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# ASYMPTOTIC PROPERTIES OF LINEAR QUANTILE EUNCTIONS

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The paper deals with some problems stated by E. Parzen (1979). Results of M. Csörgö and P. Révész (1978) for a stepwise quantile function are proved to hold also for the linear quantile function. This is used to consider the statistical hypothesis that the unknown continuous distribution function $F(x)$ is of the type $F(x)=F_0[(x-\mu)/\sigma]$, where $F_0$ is given and $\mu$ and $\sigma$ are unknown.

**1. Introduction and notations.** Let $X_1, \dots, X_n$ be an independent sample from a random variable $X$ with absolutely continuous distribution function $F(x)$; let further $X_{1:n} \leqq X_{2:n} \leqq \dots \leqq X_{n:n}$ be the order statistics of the sample and $\tilde{F}_n(x)$ the corresponding empirical distribution function, i. e.$(-\infty < x < \infty)$

$$\tilde{F}_n(x)=\begin{cases} 0 & \text{if } x<X_{1:n}, \\ k/n & \text{if } X_{k:n}\leqq x<X_{k+1:n}, \quad k=1, \dots, n-1, \\ 1 & \text{if } X_{n:n}\leqq x. \end{cases}$$

The corresponding empirical process $\tilde{\beta}_n(x)$ is $\tilde{\beta}_n(x)=n^{1/2}[\tilde{F}_n(x)-F(x)]$. Define $X_{0:n}=X_{1:n}-[2n(\log n)^2]^{-1}$. We introduce the following linear empirical distribution function $F_n(x)$ by $F_n(x)=k/n$ if $x=X_{k:n}$, $k=0, 1, \dots, n$; and linear in the intervals $[X_{k=1:n}, X_{k:n}]$, $k=1, \dots, n$; $F_n(x)=0$ if $x\leqq X_{0:n}$; $F_n(x)=1$ if $x\geqq X_{n:n}$. Denote by $\beta_n(x)$ the corresponding empirical process $\beta_n(x)=n^{1/2}[F_n(x)-F(x)]$.

Let $\tilde{Q}_n(y)$ be the quantile function corresponding to $\tilde{F}_n(x)$, i. e. $\tilde{Q}_n(y)=X_{k:n}$ if $(k-1)/n<y<y\leqq \frac{k}{n}$, $k=1, 2, \dots, n$, $\tilde{Q}_n(0)=X_{1:n}$ and the respective quantile process $\tilde{q}_n(y)=n^{1/2}[\tilde{Q}_n(y)-F^{-1}(y)]$, $0<y<1$, where $F^{-1}(y)=\inf\{x:F(x)\geqq y\}$.

In a similar way we have $Q_n(y)=X_{k:n}$ if $y=k/n$, $k=0, 1, \dots, n$, $Q_n(y)$ linear in the subintervals $[(k-1)/n, k/n]$, $k=1, \dots, n$, and $q_n(y)=n^{1/2}[Q_n(y)-F^{-1}(y)]$, $0<y<1$. One has

$$Q_n(y)=n(k/n-y)X_{k-1:n}+n(y-(k-1)/n)X_{k:n} \tag{1.1}$$

for $(k-1)/n\leq y\leq k/n$, $k=1, \dots, n$.

In the case $F(x)$ is the uniform distribution over the unit interval we shall use the following notations:

| | | | | | |
|---|---|---|---|---|---|
| $U_k$ | instead of | $X_k$ | $\alpha_n(x)$ | instead of | $\beta_n(x)$ |
| $U_{k:n}$ | " | $X_{k;n}$ | $\tilde{U}_n(y)$ | " | $\tilde{Q}_n(y)$ |
| $\tilde{E}_n(x)$ | " | $\tilde{F}_n(x)$ | $U_n(y)$ | " | $Q_n(y)$ |
| $E_n(x)$ | " | $F_n(x)$ | $\tilde{u}_n(y)$ | " | $\tilde{q}_n(y)$ |
| $\tilde{\alpha}_n(x)$ | " | $\tilde{\beta}_n(x)$ | $u_n(y)$ | " | $q_n(y)$ |

