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ON A NEW CHARACTERISTIC OF FUNCTIONS, 1

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The problem of a characterization of the best algebraical approximations in L_p and C with suitable moduli is one of the still insolved ones in the constructive theory of functions. Till now there are some modifications of the usual modulus of continuity which, as we know, do not lead to a satisfying solution of the problem. In this paper we define a new modulus, which can be successfully used as a characterization of the best algebraical approximations. The connections with some other moduli are investigated. Some properties of the new modulus are proved.

1. Introduction. Let H_n be the set of all algebraical polynomials of a degree at most n. The best L_p approximation with the weight W of the function $f \in L_p[a, b]$ by means of elements of H_n is

$$E(H_n, w; f)_{L_p[a,b]} = \inf\{\|w(f-Q)\|_p : Q \in H_n\}.$$

Here with $L_\infty[a, b]$ we denote the set of all measurable and bounded in $[a, b]$ functions, equipped with the uniform norm. We shall denote this norm with $\|\cdot\|_\infty$. Therefore $C[a, b] \subset L_\infty[a, b]$.

Let

$$\omega_k(f; t)_p = \sup\{\|\Delta_h^k f(.)\|_{p[a, b_1]} : 0 \leq h \leq t\}$$

be the k-th modulus of L_p continuity of f, where $b_1 = b$ if f is $b-a$ periodical, and $b_1 = b-kh$, if f is defined in $[a, b]$.

We shall consider only the algebraical approximations in the interval $[-1,1]$. The approximations in the other interval are evidently connected with these ones in $[-1, 1]$. We set $\Delta(d, x) = d\sqrt{1-x^2+d^2}$; $\Delta_n(x) = \Delta(1/n, x)$. The following equivalence is established as a result of the papers of Timan [12], Dzjadik [8], Freud [3] and Brudnij [7] for the case $p=\infty$:

$$E(H_n, (\Delta_n)^{-\alpha}; f)_{L_\infty[-1,1]} = O(1) \quad \Longleftrightarrow$$

$$\omega_k(f; t)_\infty = O(t^\alpha) \ (t \to 0) \quad \text{for} \quad 0 < \alpha < k.$$

This is a characterization of the moduli of continuity (or of the classical Lipschitz spaces) in terms of the best approximations. Later Devore [2] shows that the direct replacement of ∞ with p $(1 \leq p < \infty)$ in the above equivalence is impossible. So the problem of a characterization of the Lipschitz spaces in L_p is not solved till now.

But there is another natural way for the investigation of the connection between the best approximations of one function and its moduli of continuity. The following modulus provides us with a convenient tool for the characterization of the best approximations:

$$\tau_k(f, w; \delta)_{p', p[a, b]} = \|w(\cdot)\omega_k(f, \cdot; \delta(\cdot))_{p'}\|_{p[a, b]}, \tag{1.1}$$

where

$$(1.2)\qquad \omega_k(f, x; \delta(x))_{p'} = \left[\frac{1}{2\delta(x)} \int_{-\delta(x)}^{\delta(x)} |\Delta_v^k f(x)|^{p'} dv\right]^{1/p'},$$

where $\Delta_v^k f(x) = \sum_{k=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+iv)$ is defined as 0, if x or $x+kv$ are not in $[a, b]$. In (1.1) δ is an arbitrary positive function of x and w is a continuous non-negative function.

As far as we know, the local L_p modulus of continuity (1.2) is used for the first time here for $p<\infty$. The defined local L_∞ modulus is a little different from the local modulus of continuity (1.4) used till now. One other modulus using a local modulus in its definition is

$$(1.3)\qquad \tau_k(f; \delta)_p = \|\omega_k(f, \cdot\, ; \delta)\|_p; \quad \delta = \text{const},$$

where

$$(1.4)\qquad \omega_k(f, x; \delta) = \sup\{|\Delta_h^k f(t)| : t, t+kh \in [x-k\delta/2, x+k\delta/2] \cap [a, b]\}.$$

This modulus was used for the first time by Sendov [11] and Korovkin [10] in the case $k=1$. It has already some important applications (see [1], [4], [5], [6]).

We shall give now one condition over the behaviour of the weight w

$$(1.5)\qquad w(x) \leq c(\lambda) w(t) \text{ for } x, t \in [-1,1], \ |x-t| \leq \lambda \Delta(d, x),$$

Let us note, that $w(x) = \Delta^\mu(d, x)$ (real μ) and $w(x) \equiv 1$ in particular satisfy (1.5).

We are ready now to give some direct and converse theorems for the best algebraical approximations. The proofs of these theorems will be given somewhere else.

Theorem 1.1. *If w satisfies (1.5) for $d=1/n$ with $c(\lambda) = O(\lambda^c)\ (\lambda \to \infty)$ for some $c>0$, then for every $k \geq 0$ and for every f with $f^{(k)} \in L_p[-1,1]$ we have*

$$E(H_{n+k}, w; f)_{L_p[-1,1]} \leq c(k) E(H_n, w(\Delta_n)^k; f^{(k)})_{L_p[-1,1]}$$

$$E(H_{n+k}, w; f)_{L_p[-1,1]} \leq c(k) \tau_1(f^{(k)}, w(\Delta_n)^k; \Delta_n)_{1,p}$$

and in particular

$$E(H_{n+k}, 1; f)_{L_p[-1,1]} \leq c(k) E(H_n, (\Delta_n)^k; f^{(k)})_{L_p[-1,1]}$$

$$E(H_{n+k}, 1; f)_{L_p[-1,1]} \leq c(k) \tau_1(f^{(k)}, (\Delta_n)^k; \Delta_n)_{1,p}.$$

Theorem 1.2. *If w satisfies the conditions (1.5) and (1.6) $\|wQ^{(k)}(n\Delta_n)^k\|_p \leq c(k) n_1^k \|wQ\|_p$ for every $Q \in Hn_1$, $n_1 \leq n$, then for every $p' \in [1, p]$ and $f \in L_p$ we have*

$$\tau_k(f, w; \Delta_n)_{p',p} \leq \frac{c(k)}{n^k} \sum_{s=0}^{n} (s+1)^{k-1} E(H_s, w; f)_{L_p[-1,1]}.$$

Using the Konjagin's results in [9], we obtain

Corollary 1.1. *If* $f\in L_p$, $p'\in[1,p]$, $m=0,1,2,\dots$, *then*

$$\tau_k(f,(n\Delta_n)^m;\Delta_n)_{p',p}\leq\frac{c(k,m)}{n^k}\sum_{s=0}^{n}(s+1)^{k-1}E(H_s,(n\Delta_n)^m;f)_{L_p[-1,1]}$$

and in particular

$$\tau_k(f,1,\Delta_n)_{p',p}\leq\frac{c(k)}{n^k}\sum_{s=0}^{n}(s+1)^{k-1}E(H_s,1;f)_{L_p[-1,1]}$$

Theorem 1.1 and Corollary 1.1 give the following equivalence:

$$E(H_n,1;f)_{L_p[-1,1]}=O(n^{-\alpha})\Leftrightarrow$$

$$\tau_1(f,1;\Delta(d))_{p',p}=O(d^{\alpha})\ (d\to 0)\ \text{for}\ 0<\alpha<1,\ p'\in[1,p].$$

Let us note, that everywhere c is an absolute constant, $c(A,B,\dots)$ is a constant depending only on the marced parameters. In general the constants denoted by c or $c(A,B,\dots)$ are different.

2. Auxiliary results. We denote

$$J=\begin{cases}(-\infty,\infty), & \text{if we consider a periodical case;}\\ [a,b], & \text{if we consider the interval } [a,b] \text{ in a non-periodical case}\end{cases}$$

and $J(u,\psi(v))=[u-\psi(v),u+\psi(v)]\cap J$.

