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A LIMIT DISTRIBUTION RELATED TO RANDOM MAPPINGS AND ITS APPLICATION TO AN EPIDEMIC PROCESS

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In this paper we consider random graphs, corresponding to random self-mappings of th, finite set $\{1, 2, \dots, n\}$. The limit distribution of the number of cyclic graph's elements. which are contained in components of sizes exceeding $\alpha . n$, $0<\alpha<1$, is derived for $n\to\infty$. This result is obtained in terms of Laplace-Stiltjes transforms. Some limit properties of the nverse epidemic process, introduced by I. B. Gertsbakh (1977), are proved.

1. Introduction. Consider the set $\mathcal{T}_n$ of all the mappings of the finite set $X=\{1, \dots, n\}$ into itself, which satisfy $Tx\neq x$, $T\in\mathcal{T}_n$, $x\in X$. There are $(n-1)^n$ different mappings in $\mathcal{T}_n$. Each mapping $T\in\mathcal{T}_n$ is a digraph G_T, whose points belong to the set X; the points x and y are joined by an arrow iff $y=Tx$. The graph G_T may consist of disjoined components and each component includes only one cycle. We classify the components of G_T corresponding to their size, i. e. to the number of points they consist of. Let an uniform probability distribution on $\mathcal{T}_n$ be given (each mapping $T\in\mathcal{T}_n$ has probability $(n-1)^{-n}$). The random mappings just described are the second type mappings studied by B. Harris [1].

Let $\mu_{s,n}=\mu_{s,n}(T)$, $T\in\mathcal{T}_n$ be the number of cyclical points of G_T, which are contained in components of sizes exceeding $s(\geq 2)$. Suppose that $n\to\infty$, $s\sim\alpha n$, $0<\alpha<1$. We derive an asymptotic result for the distribution of $\mu_{s,n}$ in terms of Laplace-Stiltjes transforms. As an application of this limit theorem, we discover some asymptotics for the inverse epidemic process introduced by I. B. Gertsbakh [2].

2. Preliminary results. The mapping $T\in\mathcal{T}_n$ is called indecomposable iff it generates only one cycle. Let $C_{n,k}$ denote the number of indecomposable mappings in $\mathcal{T}_n$, which have exactly k cyclic elements. It is known, that for $n\geq k\geq 2$ [2, p. 432] $C_{n,k}=\binom{n}{k}k!\,n^{n-k-1}$. The number of all indecomposable mappings in $\mathcal{T}_n$ is

$$(1)\qquad B_n=\sum_{k=2}^{n} C_{n,k}=(n-1)!\sum_{k=0}^{n-2}\frac{n^k}{k!}=\sqrt{\pi n/2}\,e(n-1)^{n-1}(1+o(1)),\quad n\to\infty.$$

Assign to each indecomposable mapping T the probability B_n^{-1}, and denote by ξ_n the number of cyclic elements in T. The probability distribution of ξ_n is

$$(2)\qquad \mathsf{P}\{\xi_n=k\}=n!\,n^{n-k-1}/(n-k)!\,B_n,\quad k=2, 3, \dots, n.$$

The random variables ξ_n have generating functions

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(3) $$P_n(x)=\frac{n^{n-1}}{B_n}\sum_{k=2}^{n}\binom{n}{k}k!\,n^{-k}x^k.$$

Let $M_n(k_2,\ldots,k_n)$ be the number of all mappings $T\in\mathcal{T}_n$, which have k components of size i, $k_i\geqq 0$, $i=2,\ldots,n$, $\Sigma ik_i=n$. It is easy to see that

$$M_n(k_2,\ldots,k_n)=n!\,B_2^{k_2}\ldots B_n^{k_n}/(2!)^{k_2}\ldots(n!)^{k_n}k_2!\ldots k_n!.$$

Now define the sequence of generating functions of the numbers $M_n(k_2,\ldots,k_n)$:

$$m_0=1,\ m_1=0,\ m_n(x_2,\ldots,x_n)=\sum_{\substack{k_2,\ldots,k_n\geqq 0\\ \Sigma ik_i=n}} M_n(k_2,\ldots,k_n)x_2^{k_2}\ldots x_n^{k_n},\ n\geqq 2.$$

The following relation between the generating functions m_n and P_m is obtained in [3]:

Lemma 1. *If $|z|\leqq e^{-1}$, $z\neq e^{-1}$, then*

$$\sum_{n=0}^{\infty} m_n(P_2(x_2),\ldots,P_n(x_n))\frac{z^n}{n!}=\exp\Big(\sum_{k=2}^{\infty}\frac{B_k}{k!}P_k(x_k)z^k\Big)$$

holds for $|x_k|\leqq 1$, $k=2,\ldots,n$.

