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SEQUENTIAL ESTIMATION FOR COMPOUND POISSON PROCESS, II

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In the present paper all efficient plans of the compound Poisson process, except those which were determined in Stefanov (1982), are found.

The present paper is a continuation of the paper [5]. The terms and notations from [5] will be used without giving again their definitions.

1. Oblique plans. Theorem 1. *If the sequential plan* (τ, f, h) *is efficient for* $[a_1, b_1]\times[a_2, b_2]$, *then there exist constants* $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, *such that* $\alpha_1^2+\alpha_2^2+\alpha_3^2\neq 0$, $\alpha_4\neq 0$ *and* $\alpha_1 k+\alpha_2 x+\alpha_3 t+\alpha_4=0$ *almost surely with respect to* ν_τ.

Proof. From [5, Theorem 3] follows, that it is sufficient to show, that $\alpha_4\neq 0$. Let us suppose, that

$$\alpha_1 k+\alpha_2 x+\alpha_3 t=0 \tag{1.1}$$

almost surely with respect to ν_τ. Obviously (1.1) is impossible, when $\alpha_1\geqq 0$, $\alpha_2\geqq 0$, $\alpha_3\geqq 0$, or $\alpha_1\leqq 0$, $\alpha_2\leqq 0$, $\alpha_3\leqq 0$. Then we have the following three cases:

a) $k=\alpha_1' t+\alpha_2' x$, where $\alpha_1'>0$

b) $k=\alpha_1' t+\alpha_2' x$, where $\alpha_1'\leqq 0$, $\alpha_2'>0$

c) $0=\alpha_1' t+\alpha_2' x$, where $\alpha_1'>0$, $\alpha_2'<0$.

Consider the case a). Let $\eta_n:=\inf\{t:\ N_t=n\}$. There exists $(\lambda_0, \mu_0)\in[a_1, b_1]\times[a_2, b_2]$ such, that $\alpha_1'/\lambda_0+\alpha_2'/\mu_0:=d>1$ or $d<1$. Let us suppose at first, that $d<1$. Let

$$Z_n:=\alpha_1'\eta_{n+1}+\alpha_2' X_{\eta_n},\quad n=1, 2,\dots \tag{1.2}$$

We have

$$\begin{aligned}&\mathsf{E}_{\lambda_0,\mu_0}(Z_{n+k}-Z_n)=kd,\quad k, n=1, 2,\dots,\\ &\mathsf{D}_{\lambda_0,\mu_0}(Z_{n+k}-Z_n)=\mathsf{E}_{\lambda_0,\mu_0}[Z_{n+k}-Z_n-\mathsf{E}_{\lambda_0,\mu_0}(Z_{n+k}-Z_n)]^2=kr(\alpha_1', \alpha_2', \lambda_0, \mu_0),\end{aligned} \tag{1.3}$$

where $r(\alpha_1', \alpha_2', \lambda_0, \mu_0)$ does not depend on k. It is easy to see that for each $m>1$

$$\mathsf{P}_{\lambda_0,\mu_0}\{N_{\eta_m}-(m-1)>\alpha_1'\eta_{m+1}+\alpha_2' X_{\eta_m},\ N_t\neq\alpha_1' t+\alpha_2' X_t \text{ for } 0<t<\eta_m\}=s(m)>0. \tag{1.4}$$

Let $Z_{n,m}:=Z_n-Z_m-E_{\lambda_0,\mu_0}(Z_n-Z_m)$. Obviously for each $n>m$, $m>1$ we have

$$\mathsf{P}_{\lambda_0,\mu_0}\{\tau>\eta_n\}\geqq s(m)\mathsf{P}_{\lambda_0,\mu_0}\{Z_{m+1}-Z_m<m-1,\dots,\ Z_n-Z_m<n-m-1+m-1\}$$
$$\geqq s(m)\mathsf{P}_{\lambda_0,\mu_0}\{|Z_{m+1,m}|<m-1-d,\dots,|Z_{n,m}|<(n-m)(1-d)+(m-2)\}.$$