Let $\lambda\geq 1$ and $d\in(0,(2\lambda)^{-1}]$. We set

$$(2.1)\quad a_\lambda(d,x)=a(d,x)=a(x)=\begin{cases}\Delta(d,x), & \text{if } |x|\leq\sqrt{1-4\lambda^2d^2};\\ (2\lambda+1)d^2, & \text{if } |x|>\sqrt{1-4\lambda^2d^2}\end{cases}$$

$a(x)$ is defined for each x, while $\Delta(x)$ is defined in $[-1,1]$. Obviously $a(x)$ is continuous. The monotonousness of $\Delta(x)$ in $[-1,0]$ and $[0,1]$ and (2.1) give

$$(2.2)\qquad \Delta(d,x)\leq a_\lambda(d,x)\leq(2\lambda+1)\Delta(d,x)\ \text{for}\ |x|\leq 1$$

$a(x)$ is differentiable for $x\neq\pm\sqrt{1-4\lambda^2d^2}$,

$$a'_\lambda(x)=\begin{cases}-xd/\sqrt{1-x^2}, & \text{if } |x|>\sqrt{1-4\lambda^2d^2};\\ 0, & \text{if } |x|>\sqrt{1-4\lambda^2d^2}.\end{cases}$$

Therefore $\|a'_\lambda\|_\infty\leq d/\sqrt{1-(1-4\lambda^2d^2)}=d/2\lambda d$ or

$$(2.3)\qquad \|a'_\lambda\|_\infty\leq(2\lambda)^{-1}$$

If $|x|\leq 1$ and $|y-x|\leq\lambda\Delta(d,x)$, then

$$|a_\lambda(y)-a_\lambda(x)|\leq\|a'_\lambda\|_\infty|x-y|\leq\frac{1}{2\lambda}\lambda\Delta(x)=\frac{1}{2}\Delta(x)$$

or

$$a_\lambda(x)-\frac{1}{2}\Delta(x)\leq a_\lambda(y)\leq a_\lambda(x)+\frac{1}{2}\Delta(x).$$

The above inequality and (2.2) give

$$(2.4)\qquad \frac{1}{2}\Delta(d,x)\leq a_\lambda(d,y)\leq\left(2\lambda+\frac{3}{2}\right)\Delta(d,x)\ \text{for}\ |x-y|\leq\lambda\Delta(d,x).$$

If also $|y|\leq 1$, (2.2) and (2.4) give

$$\Delta(d, y)\leq a_\lambda(d, y)\leq(2\lambda+3/2)\Delta(d, x) \quad \text{and}$$

$$\Delta(d, x)\leq 2a_\lambda(d, y)\leq(4\lambda+2)\Delta(d, y) \quad \text{or}$$

$$\frac{1}{4\lambda+2}\Delta(d, x)\leq\Delta(d, y)\leq(2\lambda+3/2)\Delta(d, x), \tag{2.5}$$

$$\text{if } |x|, |y|\leq 1,\ |x-y|\leq\lambda\Delta(d, x),\ d\leq(2\lambda)^{-1}.$$

Lemma 2.1. *For each* $x\in[-1, 1]$ *and for each real* γ *we have*

$$I=\int\limits_{J(x,\lambda\Delta(d,x))}\Delta^\gamma(d, y)dy\leq c(\lambda, \gamma)\Delta^{\lambda+1}(dx)$$

for $d\leq(2\lambda)^{-1}$, *where* $c(\lambda, \gamma)=\begin{cases}2\lambda(2\lambda+3/2)^\gamma, & if\ \gamma\geq 0;\\ 2\lambda(4\lambda+2)^{-\gamma}, & if\ \gamma\leq 0.\end{cases}$

Proof. a) $\gamma\geq 0$. Using (2.2) and (2.4), we have

$$I\leq\int\limits_{x-\lambda\Delta(d,x)}^{x+\lambda\Delta(d,x)}(a_\lambda^\gamma(d, y)dy\leq\int\limits_{x-\lambda\Delta(d,x)}^{x+\lambda\Delta(d,x)}[(2\lambda+3/2)\Delta(d, x)]^\gamma dy=2\lambda(2\lambda+3/2)^\gamma\Delta^{\gamma+1}(d, x).$$

b) $\gamma<0$. Using (2.2) and (2.4), we have

$$I\leq\int\limits_{x-\lambda\Delta(d,x)}^{x+\lambda\Delta(d,x)}\frac{a_\lambda^\gamma(d, y)dy}{(2\lambda+1)^\gamma}\leq(2\lambda+1)^{-\gamma}\int\limits_{x-\lambda\Delta(d,x)}^{x+\lambda\Delta(d,x)}\left(\frac{1}{2}\Delta(d, x)\right)^\gamma dy=2\lambda(4\lambda+2)^{-\gamma}\Delta^{\gamma+1}(d, x).$$

Let us consider $\varphi(x)=x+\Delta(x)$. $\varphi'(x)=1-xd/\sqrt{1-x^2}$. φ' decreases in $[0, 1)$. Therefore $\varphi'>0$ in $[0, 1-d^2]$ for $\varphi'(1-d^2)=1-\sqrt{(1-d^2)^2/(2-d^2)}>0$, if $d\leq 1$. Hence φ increases strictly in $[0, 1-d^2]$. If $d\leq 1/2$, we have $\varphi(0)=d+d^2<1$ and $\varphi(1-d^2)=1+d^2\sqrt{2-d^2}>1$. Hence there is unique $x_1\in[0, 1-d^2]$ such that $x_1+\Delta(d, x_1)=1$.

Lemma 2.2. *Let* $\alpha=2+\sqrt{3}$, $d\leq 1/2$ *and*

$$\gamma_1(x)=\begin{cases}x-\Delta(d, x) & if\ -x_1\leq x\leq 1;\\ -1 & if\ -1\leq x\leq -x_1;\end{cases}\qquad \gamma_2(x)=\begin{cases}x+\Delta(d, x) & if\ -1\leq x\leq x_1;\\ 1 & if\ x_1\leq x\leq 1;\end{cases}$$

$$\beta_1(x)=\begin{cases}-1 & if\ -1\leq x\leq -1+\alpha d^2;\\ y,\ where\ \ y+\alpha\Delta(d, y)=x & if\ -1+\alpha d^2\leq x\leq 1;\end{cases}$$

$$\beta_2(x)=\begin{cases}-1 & if\ 1-\alpha d^2\leq x\geq 1;\\ y,\ where\ \ y-\alpha\Delta(d, y)=x & if\ -1\leq x\leq 1-\alpha d^2.\end{cases}$$

Then $\beta_1(x)\leq\gamma_1(x)$ *and* $\gamma_2(x)\leq\beta_2(x)$ *for each* $x\in[-1, 1]$.

Proof. We shall prove only $\beta_1\leq\gamma_1$ because the two inequality are symmetrical. First, we shall show that the definition of β_1 is correct, i. e. the equation $y+\alpha\Delta(d, y)=x$ has unique solution, if $-1+\alpha d^2\leq x\leq 1$. We set $\psi(y)=y+\alpha\Delta(d, y)$.

$$\psi'(y)=1-\alpha yd/\sqrt{1-y^2}.\ \psi'(y)>0, \text{ if } y\in[-1, (1+\alpha^2d^2)^{-1/2}].$$

Therefore, ψ increases strictly in $[-1, (1+\alpha^2d^2)^{-1/2}]$; $\psi(-1)=-1+\alpha d^2$ and $\psi((1+\alpha^2d^2)^{-1/2})=(1+\alpha^2d^2)^{1/2})+\alpha d^2>1$. Hence the definition of β_1 is correct.

1) Let $x\in[-1, -1+\alpha d^2]$. We have $\beta_1(x)=-1$ and $\gamma_1(x)\geqq -1$ for every $x\in[-1, 1]$. Hence $\beta_1(x)\leqq\gamma_1(x)$.

2) Let $x\in[-1+\alpha d^2, 1]$. Then $\gamma_1(x)=x-\Delta(x)$ because $-1+\alpha d^2-\Delta(d, -1+\alpha d^2)=-1+(1+\sqrt{3})d^2(1-(1-\alpha d^2/2)^{1/2})>-1$. We set $y=\beta_1(x)\geqq -1$. Then $\gamma_1(x)=x-\Delta(x)=y+\alpha\Delta(y)-\Delta(y+\alpha\Delta(y))$.
Hence

$$\gamma_1(x)-\beta_1(x)=\alpha(d\sqrt{1-y^2+d^2})-(d\sqrt{1-(y+\alpha d\sqrt{1-y^2}+\alpha d^2)^2+d^2})$$

$$=\alpha d\sqrt{1-y^2}+(\alpha-1)d^2-d\sqrt{1-y^2-2yd\alpha(\sqrt{1-y^2}+d)-\alpha^2d^2(\sqrt{1-y^2}+d)^2}$$

$$\geqq d\sqrt{\alpha^2(1-y^2)+2\alpha(\alpha-1)d\sqrt{1-y^2}+(\alpha-1)^2d^2}$$

$$-d\sqrt{1-y^2-2y\alpha d\sqrt{1-y^2}-2y\alpha d^2}>0$$

because $\alpha^2(1-y^2)=(2+\sqrt{3})^2(1-y^2)\geqq 1-y^2$,

$$2\alpha(\alpha-1)d\sqrt{1-y^2}=(1+\sqrt{3})2\alpha d\sqrt{1-y^2}\geqq -2\alpha yd\sqrt{1-y^2} \quad \text{and}$$

$$(\alpha-1)^2d^2=(1+\sqrt{3})^2d^2=2(2+\sqrt{3})d^2=2\alpha d^2\geqq -2\alpha yd^2 \text{ for } |y|\leqq 1.$$

This completes the proof.