We shall use also the power serieses $B(z)=\Sigma_{n=2}^{\infty}B_nz^n/n!$, $S(z)=\Sigma_{n=0}^{\infty}n^nz^n/n!$ and $\Theta(z)=\Sigma_{n=1}^{\infty}n^{n-1}z^n/n!$ ($|z|\leqq e^{-1}$, $z\neq e^{-1}$). It is well-known [4, Section 7.1 and 7.2] that the function $\Theta(z)$ satisfies the transcendental equations

(4) $$\Theta e^{-\Theta}=z,\ \Theta(ze^{-z})=z\ \ (\Theta(e^{-1})=1),$$

and

(5) $$S(z)=[1-\Theta(z)]^{-1}.$$

Using some general combinatorial results [5 p. 179] we can obtain also that

(6) $$\exp\{B(z)\}=S(z)\exp\{-\Theta(z)\}.$$

We also need the limit distribution of the random variables ξ_n (see (2)) for $n\to\infty$.

Lemma 2. *If $n\to\infty$ and $u=o(n^{1/6})$, $u>0$, then*

(7) $$\mathsf{P}\{\xi_n/\sqrt{n}=u\}=\sqrt{2/\pi n}\,e^{-u^2/2}(1+o(1)).$$

Proof. Applying the limit-relation (1) and Stirling's formula in (2) for $u=k/\sqrt{n}$, we obtain

$$\mathsf{P}\{\xi_n/\sqrt{n}=u\}=e^{-u\sqrt{n}}\sqrt{2/\pi}(1-u/\sqrt{n})^{-n+u\sqrt{n}-1/2}(1/\sqrt{n})(1+o(1)).$$

For every $u=o(n^{1/6})$ we have $(-n+u\sqrt{n}-1/2)\ln(1-u/\sqrt{n})=u\sqrt{n}-u^2/2+o(1)$. These two relations give the limit distribution (7).

3. The limit distribution of $\mu_{s,n}$. Consider the random variable $\mu_{s,n}$, defined in the introduction. In [3] is obtained the following result: If $n\to\infty$, and $s\sim an$, $0<a<1$, then

$$\mathsf{E}\,\mu_{s,n}=\sqrt{2n/\pi}\,(\pi/2-\arcsin\sqrt{a})+O(1).$$

Here we shall give an asymptotic relation for the Laplace-Stiltjes transformation of $\mu_{s,n}/\sqrt{n}$.

Theorem 1. *If $n\to\infty$ and $s\sim an$, $0<a<1$, then for a fixed $t\geqq 0$*

$$\mathsf{E}\exp(-t\mu_{s,n}/\sqrt{n})=\sum_{0\leqq k<1/a}\frac{I_k(t,a)}{2^k k!}+o(1),$$

where $I_0(t,a)=1$,

$$I_k(t,a)=\int\cdots\int_{V_{a,k}}\frac{[f(t\sqrt{x_1})-1]\ldots[f(t\sqrt{x_k})-1]dx_1\ldots dx_k}{x_1\ldots x_k\sqrt{1-x_1-\ldots-x_k}},\quad k\geqq 1,$$

$V_{a,k}=\{x_i\geqq a;\ i=1,\ldots,k,\ x_1+\ldots+x_k\leqq 1\}$, *and*

$$f(t)=\sqrt{2/\pi}\int_0^\infty\exp(-tx-x^2/2)dx. \tag{8}$$

Proof. By lemma 1, substituting $x_j=1$ for $j\leqq s$ and $x_j=x$ for $j\geqq s+1$, we have

$$\sum_{n=0}^\infty\Psi_n(x)\frac{z^n}{n!}=\exp\{B(z)+\sum_{k=s+1}^\infty\frac{B_k z^k}{k!}(P_k(x)-1)\},$$

where $\Psi_n(x)=m_n(\overbrace{1,\ldots,1}^{s},\ P_{s+1}(x),\ldots,P_n(x))$. Hence for the generating function of $\mu_{s,n}$ it follows

$$\mathsf{E}\,x^{\mu_{s,n}}=(n-1)^{-n}\Psi_n(x)=\frac{n!}{(n-1)^n}\frac{1}{2\pi i}\oint\exp\{B(z)+\sum_{k=s+1}^\infty\frac{B_k z^k}{k!}(P_k(x)-1)\}\frac{dz}{z^{n+1}}.$$