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Let further $\{B(y),\ 0\leq y\leq 1\}$ be a Brownian bridge, i. e. a separable Gaussian process on $[0, 1]$ with $\mathsf{E}\,B(y)=0$, $\mathsf{E}\,B(y_1)B(y_2)=y_1\wedge y_2-y_1y_2$ and $\{K(y,t),\ 0\leq y\leq 1,\ 0\leq t\}$ be a Kiefer process, i. e. a separable Gaussian process on $[0, 1]\times[0, \infty]$ with $\mathsf{E}K(y,t)=0$, $\mathsf{E}K((y_1,t_1)K(y_2,t_2)=(t_1\wedge t_2)(y_1\wedge y_2-y_1y_2)$.

**2. Approximation of the uniform process $u_n(y)$.**

Theorem A [4]. *For every $n$ there exists such a Brownian bridge $\{B_n(y),\ 0\leq y\leq 1\}$ that for arbitrary $z$*

$$\mathsf{P}\{\sup_{0\leq y\leq 1}|\tilde{u}_n(y)-B_n(y)|>n^{-1/2}(A\log n+z)\}\leq\tilde{B}e^{-\tilde{C}z}, \tag{2.1}$$

*where $A$, $\tilde{B}$, $\tilde{C}$ are positive constants.*

Letting $z=K\log n$ $(K.\tilde{C}>1)$ one has from (2.1) and the lemma of Borel-Cantelli $\sup_{0\leq y\leq 1}|\tilde{u}_n(y)-B_n(y)|=O(n^{-1/2}\log n)$ *a. s.*

Theorem 1. *For every $n$ and the Brownian bridge of Theorem A there exist positive constants $A$, $B$, $C$, so that for every $z$*

$$\mathsf{P}\{\sup_{0\leq y\leq 1}|u_n(y)-B_n(y)|>n^{-1/2}(A\log n+z)\}\leq Be^{-Cz},$$

*and* $\sup_{0\leq y\leq 1}|u_n(y)-B_n(y)|=O(n^{-1/2}\log n)$ *a. s.*

Proof of Theorem 1. We first prove

Lemma 1. *Let $\{B(y),\ 0\leq y\leq 1\}$ be a Brownian bridge. Then for an arbitrary $z$ we have*

$$\mathsf{P}\{\max_{1\leq k\leq n}|B(k/n)-B((k-1)/n)>n^{-1/2}(A\log n+z)\}<n^{-\varepsilon}B_1e^{-C_1z}, \tag{2.2}$$

*where $A$ and $\varepsilon$ are arbitrary positive numbers and $B_1$ and $C_1$ are positive constants depending on $A$ and $\varepsilon$.*

Proof of Lemma 1. Denote for $k=1,\dots,n$

$$G_k=[B(\tfrac{k}{n})-B(\tfrac{k-1}{n})]:[\tfrac{1}{n}(1-\tfrac{1}{n})]^{1/2}.$$

The random variable $G_k$ has a $N(0, 1)$ distribution. Further we have

$$\mathsf{P}\{|G_k|>A\log n+z\}\leq\inf_{0<t}\mathsf{E}e^{t|G_k|}e^{-t(A\log n+z)}$$

and some calculations give $\mathsf{E}e^{t|G_k|}\leq 2te^{2t^2}/\sqrt{2\pi}+e^{t^2/2}$. Choosing a $t$ with $tA\geq 1+\varepsilon$ we get

$$\mathsf{P}\{|G_k|>A\log n+z\}\leq n^{-(1+\varepsilon)}B_1\,.\,e^{-C_1z}. \tag{2.3}$$

The lemma is proved by the inequality

$$\mathsf{P}\{|B(\tfrac{k}{n})-B(\tfrac{k-1}{n})|>n^{-1/2}(A\log n+z)\}\leq\mathsf{P}\{|G_k|>A\log n+z\}$$

and (2.3).

