-finite measure ν_τ on $(U:=[0,+\infty)\times\mathcal{N}\times R, \mathcal{B}_U)$ ($\mathcal{N}$ is the set of natural numbers and zero, $\mathcal{B}_U$ — the field of Borel subsets of U) independent of the parameter (λ,μ) such that:

$$\mathsf{P}_{\lambda,\mu}\{(\tau, N_\tau, X_\tau)\in A\} = \int_A \lambda^k\mu^k e^{-\lambda t-\mu x}\, d\nu_\tau(t,k,x),\ A\in\mathcal{B}_U, \tag{2.3}$$

where τ is a Markov stopping time such that $\mathsf{P}_{\lambda,\mu}\{\tau<+\infty\}=1$, $\mathsf{P}_{\lambda_0,\mu_0}\{\tau<+\infty\}=1$. We shall consider only such τ that $\mathsf{P}_{\lambda,\mu}\{\tau>0\}=1$ for every $(\lambda,\mu)\in(0,+\infty)\times(0,+\infty)$ and $\mathsf{P}_{\lambda,\mu}\{\tau<+\infty\}=1$ for every (λ,μ) belonging to some interval $J:=[a_1,b_1]\times[a_2,b_2]\subset(0,+\infty)\times(0,+\infty)$. Let $h(\lambda,\mu)$ be a real function. We shall suppose that $h'_\lambda:=\partial h/\partial\lambda$ and $h'_\mu:=\partial h/\partial\mu$ exist for each $(\lambda,\mu)\in J$. A real function f defined on U and measurable with respect to $\mathcal{B}_U$ is called an estimator of the parameter $h(\lambda,\mu)$. We shall consider only such estimators for which the integral

$$\int_U f^2(u)\lambda^k\mu^k e^{-\lambda t-\mu x}\, d\nu_\tau(u),\quad u=(t,k,x) \tag{2.4}$$

exists and is finite for each $(\lambda,\mu)\in J$, and which are unbiased, i. e.

$$\mathsf{E}_{\lambda,\mu} f = \int_U f(u)\lambda^k\mu^k e^{-\lambda t-\mu x}\, d\nu_\tau(u) = h(\lambda,\mu),\ (\lambda,\mu)\in J. \tag{2.5}$$

We assume that there exist functions $G(u)$, $H_1(u)$, $H_2(u)$ independent of $(\lambda,\mu$ and such that for each $u\in U$ and $(\lambda,\mu)\in J$:

$$\begin{aligned} &(t^2+k^2+x^2)\lambda^k\mu^k e^{-\lambda t-\mu x} < G(u),\\ &|f(u)(k/\lambda-t)|\,\lambda^k\mu^k e^{-\lambda t-\mu x} < H_1(u),\\ &|f(u)(k/\mu-x)|\,\lambda^k\mu^k e^{-\lambda t-\mu x} < H_2(u),\end{aligned} \tag{2.6}$$

$$\int_U G(u)d\nu_\tau(u)<+\infty,\quad \int_U H_1(u)d\nu_\tau(u)<+\infty,\quad \int_U H_2(u)d\nu_\tau(u)<+\infty. \tag{2.7}$$

Consider the identity

$$\int_U \lambda^k\mu^k e^{-\lambda t-\mu x}\, d\nu_\tau(u)=1. \tag{2.8}$$

Differentiating the function under the sign of the integral in (2.8) with respect to the parameters λ and μ we get

$$\frac{1}{\lambda}\mathsf{E}_{\lambda,\mu}N_\tau-\mathsf{E}_{\lambda,\mu}\tau=0,\quad \frac{1}{\mu}\mathsf{E}_{\lambda,\mu}N_\tau-\mathsf{E}_{\lambda,\mu}X_\tau=0. \tag{2.9}$$

Differentiating the function under the sign of the integral in (2.8) twice with respect to the parameters λ and μ we get respectively

$$\begin{aligned}&\int[(k/\lambda-t)^2-k/\lambda^2]\lambda^k\mu^k e^{-\lambda t-\mu x}\, d\nu_\tau(u)=0,\\ &\int[(k/\mu-x)^2-k/\mu^2]\lambda^k\mu^k e^{-\lambda t-\mu x}\, d\nu_\tau(u)=0.\end{aligned} \tag{2.10}$$