Further, as in [5, Theorem 3] we get that there exists m_0 such, that

$$P_{\lambda_0,\mu_0}\{\tau>\eta_n\}\geq s(m_0)P_{\lambda_0,\mu_0}\{(Z_{m_0+1,m_0})^2<(1+A)(D_{\lambda_0,\mu_0}Z_{m_0+1,m_0})^{3/2},\ldots$$
$$\ldots,(Z_{n,m_0})^2<(1+A)(D_{\lambda_0,\mu_0}Z_{n,m_0})^{3/2}\},$$

where $A=\sum_{i=m_0+1}^{\infty}(D_{\lambda_0,\mu_0}Z_{i,m_0})^{-3/2}D_{\lambda_0,\mu_0}Z_{m_0+1,m_0}$. Of course, from (1.3) *we* have $0<A<+\infty$. From Hajek-Renyi-Chow inequality (see [7]) we obtain that for each $n>m_0$

$$P_{\lambda_0,\mu_0}\{\tau>\eta_n\}\geq s(m_0)[1-A/(1+A)]=s>0,$$

where s does not depend on n. Suppose now that $d>1$. Let

$$\tilde{Z}_n:=\alpha_1'\eta_{n-1}+\alpha_2'X_{\eta_n},\quad n=2,3,\ldots \tag{1.5}$$

We have that for each k

$$E_{\lambda_0,\mu_0}(\tilde{Z}_{n+k}-\tilde{Z}_n)=kd,\ D_{\lambda_0,\mu_0}(\tilde{Z}_{n+k}-\tilde{Z}_n)=kr(\alpha_1',\alpha_2',\lambda_0,\mu_0).$$

It is easy to see that for each $m>1$

$$P_{\lambda_0,\mu_0}\left\{\begin{matrix}N_{\eta_m}+m<\alpha_1'\eta_{m-1}+\alpha_2'X_{\eta_m}\\ N_{\eta_m}+m<\alpha_1'\eta_{m-1}+\alpha_2'X_{\eta_{m-1}}\end{matrix},\ N_t\neq\alpha_1't+\alpha_2'X_t\ \text{ for }\ 0<t<\eta_{m-1}\right\} \tag{1.6}$$
$$=\tilde{s}(m)>0.$$

Let $\tilde{Z}_{n,m}:=\tilde{Z}_n-\tilde{Z}_m-E_{\lambda_0,\mu_0}(\tilde{Z}_n-\tilde{Z}_m)$. Obviously, for each $n>m$ we have

$$P_{\lambda_0,\mu_0}\{\tau>\eta_n\}\geq\tilde{s}(m)P_{\lambda_0,\mu_0}\{\tilde{Z}_{m+1}-\tilde{Z}_m>-(m-1),\ldots,\ \tilde{Z}_n-\tilde{Z}_m>n-m-m\}$$
$$\geq\tilde{s}(m)P_{\lambda_0,\mu_0}\{|\tilde{Z}_{m+1,m}|<d+m-1,\ \ldots,\ |\tilde{Z}_{n,m}|<(n-m)(d-1)+m\}$$

Further, as above, we get that there exists $\tilde{m}_0$ such, that $P_{\lambda_0,\mu_0}\{\tau>\eta_n\}>\tilde{s}>0$ for each $n>\tilde{m}_0$, where $\tilde{s}$ does not depend on n.

From the obvious fact (which for instance follows from the law of large numbers) $P_{\lambda_0,\mu_0}\{\eta_n>n(1/\lambda_0-\varepsilon)\}\xrightarrow[n\to\infty]{}1$, $\varepsilon>0$, we get that there exists n_0 such, that for each $n>n_0$ $P_{\lambda_0,\mu_0}\{\tau>n\}>s'>0$ i. e. $P_{\lambda_0,\mu_0}\{\tau=+\infty\}>0$, which implies that the equality (1.1) is not possible.

The case b) can be considered in an analogous way substituting η_{n+1} in (1.2) by η_{n-1} and η_{n-1} in (1.5) by η_{n+1}.