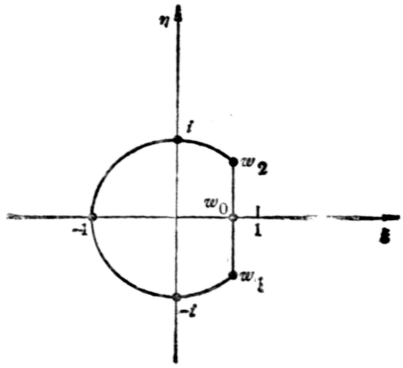

Fig. 1

Substitute in the last integral $z=we^{-w}$, $x=e^{-t/\sqrt{n}}$ $(t\geqq 0)$ and choose the path of integration C in the plane $w=\xi+i\eta$ as on Fig. 1 (see [6, p. 648]). Here $w_0=1-1/\sqrt{s}$, $w_1=1-1\sqrt{s}-i\sqrt{1-(1-1/\sqrt{s})^2}$, $w_2=1-1/\sqrt{s}+i\sqrt{1-(1-1/\sqrt{s})^2}$. By (4)—(6) we obtain

$$\mathsf{E}\exp(-t\mu_{s,n}/\sqrt{n})=\frac{n!}{(n-1)^n}\frac{1}{2\pi i}\int_C e^{(n-1)w}\exp\{\sum_{k=s+1}^{\infty}\frac{B_k w^k e^{-kw}}{k!}(P_k(e^{-t/\sqrt{n}})-1)\}\frac{dw}{w^{n+1}}\cdot$$

It is easy to see that the integral on the unit circle tends to 0, when $n\to\infty$ and $s\sim\alpha n$, $0<\alpha<1$, and the non-zero part of the integral is on the chord w_1w_2:

$$\mathsf{E}\exp(-t\mu_{s,n}/\sqrt{n}) \tag{9}$$

$$=\frac{n!}{(n-1)^n}\frac{1}{2\pi i}\int_{w_1}^{w_2} e^{(n-1)w}\exp\{\sum_{k=s+1}^{\infty}\frac{B_k w^k e^{-kw}}{k!}(P_k(e^{-t}/\sqrt{n})-1)\}\frac{dw}{w^{n+1}}+o(1).$$

Put in this integral $w=1-v/\sqrt{s}$. For the same values of s it is easy to verify that

$$n!/(n-1)^n e^{(n-1)(1-v/\sqrt{s})}(1-v/\sqrt{s})^{-(n+1)}=\sqrt{2\pi n}\,e^{v^2/2\alpha}(1+o(1)), \tag{10}$$

$$(1-v/\sqrt{s})^k e^{kv/\sqrt{s}}=e^{-kv^2/2s}(1+O(1/\sqrt{s})), \tag{11}$$

and

$$B_k e^{-k}/(k-1)!=1/2+O(1/\sqrt{k}),\quad k\to\infty. \tag{12}$$

Now we shall study the asymptotic behaviour of the sum

$$\sigma_s(v)=\sum_{k=s+1}^{\infty}\frac{B_k}{k!}(1-\frac{v}{\sqrt{s}})^k e^{-k(1-v/\sqrt{s})}(P_k(e^{-t/\sqrt{n}})-1).$$

Using (11) and (12) we obtain the representation

$$\sigma_s(v)=\frac{1}{2}\sum_{k>s}\frac{1}{k}e^{-kv^2/2s}(P_k(e^{-t/\sqrt{n}})-1)(1+O(1\sqrt{s}))$$

$$=[\sigma_s^{(1)}(v)-\sigma_s^{(2)}(v)](1+O(1/\sqrt{s})), \tag{13}$$

$$\sigma_s^{(1)}(v)=\frac{1}{2}\sum_{k>s}\frac{1}{k}e^{-kv^2/2s}P_k(e^{-t/\sqrt{n}}),\quad \sigma_s^{(2)}(v)=\frac{1}{2}\sum_{k>s}\frac{1}{k}e^{-kv^2/2s}.$$

We may consider $\sigma_s^{(2)}(v)$ as a Riemann's integral sum:

$$\sigma_s^{(2)}(v)=\frac{1}{2s}\sum_{k>s}\frac{1}{(k/s)}e^{-v^2(k/s)/2}=\frac{1}{2}\int_1^{\infty}\frac{e^{-uv^2}}{u}du+o(1)=\frac{1}{2}\int_{v^2}^{\infty}\frac{e^{-x/2}}{x}dx+o(1). \tag{14}$$