Now we are prepared to prove Theorem 1. Let $B_n(y)$ is the Brownian bridge of Theorem A. One has the following chain inequalities

$$\begin{aligned}\sup_{0\leq y\leq 1}|u_n(y)-B_n(y)|&\leq\sup_{0\leq y\leq 1}|\tilde{u}_n(y)-B_n(y)|+\sup_{0\leq y\leq 1}|u_n(y)-\tilde{u}_n(y)|\\ &\leq\sup_{0\leq y\leq 1}|\tilde{u}_n(y)-B_n(y)|+\max_{1\leq k\leq n}|u_n(\tfrac{k}{n})-u_n(\tfrac{k-1}{n})|+n^{-1/2}\\ &\leq\sup_{0\leq y\leq 1}|\tilde{u}_n(y)-B_n(y)|+2\max_{1\leq k\leq n}|\tilde{u}_n(\tfrac{k}{n})-B_n(\tfrac{k}{n})|\end{aligned} \tag{2.4}$$

$$+\max_{1\le k\le n}|B_n(\frac{k}{n})-B_n(\frac{k-1}{n})|+n^{-1/2}\le 3\sup_{0\le y\le 1}|\tilde{u}_n(y)-B_n(y)|$$

$$+\max_{1\le k\le n}|B_n(\frac{k}{n})-B_n(\frac{k-1}{n})|+n^{-1/2}.$$

Combined with (2.1) and (2.2) the last inequality (2.4) proves Theorem 1.

Remark. Theorems A and 1 imply

$$\sup_{0\le y\le 1}|u_n(y)-\tilde{u}_n(y)|=O(n^{-1/2}\log n)\text{ a.s.} \tag{2.5}$$

Theorem 2. *There exists a Kiefer process* $\{K(y,t),0\le y\le 1,0\le t\}$ *with*

$$\sup_{0\le y\le 1}|u_n(y)-n^{-1/2}K(y,n)|=O(n^{-1/4}(\log\log n)^{1/4}(\log n)^{1/2})\quad a.\ s.$$

Proof of Theorem 2. The theorem follows from Theorem B in [4] and (2.5).

**3. The distance between the general normed quantile process and the corresponding uniform process.** Here we prove an analogue of Theorem 3 in [4].

Lemma 2. *For* $\delta_n^*=25n^{-1}\log\log n+n^{-1}$ *one has*

$$\limsup_{n\to\infty}\sup_{\delta_n^*\le y\le 1-\delta_n^*}[y(1-y)\log\log n]^{-1/2}|u_n(y)|\le 4\quad a.\ s. \tag{3.1}$$

$$[y(1-y)\log\log n]^{-1/2}|un(y)|\le 4\text{ a. s.}$$

Proof of Lemma 2. Using Theorem 3.2 of Csaki [3] and the fact that $|\alpha_n(y)-\alpha_n(y)|\le n^{-1/2}$ a. s. for $y\in[0,1]$, we get

$$\limsup_{n\to\infty}\sup_{\varepsilon_n\le y\le 1-\varepsilon_n}[y(1-y)\log\log n]^{-1/2}|\alpha_n(y)|=2\text{ a. s.,} \tag{3.2}$$

where $\varepsilon_n=dn^{-1}\log\log n$, $d=0,236\ldots$ Following the proof of Theorem 2 in [4] and (3.2), we prove Lemma 2.

Theorem 3. *Let the quantile process* $q_n(y)$ *resp.* $u_n(y)$ *be defined in terms of* $X_{k:n}$ *resp.* $U_{k:n}=F(X_{k:n})$. *Let* $F(x)$ *satisfy the following assumptions:*

(3.3) *$F(x)$ is twice differentiable on $(a,b)$, where*

$-\infty\le a=\sup\{x:F(x)=0\}$, $\infty\ge b=\inf\{x:F(x)=1\}$ *and* $F'(x)=f(x)\ne 0$ *on* $(a,b)$,

$$\sup_{a<x<b}F(x)(1-F(x))\frac{|f'(x)|}{f^2(x)}\le\gamma\ \textit{for some}\ \gamma>0. \tag{3.4}$$

*Then we have*

$$\limsup_{n\to\infty}\frac{n^{1/2}}{\log\log n}\sup_{\delta_n\le y\le 1-\delta_n}|f(F^{-1}(y))q_n(y)-u_n(y)|\le L\quad a.\ s., \tag{3.5}$$

*where* $\delta_n=25n^{-1}\log\log n$ *and L depends only on* $\gamma$.