Consider the case c). There exists $(\lambda_0,\mu_0)\in[a_1,b_1]\times[a_2,b_2]$ such, that $d>0$ or $d<0$. Let us suppose at first, that $d<0$. It is easy to see that for each m

$$P_{\lambda_0,\mu_0}\{0>\alpha_1'\eta_{m+1}+\alpha_2'X_{\eta_m}+m,\ \alpha_1't+\alpha_2'X_t\neq0\ \text{ for }\ 0<t<\eta_m\}=\tilde{\tilde{s}}(m)>0.$$

For each $n>m$ we have

$$P_{\lambda_0,\mu_0}\{\tau>\eta_n\}\geq\tilde{\tilde{s}}(m)P_{\lambda_0,\mu_0}\{Z_{m+1}-Z_m<m,\ldots,\ Z_n-Z_m<m\}$$
$$\geq\tilde{\tilde{s}}(m)P_{\lambda_0,\mu_0}\{|Z_{m+1,m}|<m-d,\ldots,|Z_{n,m}|<m-(n-m)d\}.$$

Further, as above, we get $\mathsf{P}_{\lambda_0,\mu_0}\{\tau=+\infty\}>0$. Let $d>0$. It is easy to see that for each $m>1$

$$\mathsf{P}_{\lambda_0,\mu_0}\{0<\alpha_1'\eta_{m-1}+\alpha_2'X_{\eta_m}-m,\ \alpha_1't-\alpha_2'X_t\neq 0 \text{ for } 0<t<\eta_{m-1}\}=\tilde{\tilde{s}}(m)>0.$$

Further in an analogous way as above, we get $\mathsf{P}_{\lambda_0,\mu_0}\{\tau=+\infty\}>0$. It completes the proof of Theorem 1.

It is not difficult to see that some cases in Theorem 1 may be considered without using $\{\eta_n\}_{n\geq 1}$ (see [5, Theorem 3]).

Using the methods from Theorem 1 one can easily get the following

Corollary 1.1. *Let* $\tau_0:=\inf\{t:t>\eta_2,\ \ \alpha_1N_t+\alpha_2X_t+\alpha_3t+\alpha_4\leq 0\}$ *and let* $\alpha_1\lambda_0+\alpha_2\lambda_0/\mu_0+\alpha_3>0$. *Then* $\mathsf{P}_{\lambda_0,\mu_0}\{\tau_0=+\infty\}>0$.

Definition 1.1. *A sequential plan* (τ, f, h) *is called an oblique plan, if* τ *denotes the moment of the first attaining of the set*

$$\{(t,k,x):\ t=ak+bx+c\}, \tag{1.7}$$

where $a\geq 0$, $b\geq 0$, $c>0$, $a+b>0$.

Definition 1.2. *A sequential plan* (τ, f, h) *is called efficient, if there exists* $[a_1, b_1]\times[a_2, b_2]\subset(0,+\infty)\times(0,+\infty)$ *such, that* (τ, f, h) *is efficient for* $[a_1, b_1]\times[a_2, b_2]$.

Using Corollary 1.1 one can easily get the following

Theorem 2. *Efficient plans may be only the simple plans, inverse plans and oblique plans.*

Let $C_{a,b}:=\{(\lambda,\mu):\ \lambda\in(0,+\infty),\ \mu\in(0,+\infty),\ a\lambda+b\lambda/\mu<1\}$.

Theorem 3. *The oblique plan is closed for* $C_{a,b}$, i. e. $\mathsf{P}_{\lambda,\mu}\{\tau<+\infty\}=1$ *for each* $(\lambda,\mu)\in C_{a,b}$, *where* τ *denotes the moment of the first attaining of the set given by* (1.7).

Proof. Let $Y_n:=n-aN_n-bX_n$. It is sufficient to prove that almost surely ($\mathsf{P}_{\lambda,\mu}$, $(\lambda,\mu)\in C_{a,b}$) for sufficiently large n we have $n-aN_n-bX_n>c$ From the law of the iterated logarithm we have

$$\mathsf{P}_{\lambda,\mu}\{\liminf_n \frac{Y_n-(1-a\lambda-b\lambda/\mu)n}{\sqrt{\mathsf{D}_{\lambda,\mu}\,Y_1 2n\ln\ln n}}=-1\}=1. \tag{1.8}$$