For the sum $\sigma_s^{(1)}(v)$ we shall use relation (7) of lemma 2, which can be written as

$$\mathsf{P}\{\xi_k=l\}=\sqrt{\frac{2}{\pi k}}\,e^{-l^2/2k}(1+o(1)), \tag{15}$$

for $k\geq s\sim\alpha n$, $0<\alpha<1$, and $l=o(k^{2/3})$. For arbitrary l $(2\leq l\leq k)$ and $\varepsilon>0$ there exists a natural number k_0 such that

$$\mathsf{P}\{\xi_k=l\}\leq(1+\varepsilon)\sqrt{2/\pi k}\exp(-l^2/k),$$

when $k\geq k_0$. Therefore

(16) $$\sum_{k^{7/12}\leqq l\leqq k} \mathsf{P}\{\xi_k=l\}e^{-tl/\sqrt{n}}\leqq(1+\varepsilon)\sqrt{2k/\pi}e^{-k^{1/6}}\to 0,\ k\to\infty.$$

By virtue of (14)—(16) we get

$$\sigma_s^{(1)}(v)=\frac{1}{2}\sqrt{\frac{2}{\pi}}\sum_{k>s}\frac{1}{k^{3/2}}e^{-kv^2/2s}\sum_{2\leqq l<k^{7/12}}\exp\left(-\frac{tl}{\sqrt{n}}-\frac{l^2}{k}\right)(1+o(1))$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\sum_{k/s>1}\frac{1}{s}\cdot\frac{1}{(k/s)^{3/2}}e^{-kv^2/2s}\sum_{(k^{7/12}/\sqrt{s})>(l/\sqrt{s})\geqq(2/\sqrt{s})}\frac{1}{\sqrt{s}}\exp\left(-\frac{t\sqrt{\alpha}\,l}{\sqrt{s}}-\frac{l^2}{s(k/s)}\right)$$
$$(1+o(1)).$$

Using again the Riemann's integral sums we obtain

$$\sigma_s^{(1)}(v)=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_1^\infty\frac{e^{-xv^2/2}}{x\sqrt{x}}\int_0^\infty e^{-t\sqrt{\alpha}\,y}\,e^{-y^2/2x}\,dy\,dx+o(1)$$

(17) $$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{v^2}^\infty\frac{e^{-x/2}}{x}\int_0^\infty e^{-t\sqrt{\alpha x}y/v}\,e^{-y^2/2}\,dydx+o(1).$$

From (9), (10), (13), (14), and (17) it follows that

$$\mathsf{E}\exp(-t\mu_{s,n}/\sqrt{n})$$
$$=\frac{\sqrt{2\pi}}{2\pi i\sqrt{\alpha}}\int_{1-i\infty}^{1+i\infty}\exp\Big(\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{v^2}^\infty\frac{e^{-x/2}}{x}\int_0^\infty e^{-t\sqrt{\alpha x}y/v}\,e^{-y^2/2}\,dydx$$
$$-\frac{1}{2}\int_{v^2}^\infty\frac{e^{-x/2}}{x}\,dx\Big)e^{v^2/2\alpha}\,dv+o(1).$$

Putting again in the last integral $v^2=p$ and then $x/2p=u$ we get

(18) $$\mathsf{E}\exp(-t\mu_{s,n}/\sqrt{n})=\sqrt{\frac{\pi}{2\alpha}}\,\frac{1}{2\pi i}\int_\Pi\exp\Big\{\int_{1/2}^\infty e^{-up}g_\alpha(u,t)\,du\Big\}\,\frac{e^{p/2\alpha}}{\sqrt{p}}\,dp+o(1),$$

where the path of integration Π is the corresponding to the substitution parabola, and

$$g_\alpha(u,t)=\begin{cases}[f(t\sqrt{2\alpha u})-1]/2u, & \text{when } u\geqq 1/2\\ 0, & \text{when } u<1/2,\end{cases}$$