The differentiation is justified, because

$$\begin{aligned}&|(k/\lambda-t)^2-k/\lambda^2|\,\lambda^k\mu^k e^{-\lambda t-\mu x}\leq G(u)\,[(1/\lambda+1)^2+1/\lambda^2],\\ &|(k/\mu-x)^2-k/\mu^2|\,\lambda^k\mu^k e^{-\lambda t-\mu x}\leq G(u)\,[(1/\mu+1)^2+1/\mu^2],\end{aligned}$$

and $[(1/\lambda+1)^2+1/\lambda^2]$, $[(1/\mu+1)^2+1/\mu^2]$ are bounded in J. Differentiating the function under the sign of the integral in (2.8) with respect to the parameter λ, after that with respect to the parameter μ yields:

$$(2.11) \qquad \int (k^2/\lambda\mu - kx/\lambda - kt/\mu + xt)\lambda^k\mu^k e^{-\lambda t-\mu x}\, d\nu_\tau(u) = 0.$$

The differentiation is justified, because

$$|k^2/\lambda\mu - kx/\lambda - kt/\mu + xt|\, \lambda^k\mu^k e^{-\lambda t-\mu x} \leqq G(u)(1/\lambda\mu + 1/\lambda + 1/\mu + 1)$$

and $(1/\lambda\mu+1/\lambda+1/\mu+1)$ is bounded in J. Differentiating the function under the sign of the integral in (2.5) with respect to the parameters λ and μ taking into consideration (2.7), we get

$$(2.12) \qquad h'_\lambda(\lambda, \mu) = \mathsf{E}_{\lambda,\mu}\left(\frac{1}{\lambda} N_\tau f - \tau f\right), \quad h'_\mu(\lambda, \mu) = \mathsf{E}_{\lambda,\mu}\left(\frac{1}{\mu} N_\tau f - X_\tau f\right).$$

Theorem 2. *For each Markov stopping τ such that $\mathsf{P}_{\lambda,\mu}\{\tau < +\infty\} = 1$ for each $(\lambda, \mu) \in J$ and for each estimator f of the function h satisfying the assumptions made above*

$$(2.13) \qquad \mathsf{D}_{\lambda,\mu} f \geqq \{[\lambda h'_\lambda(\lambda, \mu)]^2 + [\mu h'_\mu(\lambda, \mu)]^2\}/\{\lambda\, \mathsf{E}_{\nu,\mu}\tau\},$$

where an equality in (2.13) *holds at (λ_0, μ_0) iff there exist constants β and γ, not both zero, such that $f(u) - h(\lambda_0, \mu_0) = \beta(k/\lambda_0 - t) + \gamma(k/\mu_0 - x)$ almost surely (ν_τ).*

Proof. Let us consider the random variables $(N_\tau/\lambda - \tau)$ and $(N_\tau/\mu - X_\tau)$, $(\lambda, \mu) \in J$. We get from (2.9) and (2.10)

$$\mathsf{E}_{\lambda,\mu}\left(\frac{1}{\lambda} N_\tau - \tau\right) = 0, \quad \mathsf{E}_{\lambda,\mu}\left(\frac{1}{\mu} N_\tau - X_\tau\right) = 0, \quad \mathsf{E}_{\lambda,\mu}\left(\frac{1}{\lambda} N_\tau - \tau\right)^2$$

$$2.14 \qquad = \frac{1}{\lambda^2}\mathsf{E}_{\lambda,\mu} N_\tau, \quad \mathsf{E}_{\lambda,\mu}\left(\frac{1}{\mu} N_\tau - X_\tau\right)^2 = \frac{1}{\mu^2}\mathsf{E}_{\lambda,\mu} N_\tau.$$

Let us consider the Hilbert space of all functions $\eta\colon U \to R$, $\mathsf{E}_{\lambda,\mu}\eta^2(\tau, N_\tau, X_\tau) < +\infty$ with a scalar product:

$$\langle \eta_1, \eta_2 \rangle := \int \eta_1(u)\eta_2(u)\lambda^k\mu^k e^{-\lambda t-\mu x}\, d\nu_\tau(u).$$

It follows from (2.11) that $(k/\lambda - t)$ and $(k/\mu - x)$ are orthogonal. We know every orthogonal system of vectors in a Hilbert space can be extended up to a complete orthogonal system. The representation of $[f - h(\lambda, \mu)]$ in Fourier's series with respect to the complete orthogonal system generated by $(k/\lambda - t)$, and $(k/\mu - x)$ yields:

$$\mathsf{E}_{\lambda,\mu}[f - h(\lambda, \mu)]^2 \geqq \mathsf{E}^2_{\lambda,\mu}\left[f\left(\frac{1}{\lambda} N_\tau - \tau\right)\right] / \mathsf{E}_{\lambda,\mu}\left(\frac{1}{\lambda} N_\tau - \tau\right)^2$$

$$+ \mathsf{E}^2_{\lambda,\mu}\left[f\left(\frac{1}{\mu} N_\tau - X_\tau\right)\right] / \mathsf{E}_{\lambda,\mu}\left(\frac{1}{\mu} N_\tau - X_\tau\right)^2.$$

We get (2.13) from (2.12) and (2.14), where of course an equality in (2.13) holds iff there exist constants β and γ, not both zero, such that $f(u) - h(\lambda, \mu) = \beta(k/\lambda - t) + \gamma(k/\mu - x)$ almost surely (ν_τ).