Lemma 2.3. *If* $p\geqq 1$, $d\leqq(2+\sqrt{3})^{-1}\lambda^{-1}$, $g(x)\geqq 0$, $G(x)\geqq 0$ *for* $|x|\leqq 1$, $G(x)=0$ *for* $|x|>1$ *and one of the inequalities* $g(x)\leqq c(\lambda\Delta(d, x))^{1-1/p}[\int_{x-\lambda\Delta(d,x)}^{x+\lambda\Delta(d,x)} G^p(v)\,dy]^{1/p}$ *or* $g(x)\leqq c\int_{x-\lambda\Delta(d,x)}^{x+\lambda\Delta(d,x)} G(y)\,dy$ *holds true for* $|x|\leqq 1$, *then* $[\int_{-1}^{1} g^p(x)\,dx]^{1/p}\leqq c\lambda^2[\int_{-1}^{1}\Delta^p(d, x)G^p(x)dx]^{1/p}$

Proof. The Hölder's inequality gives

$$\int_{x+\lambda\Delta}^{x-\lambda\Delta} G(y)dy\leqq(2\lambda\Delta)^{1-1/p}[\int_{x-\lambda\Delta}^{x+\lambda\Delta} G^p(y)dy]^{1/p}$$

Hence we shall consider the first inequality holds true. Then using Lemma 2.1 and Lemma 2.2, we have

$$[\int_{-1}^{1} g^p(x)dx]^{1/p}\leqq[\int_{-1}^{1} c^p[\lambda\Delta(x)]^{p-1}\int_{x-\lambda\Delta(d,x)}^{x+\lambda\Delta(d,x)} G^p(y)dy\,dx]^{1/p}$$

$$=c[\int_{-1}^{1}\int_{x-\lambda\Delta(d,x)}^{x-\lambda\Delta(d,x)} G^p(y)\lambda^{p-1}\Delta^{p-1}(d, x)dy\,dx]^{1/p}$$

$$\leqq c[\int_{-1}^{1} G^p(y)\lambda^{p-1}(\int_{J(y,\alpha\lambda\Delta(d,y))}\Delta^{p-1}(d, x)dx)dy]^{1/p}$$

$$\leqq c[\int_{-1}^{1} G^p(y)\lambda^{p-1}2\lambda(2\lambda+3/2)^{p-1}\Delta^{p-1+1}(d, y)dy]^{1/p}$$

$$\leqq c\lambda^2[\int_{-1}^{1}\Delta^p(d, y)G^p(y)dy]^{1/p},$$

Now we shall prove an identity with finite differences.

$$\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\Delta^{k}_{h+r(\varepsilon-h)/k}f(x)$$

$$=\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(x+i(h+r(\varepsilon-h)/k))$$

$$=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}f(x+ih+ri(\varepsilon-h)/k)$$

$$=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\Delta^{k}_{i(\varepsilon-h)/k}f(x+ih).$$

Separating from the left-hand side the term for $r=0$, we have

$$(2.6)\qquad (-1)^k\Delta^k_h f(x)=\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}\left\{\Delta^k_{i(\varepsilon-h)/k}f(x+ih)-\Delta^k_{h+i(\varepsilon-h)/k}f(x)\right\}$$

or in the right-hand side we have $\Delta^k_0 f(x)=0$ for $i=0$. Hence

$$(2.7)\qquad |\Delta^k_h(f(x)|\leq\sum_{i=1}^{k}\binom{k}{i}\{|\Delta^k_{i(\varepsilon-h)/k}f(x+ih)|+|\Delta^k_{h+i(\varepsilon-h)/k}f(x)|\}.$$

The identity (2.6) enables us to present the finite difference of the order k with a fix step h as a sum of finite differences of the same order:

a) with less steps (if $0<\varepsilon<h$);

b) with steps depending on a parameter running some interval.

Let $p\geq 1$. Then for u, $v\geq 0$, we have $((u+v)/2)^p\leq(u^p+v^p)/2$ or

$$(2.8)\qquad u+v\leq 2^{1-1/p}(u^p+v^p)^{1/p}$$

Also $(u+v)^{1/p}\leq u^{1/p}+v^{1/p}$ because the function $x^{1/p}$ is concave in $[0,\infty)$. Hence by induction we have

$$(2.9)\qquad \left(\sum_{i=1}^{r}u_i\right)^{1/p}\leq\sum_{i=1}^{r}u_i^{1/p}\ \text{ if } u_i\geq 0 \text{ for } i=1,2,\dots,r.$$

3. A connection between $\tau_k(f,w;\delta)_{p',p}$ and some other moduli. Exept the usual norm in L_p

$$(3.1)\qquad \|f\|_{p[a,b]}=\left[\int_a^b|f(x)|^p\,dx\right]^{1/p},$$

we shall use its following modification

$$(3.2)\qquad \|f\|'_{p[a,b]}=\left[\frac{1}{b-a}\int_a^b|f(x)|^p\,dx\right]^{1/p}.$$

It is more convenient for us to work only with (3.2), but the usual definitions of moduli of continuity are connected with (3.1). When it is necessary, we shall note outside the variable in the normalized space like this $\|f(x)\|_{(x)p[a,b]}$. Using (3.2) and Hölder's inequality, we obtain

$$\|f\|_{p_1} \leq \|f\|'_{p_2}, \text{ if } 1 \leq p_1 \leq p_2,\ f \in L_{p_2}. \tag{3.3}$$

Theorem 3.1. *If* $\delta = \text{const} > 0$ ($\delta \leq (b-a)/2k$ *in a non-periodical case*), $f \in L_p[a, b]$, $p' \in [1, p]$, *then*

$$\tau_k(f, 1; \delta)_{p', p[a,b]} \leq \omega_k(f; \delta)_{p[a,b]} \leq c(k)\tau_k(f, 1; \delta)_{p', p[a,b]}. \tag{3.4}$$

Proof. a) The periodical case: $[a, b] = [0, 2\pi]$. (2.7) for $0 \leq h \leq \delta$ gives

$$|\Delta_h^k f(x)| = \|\Delta_h^k f(x)\|'_{(\varepsilon)p'[0,\delta]} \tag{3.5}$$

$$\leq \sum_{i=1}^{k} \binom{k}{i} [\|\Delta^k_{i(\varepsilon-h)/k} f(x+ih)\|'_{(\varepsilon)p'[0,\delta]} + \|\Delta^k_{h+i(\varepsilon-h)/k} f(x)\|'_{(\varepsilon)p'[0,\delta]}]$$

$$= \sum_{i=1}^{k} \binom{k}{i} \left\{ \left[\frac{1}{\delta} \int_0^\delta |\Delta^k_{i(\varepsilon-h)/k} f(x+ih)|^{p'} d\varepsilon \right]^{1/p'} + \left[\frac{1}{\delta} \int_0^\delta |\Delta^k_{h+i(\varepsilon-h)k} f(x)|^{p'} d\varepsilon \right]^{1/p'} \right\}$$

$$= \sum_{i=1}^{k} \binom{k}{i} \binom{k}{i}^{1/p'} \left\{ \left[\frac{1}{\delta} \int_{-ih/k}^{k} |\Delta_v^k f(x+ih)|^{p'} dv \right]^{1/p'} + \left[\frac{1}{\delta} \int_{(k-i)h/k}^{h+i(\delta-h)k} |\Delta_v^k f(x)|^{p'} dv \right]^{1/p'} \right\}$$

$$\leq \sum_{i=1}^{k} c(k, i) \left\{ \left[\frac{1}{\delta} \int_{-\delta}^{\delta} |\Delta_v^k f(x+ih)|^{p'} dv \right]^{1/p'} + \left[\frac{1}{\delta} \int_0^\delta |\Delta_v^k f(x)|^{p'} dv \right]^{1/p'} \right\}$$

$$\leq \sum_{i=0}^{k} (c(k, i)\, \omega_k(f, x+ih; \delta)_{p'}$$

(3.5) for f is 2π periodical gives

$$\|\Delta_h^k f(x)\|_{(x)p[0,2\pi]} \leq c(k)\tau_k(f, 1; \delta)_{p'p,[0,2\pi]} \text{ for } 0 \leq h \leq \delta. \tag{3.6}$$

The definition of $\omega_k(f; \delta)_p$ and (3.6) give the second inequality in (3.4). Using (3.3), we get

$$\tau_k(f, 1; \delta)^p_{p',p} \leq \tau_k(f, 1; \delta)^p_{p,p} = \int_0^{2\pi} \frac{1}{2\delta} \int_{-\delta}^{\delta} |\Delta_v^k f(x)|^p dv\, dx$$

$$= \frac{1}{2\delta} \int_{-\delta}^{\delta} \left(\int_0^{2\pi} |\Delta_v^k f(x)^p dx \right) dv \leq \frac{1}{2\delta} \int_{-\delta}^{\delta} \omega_k(f; \delta)^p_p dv = \omega_k(f; \delta)^p_p.$$

The above inequality gives the first one in (3.4). (3.4) can be written in the form

$$\tau_k(f, 1; \delta)_{p',p} \asymp \omega_k(f; \delta)_p \text{ for } p' \in [1, p]. \tag{3.7}$$

As a corollaly of (3.7) we have

$$\tau_k(f, 1; \delta)_{p',p} \asymp \tau_k(f, 1; \delta)_{p'',p} \text{ for } p', p'' \in [1, p]. \tag{3.8}$$

b) The non-periodical case. We have to take care in this case because $\Delta_v^k f(x)$ is defined a priori iff $x, x+kv \in [a, b]$. Let us remember that for the other x, v we have set $\Delta_v^k f(x) = 0$.