If in addition to (3.3) and (3.4) we also assume that $f(x)$ is

(3.6) nondecreasing on an interval to the right of $a$ (or $0<f(a+0)<\infty$ if $-\infty<a$) and nonincreasing on an interval to the left of $b$ (or $0<f(b-0)<\infty$ if $b<\infty$),

then

$$\sup_{0<y<1}|f(F^{-1}(y))q_n(y)-u_n(y)|=_{\text{a.s.}}O(n^{-1/2}\log\log n)\text{ if }\gamma<1 \tag{3.7}$$

$$=_{\text{a.s.}}O(n^{-1/2}(\log\log n)^2)\text{ if }\gamma=1=_{\text{a.s.}}O(n^{-1/2}(\log\log n)\gamma(\log n)^{(1+\varepsilon)(\gamma-1)})\text{ if }\gamma>1,$$

where $\varepsilon>0$ is arbitrary.

Proof of Theorem 3. If $y\in[(k-1)/n,k/n]$, $k=1,\ldots,n$, we have the following equalities

$$f(F^{-1}(y))q_n(y)=n^{1/2}f(F^{-1}(y))$$

$$\times\{n(\frac{k}{n}-y)[F^{-1}(U_{k-1:n})-F^{-1}(y)]+n(y=\frac{k-1}{n})[F^{-1}(U_{k:n})-F^{-1}(y)]\}$$

$$=n^{1/2}f(F^{-1}(y))\{n(\frac{k}{n}-y)|[(U_{k-1:n}-y)(F^{-1}(y))'+\frac{1}{2}(U_{k-1:n}-y)^2(F^{-1}(\xi_1))'']$$

$$(3.8)\qquad +n(y-\frac{k-1}{n})[(U_{k:n}-y)(F^{-1}(y))'+\frac{1}{2}(U_{k:n}-y)^2(F^{-1}(\xi_2))'']\}$$

$$=n^{1/2}[U_n(y)-y]f(F^{-1}(y))(F^{-1}(y))'+\frac{1}{2}n^{1/2}f(F^{-1}(y))$$

$$\times\{n(\frac{k}{n}-y)(U_{k-1:n}-y)^2(F^{-1}(\xi_1))''+n(y-\frac{k-1}{n})(U_{k:n}-y)^2(F^{-1}(\xi_1))''\},$$

where $\xi_1$ is between $y$ and $U_{k-1:n}$, resp. $\xi_2$ between $y$ and $U_{k:n}=y+^{-1/2}\tilde{u}_n(y)$.

Taking into account $f(F^{-1}(y))(F^{-1}(y))'=1$ (3.8) implies

$$|f(F^{-1}(y))q_n(y)-u_n(y)|$$

$$(3.9)\qquad \leq\frac{1}{2}n^{-1/2}\tilde{u}_n^2(y)f(F^{-1}(y))[|f'(F^{-1}(\xi_2))|/f^3(F^{-1}(\xi_2))]$$

$$+\frac{1}{2}n^{1/2}(U_{k-1:n}-y)^2f(F^{-1}(y))[|f'(F^{-1}(\xi_1))|/f^3(F^{-1}(\xi_1))].$$

Now take a fixed $n\geq 21$ and a fixed $y\in[\delta_n,1-\delta_n]$. It follows that $y\in[(k-1)/n, k/n]\subset[\delta_n^*,1-\delta_n^*]$ with $\delta_n^*=\delta_n-n^{-1}$ and $\delta_n$ defined by (3.5). From Theorem 3 in [4] one has

$$\frac{1}{2}n^{-1/2}\tilde{u}_n^2(y)f(F^{-1}(y))\frac{|f'(F^{-1}(\xi))|}{f^3(F^{-1}(\xi))}\leq Kn^{-1/2}\log\log n \text{ a. s.},$$

where $y\in[\delta_n,1-\delta_n],\xi$ in between $y$ and $y+n^{-1/2}\tilde{u}_n(y)$ and $K=40\gamma 10^\gamma$. This means that the first term on the right hand side of (3.9) is less than $Kn^{-1/2}\log\log n$ If $U_{k-1:n}-y\geq 0$, the second term is less than $\frac{1}{2}n^{-1/2}\tilde{u}_n^2(y)\times f(F^{-1}(y))\times |f'(F^{-1}(\xi_1))|/f^3(F^{-1}(\xi_1))$, i.e. is also smaller than $Kn^{-1/2}\log\log n$. In consequence, in order to prove (3.5) one has to consider only the case $U_{k-1:n}-y<0$. Denote by $H_n(y)$ the second term on the right hand side of (3.9) and $y_k=(k-1)/n$. One has