Let $\varepsilon>0$. From (1.8) we obtain that

$$\mathsf{P}_{\lambda,\mu}\{\exists n_0,\ \forall n>n_0:\ Y_n>n(1-a\lambda-b\lambda/\mu)-(1+\varepsilon)\sqrt{\mathsf{D}_{\lambda,\mu}\,Y_1 2n\ln\ln n}\}=1.$$

From the facts that $\sqrt{2n\ln\ln n}/n\xrightarrow[n\to\infty]{}0$ and $a\lambda+b\lambda/\mu<1$, it follows $\mathsf{P}_{\lambda,\mu}\{\exists n_1>n_0\ \forall n>n_1:\ Y_n>c\}=1$, which ends the proof of Theorem 3.

Suppose that the oblique plan (τ, f, h) is efficient for (λ,μ). Then from $\mathsf{E}_{\lambda,\mu}N_\tau=\lambda\mathsf{E}_{\lambda,\mu}\tau$, $\mathsf{E}_{\lambda,\mu}N_\tau=\mu\mathsf{E}_{\lambda,\mu}X_\tau$ (see [5]) we have

$$\mathsf{E}_{\lambda,\mu}\tau=c/(1-a\lambda-b\lambda/\mu). \tag{1.9}$$

Of course from (1.9) we have, that (τ, f, h) is not efficient for (λ,μ), when $(\lambda,\mu)\notin C_{a,b}$.

Theorem 4. *The oblique plan is efficient for* $C_{a,b}$. *The only efficiently estimable functions are*

$$h(\lambda,\mu)=\frac{\alpha\lambda+\beta\lambda/\mu+\gamma}{1-a\lambda-b\lambda/\mu}+\delta \tag{1.10}$$

and their only efficient estimators are

(1.11) $$f(u)=\alpha k/c+\beta x/c+\gamma t/c+\delta.$$

Consider the following compound Poisson process $X_t^h:=\Sigma_{i=1}^{N_t}\xi_i^h$, $h>0$, where $\mathsf{P}\{\xi^h=j\}:=\mathsf{P}\{\xi\in[(j-1)h, jh)\}$, $j=1, 2, \ldots$ It is easy to see, that

$$\frac{d\mathsf{P}^h_{\lambda,\mu,t}}{d\mathsf{P}^h_{1,1,t}}=\lambda^{N_t}e^{-\lambda t}(1-e^{-\mu h})^{N_t}e^{-\mu h(X_t-N_t)}[e^{t+h(X_t-N_t)}(1-e^{-h})^{-N_t}],$$

where $\{\mathsf{P}^h_{\lambda,\mu,t}\}_{t\geqq 0}$ are the restrictions of $\mathsf{P}_{\lambda,\mu}$ on $\{\sigma(X_s^h, s\leq t)\}_{t>0}$. Of course (see [5]), there exists a σ-finite measure ν_τ^h on $(V=[0, +\infty)\times\mathcal{N}\times\mathcal{N}, B_V)$ such, that

(1.12) $$P_{\lambda,\mu}\{(\tau, N_\tau, X_\tau^h)\in A\}=\int_A\lambda^ie^{-\lambda t}(1-e^{-\mu h})^ie^{-\mu h(j-i)}d\nu_\tau^h\ (t, i, j),$$

where $A\in B_V$ and τ is a Markov stopping time, such that $P_{\lambda,\mu}\{\tau<+\infty\}=1$. Let $\tau_1:=\inf\{t: aN_t+bhX_t^h+c=t\}$, where a, b, c as in (1.7). Of course, for each h we have

(1.13) $$\tau_1\geqq\tau,\quad hX^h_{\tau_1}\geqq X_\tau,\ N_{\tau_1}\geqq N_\tau,$$

almost surely with respect to $\mathsf{P}_{\lambda,\mu}$ for each $(\lambda,\mu)\in(0, +\infty)\times(0, +\infty)$, where τ denotes the moment of the first attaining of the set given by (1.7) for the process $\{X_t\}_{t\geqq 0}$. Further we shall show that for each $(\lambda_0, \mu_0)\in C_{a,b}$ there exists h such, that