($f(t)$ was defined by (8)). Let us represent the first exponential function in (18) as a power-series in

$$G_\alpha(p,t)=\int_{1/2}^\infty e^{-up}g_\alpha(u,t)du.$$

The powers $[G_\alpha(p,t)]^k$ are Laplace transforms of the kth convolution $g_{\alpha,k}(u,t)$ of the function $g_\alpha(u,t)$. It is easy to see that the integral over Π in (18) gives the inverse Laplace transform. Since $g_{\alpha,k}(u,t)=0$ whenever $0\leqq t<k/2$, the power series

$$\sqrt{\frac{\pi}{2\alpha}}\,\frac{1}{2\pi i}\sum_{j=0}^\infty\int_\Pi\frac{G_\alpha^j(p,t)}{j!}\,\frac{e^{p/2\alpha}}{\sqrt{p}}\,dp$$

is a finite sum whose powers are less than $1/\alpha$. Thus for any natural number $k<1/\alpha$

$$L_{k,\alpha}(t)=\sqrt{\frac{\pi}{2\alpha}}\frac{1}{2\pi i\,k\,!}\int_{\Pi}[G_\alpha(p,\ t)]^k\frac{e^{p/2\alpha}}{\sqrt{p}}dp=\frac{1}{k\,!}\int_{k/2}^{1/2\alpha}g_{\alpha,k}(u)\frac{du}{\sqrt{1-2\alpha u}}\,.$$

Performing this integral we get $L_{k,\alpha}(t)=\frac{1}{k!\ 2^k}I_k(t,\alpha)$. In order to complete the proof we shall remark that $\mathsf{E}\exp(-t\mu_{s,n}/\sqrt{n})=\Sigma_{0\leqq k<1/\alpha}L_{k,\alpha}(t)+o(1)$.

Corollary. *If $n\to\infty$ and $s\sim\alpha n$, $0<\alpha<1$ then the sequence of random variables $\{n^{-1/2}\mu_{s,n}\}_{n=1}^{\infty}$ converges weakly to the random variable μ with Laplace transform*

$$\mathsf{E}\,e^{-t\mu}=\sum_{0\leqq k<1/\alpha}I_k(t,\alpha)/2^k\,k\,!. \tag{19}$$

The distribution function $\mathsf{H}\mu(x)$ of the random variable μ satisfy the inequality $\mathsf{H}\mu(0)\neq 0$ (see (19)). For example, let $1/2<\alpha\leqq 1$, then the sum in (19 has only two terms. Therefore

$$\mathsf{H}\mu(x)=1-\frac{1}{2}\ \ln\frac{1+\sqrt{1-\alpha}}{1-\sqrt{1-\alpha}}+\frac{1}{\sqrt{2\pi}}\int_0^x\int_\alpha^1\frac{e^{-u^2/2y}}{y^{3/2}\sqrt{1-y}}dy\,du$$

$$=1-\frac{1}{2}\ \ln\frac{1+\sqrt{1-\alpha}}{1-\sqrt{1-\alpha}}+\int_0^x\frac{e^{-u^2/2}}{u}\,\Phi_0\Big(u\sqrt{\frac{1}{\alpha}-1}\Big)du,$$

where $\Phi_0(u)=\sqrt{2/\pi}\int_0^u e^{-z^2/2}\,dz$.

4. Application to an epidemic process. We shall apply the result of theorem 1 to study an epidemic process introduced by I. B. Gertsbakh [2].

Define T^kx to be the kth iteration of $T\in\mathcal{T}_n$ on $x\in X$, where k is integer, i. e. $T^kx=T(T^{k-1}x)$ and $T^0x=x$. If for some $k\leqq 0$ $T^kx=y$, y is said to be a kth inverse of x in T. The set of all kth inverses of x in T is denoted by $T^{(k)}(x)$ and $P_T(x)=\bigcup_{k=-n}^{0}T^{(k)}(x)$ is the set of all inverses (or predecessors) of x.

Let m bacteria be placed at elements $x_1,\dots,x_m$, where $x_i\in X$, $i=1,\dots,m$. All $\binom{n}{m}$ different occupations are equally probable. An inverse epidemic process (IEP) [2] is defined by the infection being delivered from the infected points to all their predecessors. The area which will be infected is the set of all inverses of $x_1,\dots,x_m$: $P_T(m)=\bigcup_{i=1}^m P_T(x_i)$ Denote the number of distinct elements in the set $P_T(m)$ by $|P_T(m)|$. Consider the function $C_I(m)\colon\mathcal{T}_n\to\boldsymbol{R}^1$ which maps each $T\in\mathcal{T}_n$ into the integer $|P_T(m)|$. Let $0<\alpha<1$, and consider the event $\{C_I(m)\geqq\alpha n\}$ which denotes that the infected area arising from m bacteria exceeds a fixed α-ratio of all elements in X (m bacteria infect an essential part of the population).