Let $x, x+kh \in [a, b]$ and $h \in [0, \delta]$.

1) Let $x \in [a, (a+b)/2]$. We consider (2.7).
$x+ih \leq x+k\delta \leq (a+b)/2+k(b-a)/2k=b$. Also for $\varepsilon \in [0, \delta]$ we have $x+ih+ki(\varepsilon-h)/k=x+i\varepsilon \leq (b+a)/2+k(b-a)/2k=b$; $x+i\varepsilon \geq x \geq a$ and $x+k(h+i(\varepsilon-h)/k)=x+(k-i)h+i\varepsilon \leq x+k\delta \leq b$.

Hence $\Delta^k_{i(\varepsilon-h)/k} f(x+ih)$ and $\Delta^k_{h+i(\varepsilon-h)/k} f(x)$ are defined and (2.7) gives

$$(3.9)\quad |\Delta^k_h f(x)| \leq \sum_{i=1}^{k} \binom{k}{i} \{ \| \Delta^k_{i(\varepsilon-h)/k} f(x+ih) \|'_{(\varepsilon)p'[0,\delta[} + \| \Delta^k_{h+i(\varepsilon-h)/k} f(x) \|'_{(\varepsilon)p'[0,\delta]} \}$$

2) Let $x \in [(a+b)/2, b]$. From $|\Delta^k_h f(x)| = |\Delta^k_{-h} f(x+kh)|$ we get

$$(3.10)\quad |\Delta^k_h f(x)| \leq \sum_{i=1}^{k} \binom{k}{i} \{ |\Delta^k_{i(\varepsilon+h)/k} f(x+(k-i)h)| + |\Delta^k_{-h+i(\varepsilon+h)/k} f(x+kh)| \}$$

Let $\varepsilon \in [-\delta, 0]$. Then

$$y=x+(k-i)h+ki(\varepsilon+h)/k=x+kh+i\varepsilon \leq x+kh \leq b$$

$$y \geq x+k\,.\,0-k\delta \geq (a+b)/2-k(b-a)/2k=a$$

$$z=x+kh+k(-h+i(\varepsilon+h)/k)=x+i(\varepsilon+h) \leq x+ih \leq x+kh \leq b$$

$$z \geq x-k\delta \geq a.$$

Hence $\Delta^k_{i(\varepsilon+h)/k} f(x+(k-i)h$ and $\Delta^k_{-h+i(\varepsilon+h)k} f(x+kh)$ are defined and (3.10) gives

$$(3.11)\quad |\Delta^k_h f(x) \leq \sum_{i=1}^{k} \binom{k}{i} \{ \|\Delta^k_{i(\varepsilon+h)/k} f(x+(k-i)h) \|'_{(\varepsilon)p'[-\delta,0]} + \|\Delta^k_{-h+i(\varepsilon+h)/k} f(x+kh)\|'_{(\varepsilon)p'[-\delta,0]} \}$$

By analogy with (3.5) from (3.9) and (3.11) we have

$$(3.12)\quad |\Delta^k_h f(x)| \leq \sum_{i=0}^{k} c(k,i) \omega_k(f, x+ih; \delta)_{p'}.$$

From (3.12) we get

$$(3.13)\quad \omega_k(f; \delta)_{p[a,b]} = \sup_{0 \leq h \leq \delta} \|\Delta^k_h f(x)\|_{(x)p[a,b-kh]}$$

$$= \sup_{0 \leq h \leq \delta} \Big[\int_{-\infty}^{\infty} |\Delta^k_h f(x)|^p dx \Big]^{1/p} \leq \sup_{0 \leq h \leq \delta} \sum_{i=0}^{k} c(k,i) \Big[\int_{-\infty}^{\infty} \omega^p_k(f, x+ih; \delta)_{p'} dx \Big]^{1/p}$$

$$= \sup_{0 \leq h \leq \delta} \sum_{i=0}^{k} c(k,i) \Big[\int_{-\infty}^{\infty} \omega^p_k(f; x; \delta)_{p'} dx \Big]^{1/p} = c(k) \Big[\int_{-\infty}^{\infty} \omega^p_k(f, x; \delta)_{p'} dx \Big]^{1/p}$$

$$= c(k) \Big[\int_a^b \omega^p_k(f, x; \delta)_{p'} dx \Big]^{1/p} = c(k) \tau_k(f, 1; \delta)_{p', p[a,b]}$$

because for $x \bar{\in} [a, b]$ $\omega_k(f, x; \delta)_{p'} = 0$. We now obtain the first inequality in (3.4) as in the periodical case and this completes the proof. Let us note that the equivalences (3.7) and (3.8) hold also true for the non-periodical case.

Let us now consider the connection between $\tau_k(f, w;\delta)_{\infty,p}$ and $\tau_k(f;\delta)_p$ for $\delta=\text{const}$.

1) $k=1$. Using the definitions, we have

$$\omega_1(f, x;\delta)_\infty=\sup\{|f(x+t)-f(x)|:|t|\le\delta,\ x+t\in J\}$$
$$\le\sup\{|f(t_1)-f(t_2)|:t_1, t_2\in[x-\delta, x+\delta]\cap J\}=\omega_1(f, x;2\delta).$$

Therefore

$$\tau_1(f, 1;\delta)_{\infty,p}\le\tau_1(f;2\delta)_p\le2\tau_1(f;\delta)_p \tag{3.14}$$

On the other hand

$$\omega_1(f, x;\delta)=\sup\{|f(t_1)-f(t_2):t_1, t_2\in[x-\delta/2, x+\delta/2]\cap J\}$$
$$\le\sup\{|f(t_1)-f(x)|+|f(t_2)-f(x)|:t_1, t_2\in[x-\delta/2, x+\delta/2]\cap J\}$$
$$\le2\sup\{|f(x)-f(t):|t-x|\le\delta/2,\ t\in J\}=2\omega_1(f, x;\delta/2)_\infty$$

Therefore

$$\tau_1(f;\delta)_p\le2\tau_1(f, 1;\delta/2)_{\infty,p}\le2\tau_1(f, 1;\delta)_{\infty,p} \tag{3.15}$$

(3.14) and (3.15) gives

$$\frac{1}{2}\tau_1(f, 1;\delta)_{\infty,p}\le\tau_1(f;\delta)_p\le2\tau_1(f, 1;\delta)_{\infty,p}\ \text{for}\ 1\le p\le\infty, \tag{3.16}$$

2) $k\ge2$. This case is more complicated. We have

$$\omega_k(f, x;\delta)_\infty=\sup\{|\Delta_h^k f(x)|:x+kh\in[x-k\delta, x+k\delta]\cap J\}$$
$$\le\sup\{|\Delta_h^k f(t):t, t+kh\in[x-k\delta, x+k\delta]\cap J\}=\omega_k(f, x;2\delta),$$

Therefore $\tau_k(f, 1;\delta)_{\infty,p}\le\tau_k(f;2\delta)_p$ or

$$\tau_k(f, 1;\delta)_{\infty,p}\le c(k)\tau_k(f;\delta)_p. \tag{3.17}$$

We do not know what is the situation with the converse inequality, i. e. whether the inequality

$$\tau_k(f;\delta)_p\le c(k)\tau_k(f, 1;\delta)_{\infty,p}$$

is true or not for $k\ge2$ and for every $f\in L_\infty[a, b]$.

4. Properties of $\tau_k(f, w;\delta)_{p',p}$. We shall first give some obvious properties following directly from definitions, the Hölder's inequality and the linearity of the operator Δ_v^k.

$$\tau_k(f+g, w;\delta)_{p',p}\le\tau_k(f, w;\delta)_{p',p}+\tau_k(g, w;\delta)_{p',p}\ \text{for}\ f, g\in L_{\max\{p',p\}}; \tag{4.1}$$

$$\tau_k(\alpha f, w;\delta)_{p',p}=|\alpha|\,\tau_k(f, w;\delta)_{p',p}\ \text{for}\ \alpha\in\mathbb{R},\ f\in L_{\max\{p',p\}}; \tag{4.2}$$

$$\tau_k(f, w;\delta)_{p',p}\le\tau_k(f, w_2;\delta)_{p',p}\ \text{for}\ 0\le w_1\le w_2,\ f\in L_{\max\{p',p\}}; \tag{4.3}$$

$$\tau_k(f, w;\delta)_{p_1',p}\le\tau_k(f, w;\delta)_{p_2',p}\ \text{for}\ 1\le p_1'\le p_2'; \tag{4.4}$$

$$\tau_k(f, w;\delta)_{p',p_1}\le(b-a)^{\frac{1}{p_1}-\frac{1}{p_2}}\tau_k(f, w;\delta)_{p',p_2}\ \text{for}\ 1\le p_1\le p_2. \tag{4.5}$$

We shall give without proofs some properties in the case $\delta=\text{const}$. These properties follow immediately with worse constants from Theorem 3.1 and properties of $\omega_k(f;\delta)_p$. Let $w=1$.

$$\tau_1(f, 1; n\delta)_{1,p}\leq n\tau_1(f, 1; \delta)_{1,p} \qquad \text{for integer } n;$$

$$\tau_1(f, 1; \alpha\delta)_{1,p}\leq([\alpha]+3)\tau_1(f, 1; \delta)_{1,p} \quad \text{for real } \alpha\geq 1;$$

$$\liminf_{\delta\to 0} \frac{\tau_1(f, 1; \delta)_{1,p}}{\delta}\geq\frac{\tau_1(f, 1; \delta_1)_{1,p}}{\delta_1} \quad \text{for every } \delta_1>0;$$

$$\frac{\tau_1(f, 1; \delta_2)_{1,p}}{\delta_2}\leq 4\,\frac{\tau_1(f, 1; \delta_1)_{1,p}}{\tau_1} \quad \text{for } \delta_2>\delta_1;$$

$$\tau_k(f, 1; \delta)_{p',p}\leq c(k)\delta\tau_{k-1}(f', 1; \delta)_{p'',p} \text{ for every } p', p''\geq 1;\ k\geq 2;\ f'\in L_{\max\{p'',p\}}.$$

We now concentrate our attention over the case $\delta(x)=\Delta(d, x)$, which is main in the best algebraical approximations as Theorem 1.1 and Theorem 1.2 show.