(1.14) $$\mathsf{E}_{\lambda_0,\mu_0}\tau_1^k<+\infty,\ \mathsf{E}_{\lambda_0,\mu_0}N^k_{\tau_1}<+\infty,\ \mathsf{E}_{\lambda_0,\mu_0}(X^h_{\tau_1})^k<+\infty,\quad k=1, 2, \ldots$$

Let $\mathsf{P}_{i,j}(c):=\mathsf{P}_{\lambda,\mu}\{(N_{\tau_1}, X^h_{\tau_1})=(i, j)\}$, $j\geqq i$, $i=1, 2, \ldots$ It is easy to see that

(1.15) $$p_{i,j}(c+\Delta c)=e^{-\lambda\Delta c}p_{i,j}(c)$$
$$+\lambda\Delta ce^{-\lambda\Delta c}\sum_{s=1}^{j-i+1}p_{i-1,j-s}(c+a+bhs)e^{-\mu h(s-1)}(1-e^{-\mu h})+o(\Delta c)$$

From (1.15) we obtain

(1.16) $$p'_{i,j}(c)=-\lambda p_{i,j}(c)$$
$$+\lambda\sum_{s=1}^{j-i+1}p_{i-1,j-s}(c+a+sbh)e^{-\mu h(s-1)}(1-e^{-\mu h}),\quad j\geqq i,\quad i=1, 2, \ldots$$
$$p'_{0,0}(c)=-\lambda p_{0,0}(c)$$

with initial conditions $p_{0,0}(0)=1$, $p_{i,j}(0)=0$, $j\geqq i, i=1, 2, \ldots$ The equality (1.12) shows that the solution of (1.16) is of the form

(1.17) $$p_{i,j}(c)=\lambda^ie^{-\lambda(ai+bhj+c)}(1-e^{-\mu h})^ie^{-\mu h(j-i)}q_{i,j}(c),$$

where $q_{i,j}(c)$ does not depend on (λ,μ). From (1.16) and (1.17) we get

$$(1.18) \qquad q'_{i,j}(c)=\sum_{s=1}^{j-i+1} q_{i-1,j-s}(c+a+sbh), \quad j\geq i, \quad i=1,2,\ldots$$

$$q_{0,0}(c)=1, \quad q_{i,j}(0)=0, \quad j\geq i, \quad i=1,2,\ldots, c\in(0,+\infty).$$

It is easy to show by induction that $q_{i,j}(c)=(j-1)!\,c(c+ai+bhj)^{i-1}/((j-i)!\,(i-1)!\,i!)$, $j\geq i$, $i=1,2,\ldots$, i. e.

$$p_{i,j}(c)=\lambda^i e^{-\lambda(ai+bhj+c)}(1-e^{-\mu h})^i e^{-\mu h(j-i)}(j-1)!\,c(c+ai+bhj)^{i-1}/(j-i)!\,(i-1)!\,i!$$

(1.19)

$$p_{0,0}(c)=e^{-\lambda c}, \quad c\in(0,+\infty), \quad j\geq i, \quad i=1,2,\ldots$$

Let $(\lambda_0,\mu_0)\in C_{a,b}$. It follows that there exists h_1 such, that for each $h\leq h_1$ we have $a\lambda_0+b\lambda_0/\mu_0+h\lambda_0/\mu_0<1$. In an analogous way as in Theorem 3 one can easily show, that for each $h\leq h_1$ we have $\mathsf{P}_{\lambda_0,\mu_0}\{\tau_1<+\infty\}=1$. To show that! (1.14) holds true, it suffices to show that for (λ_0,μ_0) there exists h, $h\leq h_1$ such, that

$$(1.20) \qquad \limsup_{i+j\to\infty}[p_{i,j}(c)]^{1/(i+j)}<1.$$

Let us find all boundary points of the set $P:=\{[p_{i,j}(c)]^{1/(i+j)}: \; j\geq i, \; i=1,2,\ldots\}$. Let $i+j\to\infty$ in such way, that $i/(i+j)\to r$, where $r\in[0,1/2]$. Applying the Stirling's formula $n!\sim(2\pi)^{1/2}n^{n+1/2}e^{-n}$, one can easily see, that

$$\lim_{i+j\to\infty}[p_{i,j}(c)]^{1/(i+j)}:=\varphi_h(r)$$

$$=\lambda^r e^{-\lambda[ar+bh(1-r)]-\mu h(1-2r)}(1-e^{-\mu h})^r[ar+bh(1-r)]^r e^r(1-r)^{1-r}/(1-2r)^{1-2r}r^{2r}.$$