We shall assume that $h'_\lambda(\lambda, \mu) \neq 0$ or $h'_\mu(\lambda, \mu) \neq 0$ for at least one $(\lambda, \mu) \in J$. It follows that the function $f(u) =$ const. $u \in U$, cannot be an estimator for the parameter $h(\lambda, \mu)$.

Definition 2.1. *By a sequential plan we shall mean the triplet* (τ, f, h) *for which* (2.5) *holds.*

Definition 2.2. *A sequential plan is called efficient for* (λ_0, μ_0), *if the inequality* (2.13) *becomes an equality for* (λ_0, μ_0). *The estimator f is called efficient for* (λ_0, μ_0) *and the function h is called efficiently estimable or* (λ_0, μ_0).

Let $\mathsf{P} \subset (0, +\infty) \times (0, +\infty)$.

Definition 2.3. *A sequential plan* (τ, f, h) *is called efficient for* P, *if the inequality* (2.13) *becomes an equality for each* $(\lambda, \mu) \in \mathsf{P}$. *The estimator f is called efficient for* P *and the function h is called efficiently estimable for* P.

Theorem 3. *If the sequential plan* (τ, f, h) *is efficient for* $(0, +\infty) \times (0, +\infty)$, *then there exists constants,* $a_1, a_2\ a_3, a_4$ *such that* $a_1^2 + a_2^2 + a_3^2 \neq 0$, $a_4 \neq 0$ *and*

$$a_1 k + a_2 x + a_3 t + a_4 = 0 \tag{2.15}$$

almost surely with respect to ν_τ.

Proof. Let (λ_1, μ_1) and (λ_2, μ_2) belong to $((0, +\infty) \times (0, +\infty)$ and $\lambda_1 = \lambda_2$, $\mu_1 \neq \mu_2$. It follows from the Theorem 2 there exist constants $\beta_1, \gamma_1, \delta_1, \beta_2, \gamma_2, \delta_2$ such that almost surely (ν_τ)

$$f(u) = \beta_1(k/\lambda_1 - t) + \gamma_1(k/\mu_1 - x) + \delta_1, \quad f(u) = \beta_2(k/\lambda_2 - t) + \gamma_2(k\ \mu_2 - x) + \delta_2. \tag{2.16}$$

We obtain from (2.16) that:

$$(\beta_1/\lambda_1 - \beta_2/\lambda_2 + \gamma_1/\mu_1 - \gamma_2/\mu_2)k + (\beta_2 - \beta_1)t + (\gamma_2 - \gamma_1)x + \delta_1 - \delta_2 = 0$$

almost surely (ν_τ). It follows from $\lambda_1 = \lambda_2$, $\mu_1 \neq \mu_2$ that when $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$, then $\beta_1/\lambda_1 - \beta_2/\lambda_2 + \gamma_1/\mu_1 - \gamma_2/\mu_2 \neq 0$. Now we have to show $\delta_1 - \delta_2 \neq 0$. Let us suppose that $\delta_1 - \delta_2 = 0$. It follows that

$$a_1 k + a_2 x + a_3 t = 0 \tag{2.17}$$

almost surely (ν_τ), where a_1, a_2, a_3 are real numbers $(a_1^2 + a_2^2 + a_3^2 \neq 0)$. It is easy to see that (2.17) is impossible when $a_1 \geq 0$, $a_2 \geq 0$, $a_3 \geq 0$ or $a_1 \leq 0$, $a_2 \leq 0$, $a_3 \leq 0$. Let us consider the case when $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$ and $\operatorname{sign} a_i \neq \operatorname{sign} a_j$, where $(i, j) = (1,2)$ or $(i, j) = (1,3)$ or $(i, j) = (2,3)$. We can write (2.17) as follows:

$$t = a_1' k + a_2' x \tag{2.18}$$

a. s. (ν_τ), where $a_1' > 0$ or $a_2' > 0$. Let $c := \max(|a_1'|, |a_2'|)$ and let $\tau_0 := \inf\{t > 0 : t \leq c(N_t + X_t)\}$. To show that (2.18) is impossible it is sufficient to prove that there exists (λ_0, μ_0) such that $\mathsf{P}_{\lambda_0, \mu_0}\{\tau_0 = +\infty\} > 0$. Let $Z_t := c(N_t + X_t)$ and let $t_n := nc$. There exists $(\lambda_0, \mu_0) \in (0, +\infty) \times (0, +\infty)$ such that $d := c(\lambda_0 + \lambda_0/\mu_0) < 1$. It is easy to see that for each m and n, $0 < m < n$, we can write

$$(2.19)\qquad \mathsf{P}_{\lambda_0,\mu_0}\{\tau_0>t_n\}\geqq \mathsf{P}_{\lambda_0,\mu_0}\{Z_{t_1}=0,\ Z_{t_2}<c,\ldots,\ Z_{t_n}<(n-1)c\}$$
$$\geqq \mathsf{P}_{\lambda_0,\mu_0}\{Z_{t_m}=0\}\mathsf{P}_{\lambda_0,\mu_0}\{Z_{t_{m+1}-t_m}<mc,\ldots,\ Z_{t_n-t_m}<(n-1)c\}$$
$$\geqq \mathsf{P}_{\lambda_0,\mu_0}\{Z_{t_m}=0\}\mathsf{P}_{\lambda_0,\mu_0}\{|Z_{t_{m+1}-t_m}-\mathsf{E}_{\lambda_0,\mu_0}(Z_{t_{m+1}-t_m})|<mc-cd,$$
$$\ldots,\ |Z_{t_n-t_m}-\mathsf{E}_{\lambda_0,\mu_0}(Z_{t_n-t_m})|<(n-1)c-(n-m)cd\}.$$

But for each n, $n>m$, we have:

$$(2.20)\qquad (n-1)c-(n-m)cd=(m-1)c+(n-m)c(1-d)$$
$$\geqq[4c(m-1)]^{1/4}\left[\frac{4}{3}\,c(1-d)(n-m)\right]^{3/4}.$$

It is easy to show that

$$\mathsf{D}_{\lambda_0,\mu_0}Z_t=\mathsf{E}_{\lambda_0,\mu_0}(Z_t-\mathsf{E}_{\lambda_0,\mu_0}Z_t)^2=c^2\lambda_0 t(2/\mu_0^2+2/\mu_0+1).$$

Let $A:=[\lambda_0(2/\mu_0^2+2/\mu_0+1)]^{-1/2}\sum_{n=1}^{\infty}n^{-3/2}$. Of course $A<+\infty$ and there exists m_0 such that for each n, $n>m_0$:

$$(2.21)\qquad [4c(m_0-1)]^{1/4}\left[\frac{4}{3}\,c(1-d)(n-m_0)\right]^{3/4}\geqq\sqrt{1+A}\,[\mathsf{D}_{\lambda_0,\mu_0}(Z_{t_n-t_{m_0}})]^{3/4}.$$

We can write from (2.19), (2.20) and (2.21) that

$$\mathsf{P}_{\lambda_0,\mu_0}\{\tau_0>t_n\}\geqq\mathsf{P}_{\lambda_0,\mu_0}\{Z_{t_{m_0}}=0\}\mathsf{P}_{\lambda_0,\mu_0}\{(Z_{t_{m_0+1}-t_{m_0}}-\mathsf{E}_{\lambda_0,\mu_0}Z_{t_{m_0+1}-t_{m_0}})^2$$
$$<(1+A)(\mathsf{D}_{\lambda_0,\mu_0}Z_{t_{m_0+1}-t_{m_0}})^{3/2},\ldots,\ (Z_{t_n-t_{m_0}}-\mathsf{E}_{\lambda_0,\mu_0}Z_{t_n-t_{m_0}})^2$$
$$<(1+A)(\mathsf{D}_{\lambda_0,\mu_0}Z_{t_n-t_{m_0}})^{3/2}\}.$$

We obtain form Hájek-Rényi-Chow inequality (see [5]) that for each n, $n>m_0$:

$$\mathsf{P}_{\lambda_0,\mu_0}\{\tau_0>t_n\}\geqq\mathsf{P}_{\lambda_0,\mu_0}\{Z_{t_{m_0}}=0\}(1-A/(1+A))>0,$$

i. e. $\mathsf{P}_{\lambda_0,\mu_0}\{\tau_0=+\infty\}>0$. We can consider all other cases in analogous way. It completes the proof of the Theorem 3.