In the following theorems w will satisfy (1.5). Obviously $w=\text{const}$ satisfies (1.5) with $c(\lambda)=1$. (2.5) shows that $w(x)=\Delta^{\mu}(d, x)$ satisfies (1.5) with $c(\lambda, \mu)=(4\lambda+2)^{\mu}$, if $\mu\geq 0$ and $(2\lambda+3/2)^{-\mu}$, if $\mu<0$. The condition (1.5) enables w to depend on the parameter d. w can be more irregular, when d decreases, but always w must be greater than 0.

We set $\Delta(d, x)=d^2$, if $|x|>1$ just to take care whether $|x|$ is greater or less than 1.

Theorem 4.1. *If w satisfies (1.5), $d\leq(2+\sqrt{3})^{-1}k^{-1}$, $1\leq p'\leq p$, $f\in L_p$ then*

$$\tau_k(f, w; \Delta(d))_{p',p}\leq c(k)\,\|wf\|_p. \tag{4.6}$$

Proof. We set

$$\tilde g(x)=\begin{cases} f(x), & \text{if } |x|\leq 1;\\ 0, & \text{if } |x|>1;\end{cases} \qquad \tilde w(x)=\begin{cases} w(x), & \text{if } |x|\leq 1;\\ w(1), & \text{if } x>1;\\ w(-1), & \text{if } x\leq -1;\end{cases}$$

Then $\tilde w$ satisfies (1.5) for every x, t such, that $|x-t|\leq\lambda\Delta(d, x)$.

$$\tau_k(f, w; \Delta(d))_{p',p}\leq\|\, w(x)\,\|\,\Delta_v^k\tilde g(x)\,\|'_{(v)p[-\Delta(d,x),\Delta(d,x)]}\,\|_{p[-1,1]} \tag{4.7}$$

$$=\|\, w(x)\,\|\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\tilde g(x+rv)\,\|'_{(v)p[-\Delta(x),\Delta(x)]}\,\|_{p[-1,1]}$$

$$\leq\sum_{r=0}^{k}\binom{k}{r}\|\, w(x)\,\|\,\tilde g(x+rv)\,\|'_{(v)p[-\Delta(x),\Delta(x)]}\,\|_{(x)p[-1,1]}\,.$$

For $r\neq 0$ we have

$$\|\,\tilde g(x+rv)\,\|'_{(v)p[-\Delta,\Delta]}=\left[\frac{1}{2\Delta}\int_{-\Delta}^{\Delta}|\tilde g(x+rv)|^p\,dv\right]^{1/p}=\left[\frac{1}{2r\Delta}\int_{-r\Delta}^{r\Delta}|\tilde g(x+t)|^p dt\right]^{1/p} \tag{4.8}$$

$$=\|\,\tilde g(x+t)\,\|'_{(t)p[-r\Delta,r\Delta]}\leq\frac{k}{r}\,\|\,\tilde g(x+t)\,\|'_{(t)p[-k\Delta,k\Delta]}$$

Using (4.7), (4.8) and (1.5) for $\tilde{w}$ and Δ^{-1}, we have

$$(4\text{-}9)\quad \tau_k(f, w;\ \Delta(d))_{p',p} \leqq \| w\tilde{g}\|_{p[-1,1]} + c(k)\,\|\,\tilde{w}(x)\,\|\tilde{g}(x+t)\|'_{(t)p[-k\Delta(x),k\Delta(x)]}\,\|_{(x)p[-1,1]}$$

$$\leqq \| wf\|_p + c(k)\,\|\,k\Delta(d,x)\,\|\frac{\tilde{w}(x+t)}{k\Delta(d,x+t)}\tilde{g}(x+t)\|'_{(t)p[-k\Delta(x),k\Delta(x)]}\,\|_{(x)p[-1,1]}$$

$$= \| wf\|_p + c(k)\,\|\,k\Delta(d,x)\,\|\frac{\tilde{w}(y)\tilde{g}(y)}{k\Delta(d,y)}\|'_{(y)p[x-k\Delta(x),x+k\Delta(x)]}\,\|_{(x)p[-1,1]}.$$

Applying in (4.9) Lemma 2.3 with $G=\frac{\tilde{w}\,|\,\tilde{g}\,|}{k\,.\,\Delta(d)}$, $\lambda=k$ and $g(x)=k\Delta(d,x)\,\|G\|'_{p[x-k\Delta(x),x+k\Delta(x)]}$, we obtain

$$\tau_k(f, w;\ \Delta(d))_{p',p} \leqq \| wf\|_{p[-1,1]} + c(k)\,\|\,\Delta(d,x)\,\frac{\tilde{w}(x)\,|\,\tilde{g}(x)\,|}{k\Delta(d,x)}\|_{p[-1,1]} = c(k)\,\| wf\|_{p[-1,1]}.$$

Lemma 2.3 limits d as follows $d\leqq(2+\sqrt{3})^{-1}\lambda^{-1}=(2+\sqrt{3})^{-1}k^{-1}$.
This completes the proof.

Corollary 4.1. *If* $f\in L_p$, $1\leqq p'\leqq p$, $\mu\in\mathbf{R}$, $d\leqq 1/(2+\sqrt{3})k$ *then* $\tau_k(f, \Delta^{\mu}(d)\ \Delta(d))_{p',p}\leqq c(k,\mu)\,\|\Delta^{\mu}(d)f\|_p$.

Theorem 4.2. *If* w *satisfies (1.5),* $f'\in L_p$, $1\leqq p'\leqq\infty$, *and* $d\leqq(2+\sqrt{3})^{-1}$ *then*

$$(4.10)\qquad \tau_1(f, w;\ \Delta(d))_{p',p}\leqq c\,\| w\Delta(d)f'\|_{p[-1,1]}.$$

Proof. Let $f'(x)=0$ for $|x|>1$. Using (1.2) and properties of the finite differences, we have

$$(4.11)\quad \omega_1(f_1, x;\ \Delta(d,x))_{p'}=\|\Delta_v f(x)\|'_{(v)p'[-\Delta,\Delta]}\leqq\|\int_0^v f'(x+t)dt\|'_{(v)p'[-\Delta,\Delta]}$$

$$\leqq\|\int_{-\Delta(d,x)}^{\Delta(d,x)}|f'(x+t)|\,dt\|'_{(v)p'[-\Delta(x),\Delta(x)]}=\int_{x-\Delta(d,x)}^{x+\Delta(d,x)}|f'(y)|dy,$$

Lemma 2.3 with $\lambda=1$ and (4.11) prove (4.10).

Corollary 4.2. *If* $f'\in L_p$, $\mu\in\mathbf{R}$, $d\leqq(2+\sqrt{3})^{-1}$, $1\leqq p'\leqq\infty$ *then* $\tau_k(f, \Delta^{\mu}(d);\ \Delta(d))_{p',p}\leqq c(\mu)\,\|\Delta^{\mu+1}(d)f'\|_{p[-1,1]}$.