It is easy to see, that $\{\varphi_h(r): r\in[0,1/2]\}$ is the set of all boundary points of the set P. Let us fix r and consider $\varphi_h(r)$ as a function of (λ,μ). Obviously $\varphi_h(r)$ has max only at

$$(1.21) \qquad \lambda^*=r/(ar+bh(1-r)), \quad \mu^*=[\ln(1-r)/(1-2r)]/h, \quad r\in(0,1/2),$$

and that max equals 1. The equalities in (1.21) may be obtained for instance from $\partial\varphi_h(r,\lambda,\mu)/\partial\lambda=0$, $\partial\varphi_h(r,\lambda^*,\mu)/\partial\mu=0$. Consider

$$(1.22) \qquad a\lambda^*+b\lambda^*/\mu^*=\{a+bh/[\ln(1-r)/(1-2r)]\}/(a+bh(1-r)/r).$$

Investigating (1.22), one can easily see, that there exists h_2, $h_2<h_1$ such that for each $r\in(0,1/2)$ we have $a\lambda^*+b\lambda^*/\mu^*>a\lambda_0+b\lambda_0/\mu_0$. It follows that for each $r\in[0,1/2]$ we have $\varphi_{h_2}(r,\lambda_0,\mu_0)<1$. Because $\varphi_h(r)$ is a continuous function (with respect to r), defined on a compact set, it follows that for (λ_0,μ_0) (1.20) holds true, when $h=h_2$.

P r o o f of T h e o r e m 4. We have that

$$(1.23) \qquad t=ak+bx+c$$

almost surely with respect to ν_τ. But (1.23) gives the only linear dependence between t, k, x almost surely with respect to ν_τ, which follows as one can easily see from that for each k_1 and $[x_1,x_2)\subset(0,+\infty)$ we have $\nu_\tau\{(t,k,x): k=k_1, \; x\in[x_1\; x_2), \; t=ak+bx+c\}>0$. From [5, Theorem 2] it is aesy to see, that a necessary condition for that f is efficient estimator is

(1.24) $$f(u)=\alpha k+\beta x+\gamma t+\delta,$$

where α, β, γ, δ are real numbers. We shall show that this condition is sufficient.

From (1.13) and (1.14) it follows, that the regularity conditions (2.6) and (2.7) in [5] hold, when τ is the moment of the first attaining of the set given by (1.7) and f is of the form (1.24). Let $U_1:=N_\tau/\lambda-\tau$, $U_2:=N_\tau/\mu-X_\tau$, where τ denotes the moment of the first attaining of the set given by (1.7). From (1.23) we have $U_1=(1/\lambda-a)N_\tau-bX_\tau-c$, $U_2=N_\tau/\mu-X_\tau$. But for $(\lambda,\mu)\in C_{a,b}$ we have

(1.25) $$\begin{vmatrix} 1/\lambda-a, & -b \\ 1/\mu \quad , & -1 \end{vmatrix}=(a\lambda+b\lambda/\mu-1)/\lambda\neq 0.$$

From (1.25) it follows that the estimators given by (1.24) can be represented as a linear function of U_1, U_2 and from [5, Theorem 2] we get, that they are efficient. We can get (1.10) and (1.11) using (1.9). It completes the proof of Theorem 4.

2. The case, when ξ has a common exponential distribution. Definition 2.1. *By a common exponential distribution we shall mean any distribution which density function is given by the formula* (see [4])

(2.1) $$p(x,Q)=\psi(x)\exp[w_1(Q)+w_2(Q)x],\quad x\in R,$$

with respect io a σ-finite measure, where $Q\in L\subset R$ and L is the widest set for which (2.1) *has sense.*

We shall suppose that

1) $w_1(Q)$ and $w_2(Q)$ are twice continuously differentiable in L,
2) The derivative $w_2'(Q)$ is strictly positive for all Q,
3) $w_1'(Q)/w_2'(Q)$ is strictly decreasing in the whole L.