In the paper [2] I. B. Gertsbakh has shown that $\mathsf{P}\{C_I(m)\geqq\alpha n\}\to 0$ for $m=0(\sqrt{n})$. In [3] is proved that $\mathsf{P}\{C_I(m)\geqq\alpha n\}\to 1$ for $\sqrt{n}=o(m)$.

Theorem 1 can be applied to study the limiting behaviour of the probability $\mathsf{P}\{C_I(m)\geqq\alpha n\}$ for $m\sim\gamma\sqrt{n}$, $\gamma>0$.

Theorem 2. *If* $n\to\infty$, $m\sim\gamma\sqrt{n}$, $\gamma>0$, *and* $\alpha\in(0, 1)$ *is fixed, then*

$$M_1(\alpha, \gamma)\leqq \mathsf{P}\{C_I(m)\geqq \alpha n\}\leqq M_2(\gamma), \tag{20}$$

where $M_1(\alpha, \gamma)=1-\mathsf{E}\, e^{-\gamma\mu}$, *and*

$$M_2(\gamma)=\begin{cases} \gamma\sqrt{\pi/2} & \text{for} \quad \gamma\leqq\sqrt{2/\pi}, \\ 1 & \text{for} \quad \gamma>\sqrt{2/\pi}. \end{cases}$$

Proof. In [2, pp. 434—435] Gertsbakh derives the inequality $\mathsf{P}\{C_I(m)\geqq \alpha n\}\leqq m\sqrt{\pi/2n}$ for arbitrary m. Letting in this inequality $m\sim\gamma\sqrt{n}$ we obtain the upper bound $M_2(\gamma)$.

For the investigation of the lower bound of the probability $\mathsf{P}\{C_I(m)\geqq \alpha n\}$ we shall consider the random variable $\eta_{m,s}$ [3], which is equal to the number of bacteria, placed in cyclic elements belonging to components with sizes not less than s. According to the formula of total probability for the distribution of $\eta_{m,s}$, we obtain

$$\mathsf{P}\{\eta_{m,s}=k\}=\sum_{l=0}^{n}\frac{\binom{l}{k}\binom{n-l}{m-k}}{\binom{n}{m}}\mathsf{P}\{\mu_{s,n}=l\}. \tag{21}$$

Using relations $\{\eta_{m,s}\geqq 1\}\subset\{C_I(m)\geqq \alpha n\}$, $\mathsf{P}\{\eta_{m,s}\geqq 1\}=1-\mathsf{P}\{\eta_{m,s}=0\}$, and letting in (21) $k=0$ we obtain, that

$$\mathsf{P}\{C_I(m)\geqq \alpha n\}\geqq 1-\sum_{l=0}^{n}\binom{n-l}{m}\binom{n}{m}^{-1}\mathsf{P}\{\mu_{s,n}=l\}$$

$$=1-\sum_{l=2}^{n}(1-\frac{m}{n})\dots(1-\frac{m}{n-l+1})\mathsf{P}\{\mu_{s,n}=l\}$$

$$>1-\sum_{l=0}^{n}(1-\frac{m}{n})^{l}\mathsf{P}\{\mu_{n,n}=l\}=1-\mathsf{E}(1-\frac{m}{n})^{\mu_{s,n}}=1-\mathsf{E}(a_{m,n})^{\mu_{s,n}/\sqrt{n}},$$

where $a_{m,n}=(1-m/n)^{\sqrt{n}}=(1-\gamma/\sqrt{n}+o(1/\sqrt{n}))^{\sqrt{n}}\to e^{-\gamma}$, $n\to\infty$. The Lebesgue dominated convergence theorem gives the first inequality in (20).

REFERENCES

1. B. Harris. Probability distributions related to random mappings. *Ann. Math. Statist.*, **31**, 1960, 1045—1062.
2. I. B. Gertsbakh. Epidemic process on a random graph: some preliminary results *J. Appl. Probab.*, **14**, 1977, 427—438.
3. L. R. Mutafčiev. Epidemic processes on random graphs and their threshold function. *Serdica*, **7**, 1981, 153—159.
4. А. Гурвиц, Р. Курант. Теория функций. Москва, 1968.
5. J. Riordan. Enumeration of linear graphs for mappings of finite sets. *Ann. Math. Statist.*, **33**, 1962, 178—185.
6. В. Е. Степанов. Предельные распределения некоторых характеристик случайных отображений. *Теория вероятностей и ее применения*, **14**, 1969, 639—653.

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