Theorem 4.3. *If* w *satisfies (1.5),* $f'\in L_{\max\{p',p\}}$, $p, p', p''\geqq 1$, $k\geqq 2$, $d\leqq(2k)^{-1}$, *then*

$$(4.12)\qquad \tau_k(f, w;\ \Delta(d))_{p'',p}\leqq c(k)\tau_{k-1}(f', w\Delta(d),\ \Delta((4k+2)d))_{p',p}.$$

Lemma 4.1. *If* $f'\in L_1$, $(\delta\leqq(b-a)/2k$ *in the non-periodical case), then*

$$\omega_k(f, x;\ \delta)_\infty\leqq c(k)\int_{x-k\delta}^{x+k\delta}\omega_{k-1}(f', y;\ \delta)_1 dy.$$

Proof. $\Delta_h^k f(x)=\int_0^h\Delta_h^{k-1}f'(x+t)dt$. We set $\Delta_h^{k-1}f'(y)=0$ if $y,\ y+(k-1)h\,\bar{\in}\,[a,b]$ and $m=k-1$.

a) $0 \leq h \leq \delta$. Then

$$(4.13)\qquad |\Delta_h^k f(x)| \leq \int_0^\delta |\Delta_h^m f'(x+t)|\, dt.$$

1) $x+t \in [a, (a+b)/2]$. Then the formula of the type (2.7) holds true for $\Delta_h^m f'(x+t)$, i. e.

$$|\Delta_h^m f'(x+t)| \leq \sum_{i=1}^m \binom{m}{i} \{ |\Delta_{i(\varepsilon-h)/m}^m f(x+t+ih)| + |\Delta_{h+i(\varepsilon-h)/m)}^m f(x+t)| \}.$$

We set $\chi_1(x) = \begin{cases} 1 & \text{for } x \in [a, (a+b)/2]; \\ 0 & \text{otherwise.} \end{cases}$ Then

$$(4.14)\qquad \delta \int_0^\delta \chi_1(x+t) |\Delta_h^{k-1} f'(x+t)|\, dt = \int_0^\delta \int_0^\delta \chi_1(x+t) |\Delta_h^m f'(x+t)|\, dt\, d\varepsilon$$

$$\leq \sum_{i=1}^m \binom{m}{i} \int_0^\delta \int_0^\delta \{ |\Delta_{i(\varepsilon-h)/m}^m f'(x+t+ih)| + |\Delta_{h+i(\varepsilon-h)/m}^m f'(x+t)| \} + d\varepsilon\, dt$$

$$= \sum_{i=1}^m \binom{m}{i} \frac{m}{i} \{ \int_{x+ih}^{x+ih+\delta} \int_{-ih/m}^{i(\delta-h)/m} + \int_x^{x+\delta} \int_{(m-i)h/m}^{(m-i)h/m+i\delta/m} \} |\Delta_v^m f'(y)|\, dv\, dy$$

$$\leq c(m) \int_x^{x+k\delta} \int_{-\delta}^{\delta} |\Delta_v^m f'(y)|\, dv\, dy = 2\delta\, c(m) \int_x^{x+k\delta} \omega_{k-1}(f', y;\ \delta)_1 dy$$

2) $x+t \in [(a+b)/2, b]$ and $x+t+(k-1)h \leq b$ ((4.16) is obviously true, if $x+t+(k-1)h > b$). From (4.13) we obtain

$$(4.15)\qquad |\Delta_h^k f(x)| \leq \int_0^\delta |\Delta_{-h}^m f'(x+(k-1)h+t)|\, dt$$

From (2.7) we have

$$(4.16)\qquad |\Delta_{-h}^m f'(x+mh+t)| \leq \sum_{i=1}^m \{ |\Delta_{i(\varepsilon+h)/m}^m f'(x+(m-i)h+t)| + |\Delta_{-h+i(\varepsilon+h)/m}^m f'(x+mh+t)| \}$$

We set $\chi_2(x) = \begin{cases} 1 & \text{for } x \in [(a+b)/2, b] \\ 0 & \text{otherwise} \end{cases}$. Then

$$(4.17)\qquad \delta \int_0^\delta \chi_2(x+t) |\Delta_{-h}^m f'(x+mh+t)|\, dt$$

$$\leq \sum_{i=1}^m \binom{m}{i} \int_0^\delta \int_{-\delta}^0 \{ |\Delta_{i(\varepsilon+h)/m}^m f'(x+(m-i)h+t)| + |\Delta_{-h+i(\varepsilon+h)/m}^m f'(x+mh+t)| \} d\varepsilon\, dt$$

$$= \sum_{i=1}^m \binom{m}{i} \frac{m}{i} [\int_{x+(m-i)h}^{x+(m-i)h+\delta} \int_{i(-\delta+h)/m}^{ih/m} + \int_{x+mh}^{x+mh+\delta} \int_{(-m+i)h/m-i\delta/m}^{(-m+i)h/m}] |\Delta_v^m f'(y)|\, dv\, dy$$

$$\leq c(m) \int_x^{x+k\delta} \int_{-\delta}^{\delta} |\Delta_v^m f'(y)|\, dy = 2\delta c(m) \int_x^{x+k\delta} \omega_{k-1}(f', y;\ \delta)_1 dy$$

(4.13), (4.14), (4.15) and (4.17) give

$$|\Delta_h^k f(x)| \leqq c(k) \int_x^{x+k\delta} \omega_{k-1}(f', y;\ \delta)_1 dy \quad \text{for } 0 \leqq h \leqq \delta.$$

b) $-\delta \leqq h \leqq 0$. We consider this case as the case a) and we obtain

$$|\Delta_h^k f(x)| \leqq c(k) \int_{x-k\delta}^{x} \omega_{k-1}(f', y;\ \delta)_1 dy.$$

This completes the proof of Lemma 4.1.

Proof of Theorem 4.3. (4.4) shows that it is sufficient to prove the theorem in the case $p''=\infty$, $p'=1$, Lemma 4.1 with $\delta=\Delta(d, x)$ gives

$$\omega_k(f, x;\ \Delta(d, x))_\infty \leqq c(k) \int_{x-k\Delta(d,x)}^{x+k\Delta(d,x)} \omega_{k-1}(f', y;\ \Delta(d, x))_1 dy. \tag{4.18}$$

Using (1.2), (2.5) and the condition $|x-y| \leqq k\Delta(d, x)$, we have

$$\omega_{k-1}(f', y, \Delta(d, x))_1 = \frac{1}{2\Delta(d, x)} \int_{-\Delta(d,x)}^{\Delta(d,x)} |\Delta_v^{k-1} f'(y)|\, dv \tag{4.19}$$

$$\leqq \frac{2k+3/2}{2\Delta(d, y)} \int_{-(4k+2)\Delta(d,y)}^{(4k+2)\Delta(d,y)} \Delta_v^{k-1} f'(y)|\, dv \leqq \frac{(2k+3/2)\,(4k+2)^2}{2\Delta((4k+2)d, y)} \int_{-\Delta((4k+2)d,y)}^{\Delta((4k+2)d,y)} |\Delta_v^{k-1} f'(y)|\, dy$$

$$= c(k)\, \omega_{k-1}(f', y;\ \Delta((4k+2)d, y))_1.$$

(1.5), (4.18) and (4.19) gives

$$w(x)\omega_k(f, x;\ \Delta(d, x))_\infty \leqq c(k) \int_{x-k\Delta(d,x)}^{x+k\Delta(d,x)} w(y)\omega_{k-1}(f', y;\ \Delta((4k+2)d, y))_1 dy. \tag{4.20}$$

Using Lemma 2.3 with $\lambda=k$, $g(x)=w(x)\omega_k(f, x;\ \Delta(d, x))_\infty$ and $G(y) = w(y)\omega_{k-1}(f', y;\ \Delta((4k+2)d, y))_1$, we get (4.12). The estimate $d \leqq (2k)^{-1}$ comes from the application of (2.5) in (4.19).

Corollary 4.3. *If w satisfies (1.5), $f^{(k)} \in L_p$, $d \leqq [(2+\sqrt{3}) \prod_{i=1}^{k} (4i+2)]^{-1}$, then $\tau_k(f, w;\ \Delta(d))_{p',p} \leqq c(k)\, \| w\, \Delta^k(d) f^{(k)} \|_p$ for each $p' \in [1, \infty]$.*

Proof. If w satisfies (1.5), then for (2.5) $w\Delta(d)$ will satisfy (1.5). Using $k-1$-st times Theorem 4.3 and after that Theorem 4.2, we get

$$\tau_k(f, w, \Delta(d))_{p',p} \leqq c(k)\tau_{k-1}(f', w\Delta(d));\ \Delta((4k+2)d))_{p',p}$$

$$\leqq c(k)\tau_{k-2}(f'', w\Delta(d)\Delta((4k+2)d);\ \Delta((4k+2)(4k-2)d))_{p',p}$$

$$\leqq c(k)\tau_{k-2}(f'', w\Delta^2(d);\ \Delta((4k+2)(4k-2)d))_{p',p}$$

$$\leqq \cdots \leqq c(k)\tau_1(f^{(k-1)}, w\Delta^{k-1}(d);\ \Delta(\prod_{i=2}^{k} (4i+2)d))_{p',p}$$

$$\leqq c(k)\, \| f^{(k)} w \Delta^{k-1}(d)\Delta(\prod_{i=2}^{k} (4i+2)d) \|_p \leqq c(k)\, \| f^{(k)} w \Delta^k(d) \|_p.$$

if $d \prod_{i=2}^{k} (4i+2) \leqq \frac{1}{2+\sqrt{3}}$.