In the sequel we suppose, that for all $Q\in L$ the relation $-w_1'(Q)/w_2'(Q)=Q$ holds. We shall suppose, that ξ has a common exponential distribution. Under the above assumption we have (see [4]) $E_Q\xi=Q$, $D_Q\xi=1/w_2'(Q)$. It is easy to see, that there exists a σ-finite measure ν_τ on $(U=[0,+\infty)\times\mathcal{N}\times R, B_U)$ independing of (λ, Q) such, that

$$\mathsf{P}_{\lambda,Q}\{(\tau,N_\tau,X_\tau)\in A\}=\int_A\lambda^k e^{-\lambda t}e^{kw_1(Q)+xw_2(Q)}\,d\nu_\tau(t,k,x),$$

where $A\in B_U$ and τ is a Markov stopping time such, that $\mathsf{P}_{\lambda,Q}\{\tau<+\infty\}=1$.

We shall suppose that the function $h(\lambda, Q)$ and the estimator f satisfy the same regularity conditions as those in [5]. In the same way as it is done in [5] we can obtain

$$\mathsf{D}_{\lambda,Q}f\geq\{[\lambda h_\lambda'(\lambda,Q)]^2+[h_Q'(\lambda,Q)]^2/w_2'(Q)\}/\lambda\mathsf{E}_{\lambda,Q}\tau.$$

Equality holds at (λ_0, Q_0), if and only if there exist constants β and γ, not both zero, such that $f(u)-h(\lambda_0,Q_0)=\beta(k/\lambda_0-t)+\gamma(Q_0k-x)$.

Definition 2.1. *A sequential plan (τ,f,h) is called an oblique plan, if τ denotes the moment of the first attaining of the set $\{(t,k,x): t=ak+bx+c\}$, where*

1) $a>0$, $b=0$, $c>0$ if $\mathsf{P}_Q\{\xi>0\}>0$ *and* $\mathsf{P}_Q\{\xi<0\}>0$;
2) $a\geq 0$, $b\geq 0$, $c>0$, $a^2+b^2>0$ *if* $\mathsf{P}_Q\{\xi\geq 0\}=1$;
3) $a\geq 0$, $b\leq 0$, $c>0$, $a^2+b^2>0$ *if* $\mathsf{P}_Q\{\xi\leq 0\}=1$.

Let $C_{a,b}:=\{(\lambda, Q): \lambda \in (0, +\infty), Q \in L, a\lambda+b\lambda Q<1\}$.

Strictly in the same way as it is done in [5] and in the first part of this paper we can get the following theorems:

Theorem 5. *Efficient plans may be only the simple plans, inverse plans and oblique plans.*

Theorem 6. *The simple and inverse plans are efficient for* $(0, +\infty)\times L$. *The oblique plans are not efficient for any* (λ_0, Q_0), *which does not belong to* $C_{a,b}$. *The oblique plans are efficient for* $C_{a,b}$. *The following are the only efficiently estimable functions*

a) *for a simple plan* $h(\lambda, Q)=\alpha\lambda+\beta\lambda Q+\delta$ *and* $f(u)=\alpha k/t_0+\beta x/t_0+\delta$ *is its only efficient estimator;*

b) *for an inverse plan* $h(\lambda, Q)=\alpha/\lambda+\beta Q+\delta$ *and* $f(u)=\alpha t/k_0+\beta x/k_0+\delta$ *is its only efficient estimator;*

c) *for an oblique plan* $h(\lambda, Q)=(\alpha\lambda+\beta\lambda Q+\gamma)/(1-a\lambda-b\lambda Q)+\delta$ *and* $f(u)=\alpha k/c+\beta x/c+\gamma t/c+\delta$ *is its only efficient estimator.*

3. Conclusions. The methods used in this paper and in [5] are rather general and can be applied successfully to determining the efficient plans of many processes. In particular, it is not difficult to show that it holds true for the processes investigated in [1-4; 6].

Acknowledgment. For the introduction into the theory of efficient sequential estimation the author is grateful to S. Trybuła.

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Received 3. 6. 1980