Definition 2.4. *A sequential plan* (τ, f, h) *is called a simple plan if* τ *denotes the moment of the first attaining of the set* $t=t_0$, $[t_0\in(0,+\infty)]$ *by the process* $(t, N_t, X_t)_{t\geqq 0}$.

Definition 2.5. *A sequential plan* (τ, f, h) *is called an inverse plan if* τ *denotes the moment of the first attaining of the set* $k=k_0$, (k_0*—fixed* $k_0=1,2,\ldots$) *by the process* $(t, N_t, X_t)_{t\geqq 0}$.

Theorem 4. *Only the simple and inverse plans may be efficient sequential plans for* $(0,+\infty)\times(0,+\infty)$.

Proof. If the sequential plan (τ, f, h) is efficient for $(0,+\infty)\times(0,+\infty)$, then it follows from the Theorem 3 that ν_τ is accumulated on the set $a_1k+a_2x+a_3t+a_4=0$, where $a_1^2+a_2^2+a_3^2\neq 0$ and $a_4\neq 0$. Without loss of generality we can assume that $a_4>0$. If $a_1\geqq 0$, $a_2\geqq 0$, $a_3\geqq 0$ then $\mathsf{P}_{\lambda,\mu}\{\tau<+\infty\}<1$. If $a_1\leqq 0$, $a_2\leqq 0$, $a_3\leqq 0$, then it is easy to see that only in the cases $t=t_0$, $[t_0\in(0,+\infty)]$ and $k=k_0$, $(k_0=1,2,3,\ldots)$ we get $\mathsf{P}_{\lambda,\mu}\{\tau<+\infty\}=1$ for each $(\lambda,\mu)\in(0,+\infty)\times(0,+\infty)$. One can see that for each another case using the results of [7] and the methods in the Theorem 3 that there exists (λ_0, μ_0)

$(0, +\infty)\times(0, +\infty)$ such that $\mathsf{P}_{\lambda_0, \mu_0}\{\tau < +\infty\} < 1$. It follows that ν_τ is accumulated on the set $t=t_0$, $[t_0 \in (0, +\infty)]$ or $k=k_0$, $(k_0=1, 2, 3, \ldots)$. The fact that τ is a moment of the first attaining we can obtain in analogous way as it is done in [7].

Theorem 5. *The simple and inverse plans are efficient for* $(0, +\infty)\times(0, +\infty)$. *The following are the only efficiently estimable functions:*

a) *for the simple plan*

$$h(\lambda, \mu)=a\lambda+b\lambda/\mu+c \tag{2.22}$$

and $f(u)=ak/t_0+bx/t_0+c$ *is its only efficient estimator.*

b) *for the inverse plan*

$$h(\lambda, \mu)=a/\lambda+b/\mu+c \tag{2.23}$$

and $f(u)=at/k_0+bx/k_0+c$ *is its only efficient estimator.*

Proof. We get from the Theorem 2 that the necessary condition for f being an efficient estimator is:

$$f(u)=\alpha k+\beta x+\gamma t+\delta, \tag{2.24}$$

where α, β, γ, δ are real numbers. We shall show that this condition is sufficient. To show that (2.6) and (2.7) hold for the estimators (2.24) it is sufficient to prove that for each $(\lambda, \mu)\in(0, +\infty)\times(0, +\infty)$:

$$\mathsf{E}_{\lambda,\mu} N_\tau^2 < +\infty,\ \mathsf{E}_{\lambda,\mu}\tau^2 < +\infty,\ \mathsf{E}_{\lambda,\mu}X_\tau^2 < +\infty. \tag{2.25}$$

If τ denotes the moment of the first attaining of the set $t=t_0$, $[t_0 \in (0, +\infty)$ or $k=k_0$, $k_0=1, 2, \ldots)$, then one can see using the results of [7] and the lemmas in [3, p. 262—263] and [5, p. 44] that (2.25) holds. We can write the following equalities:

$$k=\lambda(k/\lambda-t_0)+\lambda t_0,$$

$$x=-(k/\mu-x)+\lambda(k/\lambda-t_0)/\mu+\lambda t_0/\mu,$$

$$t=-(k_0/\lambda-t)+k_0/\lambda,$$

$$x=-(k_0/\mu-x)+k_0/\mu.$$

From then and from Theorem 2 we obtain that the estimators (2.24) are efficient, when τ denotes the moment of the first attaining of the set $t=t_0$ $[t_0 \in (0, +\infty)]$ or $k=k_0$, $(k_0=1, 2, 3, \ldots)$. (2.22) and (2.23) can be got using (2.9) This ends the proof of Theorem 5.

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