Corollary 4.4. *If* $f^{(k)}\in L_p$, $d\leq[(2+\sqrt{3})\prod_{i=2}^{k}(4i+2)]^{-1}$, $k\geq 2$, $p', p\geq 1$, $\mu\in\mathbf{R}$, *then*

$$\tau_k(f, \Delta^{\mu}(d); \Delta(d))_{p',p}\leq c(k,\mu)\,\|w\Delta^{\mu+k}(d)f^{(k)}\|_p.$$

Lemma 4.2. *If* $h>0$, $x_j=x+jh$, $n\in\mathbf{N}$ *then*

$$\omega_1(f, x; nh)_p\leq 2^{1-1/p}n^{-1/p}\sum_{j=1-n}^{n-1}(n-|j|)\omega_1(f, x_j; h)_p. \tag{4.21}$$

Proof. (4.21) is evidently true if $p=\infty$. Let $p<\infty$. Then

$$\int_0^{nh}|f(x)-f(x+t)|^p dt\leq\sum_{i=0}^{n-1}\int_{ih}^{(i+1)h}|f(x)-f(x+t)|^p dt \tag{4.22}$$

$$\begin{aligned}[\int_{ih}^{(i+1)h}|f(x)-f(x+t)^p dt]^{1/p}&\leq[\int_{ih}^{(i+1)h}[|f(x_0)-f(x_i)|+|f(x_i)-f(x+t)|]^p dt]^{1/p}\\ &\leq h^{1/p}|f(x_0)-f(x_i)|+[\int_{ih}^{(j+1)h}|f(x_i)-f(x_i+(t-ih))|^p dt]^{1/p}\\ &\leq h^{1/p}\sum_{j=0}^{i-1}|f(x_j)-f(x_{j+1})|+[\int_0^h|f(x_i)-f(x_i+t)|^p dt]^{1/p}\\ &=\sum_{j=0}^{i-1}[h|f(x_j)-f(x_{j+1})|^p]^{1/p}+[\int_0^h|f(x_i)-f(x_i+t)^p dt]^{1/p}\\ &\leq\sum_{j=0}^{i-1}\{[\int_0^h|f(x_j)-f(x_j+t)|^p dt]^{1/p}+[\int_0^h f(x_{j+1})-f(x_j+t)^p dt]^{1/p}\}\\ &\quad+[\int_0^h|f(x_i)-f(x_i+t)|^p dt]^{1/p}=[\int_0^h|f(x_0)-f(x_0+t)^p dt]^{1/p}\\ &\quad+\sum_{j=1}^{i}\{[\int_0^h|f(x_j)-f(x_j+t)^p dt]^{1/p}+[\int_{-h}^0|f(x_j)-f(x_j+t)|^p dt]^{1/p}\}.\end{aligned} \tag{4.23}$$

From (4.23) and (2.8) we get

$$\begin{aligned}[\int_{ih}^{(i+1)h}|f(x)-f(x+t)|^p dt]^{1/p}&\leq[\int_0^h|f(x_0)-f(x_0+t)^p dt]^{1/p}\\ &\quad+\sum_{j=1}^{i}2^{1-1/p}[\int_{-h}^{h}|f(x_j)-f(x_j+t)|^p dt]^{1/p}.\end{aligned} \tag{4.24}$$

By analogy with (4.22) and (4.24) we have

$$\int_{-nh}^{0}|f(x)-f(x+t)|^p dt\leq\sum_{i=1-n}^{0}\int_{(i-1)h}^{ih}|f(x)-f(x+t)|^p dt \tag{4.25}$$

$$(4.26)\qquad \Big[\int_{(i-1)h}^{ih} |f(x)-f(x+t)|^p dt\Big]^{1/p} \le \Big[\int_{-h}^{0} |f(x_0)-f(x_0+t)|^p dt\Big]^{1/p}$$
$$+2^{1-1/p}\sum_{j=i}^{-1}\Big[\int_{-h}^{h}|f(x_j)-f(x_j+t)|^p dt\Big]^{1/p}$$

(4.22), (4.24), (4.25), (4.26), (2.8) and (2.9) give

$$(4.27)\qquad \Big[\int_{-nh}^{nh}|f(x)-f(x+t)|^p dt\Big]^{1/p} \le 2^{1-1/p}\sum_{j=-n+1}^{n-1}(n-|j|)\Big[\int_{-h}^{h}|f(x_j)-f(x_j+t)|^p dt\Big]^{1/p}.$$

We get (4.21) from (4.27) and (1.2).

Theorem 4.4. *If w satisfies (1-5), $p', p \ge 1$, $f \in L_{\max\{p',p\}}$, $A \ge 1$, $d \le (2A)^{-1}$, then*

$$\tau_1(f, w;\ A\Delta(d))_{p',p} \le 12(2A+1)^4 c(2A^2+A)\tau_1(f, w;\ \Delta(d))_{p',p},$$

where $c(2A^2+A)$ is the constant in (1.5).

Proof. We set $N=[2A(2A+1)]+1$. Lemma 4.2 with $p=p'$, $n=N$, $h=Aa_A(d,x)/N$ gives

$$(4.28)\qquad \omega_1(f, x, Aa_A(d,x))_{p'} \le 2(2N)^{-1/p'}\sum_{j=1-N}^{N-1}(N-|j|)\omega_1(f, x_j;\ Aa_A(d,x)/N)_{p'}$$

where $x_j = x + jAa_A(d,x)/N$. From (2.2) and (2.3) for $|x-x_j| = |j|Aa_A(d,x)/N \le Aa_A(d,x)$ we have

$$(4.29)\qquad \Delta(d, x_j) \ge \frac{1}{2A+1}a_A(d, x_j) = \frac{1}{2A+1}[a_A(d,x)+(a_A(d,x_j)-a_A(d,x))]$$
$$\ge \frac{1}{2A+1}A\Big[a(d,x)-\frac{1}{2A}|x-x_j|\Big] \ge \frac{a_A(d,x)}{2(2A+1)} \ge \frac{Aa_A(d,x)}{N}$$

and

$$(4.30)\qquad \Delta(d, x_j) \le a_A(d, x_j) \le |a_A(d,x_j)-a_A(d,x)| + a_A(d,x) \le \frac{3}{2}a_A(d,x)$$
$$=\frac{3N}{2A}\,\frac{Aa_A(d,x)}{N} \le 3\,\frac{2A(2A+1)+1}{2A}\cdot\frac{Aa_A(d,x)}{N} < 6(A+1)\,\frac{Aa_A(d,x)}{N}\,.$$

Using (4.29), (4.30) and (1.2), we have

$$(4.31)\qquad \omega_1(f, x_j;\ Aa_A(d,x)/N)_{p'} = \Big[\frac{1}{2Aa_A(d,x)/N}\int_{-Aa_A(d,x)/N}^{Aa_A(d,x)/N}|f(x_j)-f(x_j+t)|^{p'}dt\Big]^{1/p'}$$
$$\le \Big[\frac{6(A+1)}{2\Delta(d,x_j)}\int_{-\Delta(d,x_j)}^{\Delta(d,x_j)}|f(x_j)-f(x_j+t)|^{p'}dt\Big]^{1/p'} = (6A+6)^{1/p'}\omega_1(f, x_j;\ \Delta(d,x_j))_{p'}.$$

Using (4.28), (4.31), (1.2) and (2.2), we get

$$(4.32)\qquad \omega_1(f, x;\ A\Delta(d,x))_{p'} \le (2A+1)^{1/p'}\omega_1(f, x;\ Aa_A(d,x))_{p'}$$
$$\le 2\Big[\frac{(2A+1)(6A+6)}{2N}\Big]^{1/p'}\sum_{j=1-N}^{N-1}(N-|j|)\,\omega_1(f, xj;\ \Delta(d, x_j))_{p'}.$$

From (4.32), (1.1) and (1.5) with $\lambda=2A^2+A$ because $|x-x_j|\leq Aa_A(d,x)$ $\leq A(2A+1)\Delta(d,x)$, we obtain

$$(4.33)\quad \tau_1(f,w;\,A\Delta(d))_{p',p}\leq 2\left(\frac{(2A+1)3(A+1)}{(2A+1)2A}\right)^{1/p'}\sum_{j=1-N}^{N-1}(N-|j|)\,\|w(x)\omega_1(f,x_j;$$

$$\Delta(d,x_j))_{p'}\|_p\leq 6\sum_{i=1-N}^{N-1}(N-|j|)\,\|c(2A^2+A)w(x_j)\omega_1(f,x_j;\,\Delta(d,x_j))_{p'}\|_{(x)p}.$$

Interchanging the variables in (4.33)

$$(y=x_j)\ |dx/dy|=1/\left|\frac{dy}{dx}\right|=\frac{1}{|1+jAa'_A(d,x)/N|}\leq 1/\left(1-NA\frac{1}{2A}N^{-1}\right)=2)$$

we prove the theorem because

$$\sum_{j=1-N}^{N-1}(N-|j|)=N^2\leq(2A(2A+1)+1)^2\leq(2A+1)^4.$$

Corollary 4.5. *If* $d\leq(2A)^{-1}$, $A\geq 1$, $p,p'\geq 1$, $f\in L_{\max\{p',p\}}$, then

$$\tau_1(f,\Delta^\mu(d);\,A\Delta(d))_{p',p}\leq 12(2A+1)^4[c(2A^2+A)]^{|\mu|}\tau_1(f,\Delta^\mu(d);\,\Delta(d))_{p',p}$$

for each real μ *and in particular*

$$\tau_1(f,1;\,A\Delta(d))_{p',p}\leq 12(2A+1)^4\tau_1(f,1;\,\Delta(d))_{p',p}.$$

We shall prove now an analogue of (3.8) with the weight satisfying some condition in the cases $\delta=d=\text{const}$ and $\delta=\Delta(d)$.

Theorem 4.5. *If* $f\in L_p$, $1\leq p'\leq p''\leq p$, *then*

a) *if* w *satisfies (1.5) and* $d\leq(2+\sqrt{3})^{-1}k^{-1}$, *we have*

$$(4.34)\quad \tau_k(f,w;\,\Delta(d))_{p',p}\leq\tau_k(f,w;\,\Delta(d))_{p'',p}\leq c(k)\tau_k(f,w;\,(4k+2)\Delta(d))_{p',p};$$

b) *if* w *satisfies the condition* $w(t)\leq c(\lambda)w(x)$ *for each* $x,t\in[-1,1]$, $|x-t|$ $\leq\lambda d$ *and* $d\leq(b-a)/(2k)$, *then*

$$\tau_k(f,w:\,d)_{p',p}\leq\tau_k(f,w;\,d)_{p'',p}\leq c(k)\tau_k(f,w;\,d)_{p',p}.$$

Proof. a) (4.4) shows that it is sufficient to prove only the second inequality in (4.34). From (3.12) we get

$$(4.35)\quad \omega_k(f,x;\,\Delta(d,x))_{p''}\leq\sum_{i=0}^{k}c(k,i)\,\|\omega_k(f,x+ih;\,\Delta(d,x))_{p'}\|'_{(h)p''[-\Delta(d,x),\,\Delta(d,x)]}$$

$$\leq c(k)\,\|\omega_k(f,x+h;\,\Delta(d,x))_{p'}\|'_{(h)p''[-k\Delta,\,k\Delta]}+c(k)\omega_k(f,x;\,\Delta(d,x))_{p'}.$$

Using (1.5) and (2.5), we obtain

$$(4.36)\qquad w(x)\,\|\omega_k(f,x+h;\,\Delta(d,x))_{p'}\|'_{(h)p''[-k\Delta(d,x),\,k\Delta(d,x)]}$$

$$\leq c(k)\Delta(d,x)\|w(x+h)\Delta^{-1}(d,x+h)\omega_k(f,x+h;\,(4k+2)\Delta(d,x+h))_{p'}\|'_{(h)p''[-k\Delta(x),\,k\Delta(x}$$

$$=c(k)\Delta(d,x)\,\|w(y)\Delta^{-1}(d,y)\omega_k(f,y;\,(4k+2)\Delta(d,y))_{p'}\|'_{(y)p''[x-k\Delta(d,x),\,x+k\Delta(d,x)]}$$

(4.35), (4.36) and Lemma 2.3 give (4.34).

b) We proceed in this case as in the case a) with the necessary varitions It is sufficient to use the Fubiny's theorem instead of Lemma 2.3.

Theorem 4.4 and Theorem 4.5 give

Corollary 4.6. *If* w *satisfies (1.5),* $f\in L_p$, $1\le p'\le p''\le p$; $d\le 1/2+\sqrt{3}$, *then* $\tau_1(f, w; \Delta(d))_{p',p} \asymp \tau_1(f, w; \Delta(d))_{p'',p}$ *an in particular* $\tau_1(f, \Delta^{\mu}(d); \Delta(d))_{p',p} \asymp \tau_1(f, \Delta^{\mu}); \Delta(d))_{p'',p}$ *for each real* μ.

Remark. We have the restrictions of the type $d\le c(c<1)$ or $d\le c(k)$ $(c(k)<1)$ in the conditions of the assertions of this paragraph. But this assertions are true, when $d\le 1$. We can prove them if $c(k)\le d\le 1$ by the following plan: 1) we replace $\Delta(d, x)$ and $w(x)$ with 1 and $\|w\|_\infty$ (d is bounded with $c(k)$ and 1); 2) we use Theorem 3.1; 3) we apply the corresponding property of $\omega_p(f; \delta)_p$; 4) we use Theorem 3.1; 5) we put $\Delta(d, x)$ and $w(x)$ on their places.

Using the main idea of the proof of Theorem 4.3, we shall establish the following connection between $\tau_k(f; \delta)_p$ and $\omega_{k-1}(f'; \delta)_p$.

Theorem 4.6. *If* $k\ge 2$, $\delta=\text{const}$, $p\ge 1$, $f'\in L_p$, *then*

$$\tau_k(f; \delta)_p\le c(k)\delta\omega_{k-1}(f'; \delta)_p.$$

This inequality is proved by V. A. Popov, when $k=2$.

Proof. For $\delta>0$, $h\in[0, \delta]$, $a>0$, from (3.5) in the periodical case and from (3.12) in the non-periodical case with $p'=1$ and $k=s$ we have

$$\int_0^a |\Delta_h^s f(x)|\,dx\le \int_0^a \sum_{i=0}^{s} c(s, i)(2\delta)^{-1}\int_{-\delta}^{s} |\Delta_v^s f(x+ih)|\,dv\,dx \tag{4.36}$$

$$\le \sum_{i=0}^{\delta} c(s, i)\delta^{-1}\int_0^{a+s\delta}\int_{-\delta}^{\delta} |\Delta_v^s f(y)|\,dv\,dy=c(s)\delta^{-1}\int_{-\delta}^{\delta}\int_0^{a+s\delta} |\Delta_v^s f(y)|\,dy\,dv.$$

Using (4.36) with $a=k\delta$, $s=k-1$ and f' instead of f, we obtain

$$\omega_k(f, x; \delta)=\sup\{|\Delta_h^k f(t)|: t, t+kh\in[x-k\delta/2, x+k\delta/2]\cap J, h\ge 0\} \tag{4.37}$$

$$\le\sup\{\int_0^h |\Delta_h^{k-1} f'(t+u)|\,du: t, t+kh\in[x-k\delta/2, x+k\delta/2]\cap J, h\ge 0\}$$

$$\le\sup\{\int_0^{k\delta} |\Delta_h^{k-1} f'(x-k\delta/2+u)|\,du;\ 0\le h\le\delta\}$$

$$\le c(k)\,\delta^{-1}\int_{-\delta}^{\delta}\int_0^{(2k-1)\delta} |\Delta_v^{k-1} f'(x-k\delta/2+u)|\,du\,dv.$$

a) the periodical case. From (4.37) we have

$$\tau_k(f; \delta)_{p[0,2\pi]}=\|\omega_k(f, x; \delta)\|_{(x)p[0,2\pi]}$$

$$\le ck\delta^{-1}\int_{-\delta}^{\delta}\int_0^{(2k-1)\delta} \|\Delta_v^{k-1} f'(x-k\delta(2+u)\|_{(x)p[0,2\pi]}\,du\,dv$$

$$=ck\delta^{-1}\int_{-\delta}^{\delta}\int_0^{(2k-1)\delta} \|\Delta_v^{k-1} f'(y)\|_{(y)p[0,2\pi]}\,du\,dv$$

$$\leqq c(k)\delta^{-1}\int_{-\delta}^{\delta}\int_{0}^{(2k-1)\delta}\omega_{k-1}(f'\,;\,\delta)_{p[0,2\pi]}\,du\,dv=c(k)\delta\omega_{k-1}(f'\,;\,\delta)_{p[0,2\pi]};$$

b) the non-periodical case. Using (4.37), we get

$$\tau_k(f\,;\,\delta)_{p[a,b]}=\|\,\omega_k(f,\,x\,;\,\delta)\,\|_{(x)p[a,b]}$$

$$\leqq c(k)\delta^{-1}\int_{-\delta}^{\delta}\int_{0}^{(2k-1)\delta}\|\Delta_v^{k-1}f'(x-k\delta/2+u\,\|_{(x)p[a,b]}du\,dv$$

$$\leqq c(k)\delta^{-1}\int_{-\delta}^{\delta}\int_{0}^{(2k-1)\delta}\|\Delta_v^{k-1}f'(x-k\delta/2+u)\|_{(x)p(-\infty,\,\infty)}\,du\,dv$$

$$=c(k)\delta^{-1}\int_{-\delta}^{\delta}\int_{0}^{(2k-1)\delta}\|\,\Delta_v^{k-1}f'(x)\,\|_{(x)p(-\infty,\,\infty)}\,du\,dv$$

$$\leqq c(k)\delta^{-1}\int_{-\delta}^{\delta}\int_{0}^{(2k-1)\delta}\omega_{k-1}(f';\,\delta)_{p[a,b]}\,du\,dv=c(k)\delta\omega_{k-1}(f'\,;\,\delta)_{p[a,b]}.$$

This completes the proof.

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