$$H_n(y)\leq n^{-1/2}u_n^2(y_k)f(F^{-1}(y))[|f'(F^{-1}(\xi_1))|/f^3(F^{-1}(\xi_1))]$$

$$+n^{-3/2}f(F^{-1}(y))[|f'(F^{-1}(\xi_1))|/f^3(F^{-1}(\xi_1))]:=h_{n_1}(y)+h_{n_2}(y).$$

Because of $y_k\in[\delta_n^*,1-\delta_n^*]$ from Lemma 2 it follows that

$$h_{n_1}(y)\leq(16+o(1))(n^{-1/2}\log\log n)[\frac{y_k(1-y_k)}{\xi_1(1-\xi_1)}]\times[\xi_1(1-\xi_1)\cdot\frac{|f'(F^{-1}(\xi_1))|}{f^2(F^{-1}(\xi_1))}][\frac{f(F^{-1}(y))}{f(F^{-1}(\xi_1)(}]\text{ a. s.}$$

and

$$(3.10)\qquad \begin{aligned}&|\xi_1-y_k|\leq 1/n+|\xi_1-y|\leq 1/n+|U_{k-1:n}-y|\\&\leq 2/n+n^{-1/2}|u_n(y_k)|\leq 2/n+(4+o(1))(y_k(1-y_k)n^{-1}\log\log n)^{1/2}.\end{aligned}$$

From $y_k \geq \delta_n^*$ and (3.10) for $n \geq 21$ we get

$$y_k/\xi_1 \leq 1+2n^{-1}\,[\,y_k-2/n-(4+o(1))(y_k(1-y_k)n^{-1}\log\log n)^{1/2}]^{-1}$$

$$+\frac{(4+o(1))\,(y_k(1-y_k)n^{-1}\log\log n)^{1/2}}{y_k-2/n-(4+o(1))(y_k(1-y_k)n^{-1}\log\log n)^{1/2}}\leq 10$$

and similarly from $(1-y_k)\geq\delta_n^*$ and (3.10) for $n\geq 21$ it follows that $(1-y_k)/(1-\xi_1)\leq 10$. This implies that the content of the second rectangular bracket is smaller than 10. Using Lemma 1 of [4] and our Lemma 2, one can similarly prove that the number in the third bracket is less than $20^\gamma$. All this gives that $h_{n_1}(y)$ is bounded by $(16+o(1))10\gamma 20^\gamma n^{-1}\log\log n$. On the other hand, one can prove that $h_{n_2}(y)=o(n^{-1/2}\log\log n)$. Thus (3.5) is proved with $L=40\gamma 10+160\gamma 20^\gamma+1$. Finally using Lemma 2, like in the proof of (3.5) of Theorem 3 in [4], one gets (3.7), where the condition (3.6) is less restrictive compared to (3.4) of [4].

From theorems 1, 2 and 3 follows an important

Corollary. *Under the conditions* (3.3), (3.4) *and* (3.6) *there exists a Brownian bridge* $\{B_n(y), 0\leq y\leq 1\}$ *and a Kiefer process* $\{K(y,t), 0\leq y\leq 1, 0\leq t\}$ *with*

$$\sup_{0<y<1}|f(F^{-1}(y))q_n(y)-B_n(y)|$$

$$=_{a.s.}O(n^{-1/2}\log n)\ \ \textit{if}\ \ \gamma<2$$

$$=_{a.s.}O(n^{-1/2}(\log\log n)^\gamma(\log n)^{(1+\varepsilon)(\gamma-1)})\ \ \textit{if}\ \ \gamma\geq 2,$$

*where* $\gamma$ *is defined by* (3.4) *and* $\varepsilon$ *is arbitrary and positive,*

(3.11) $$\sup_{0<y<1}|f(F^{-1}(y))q_n(y)-n^{-1/2}K(y,n)|=_{a.s.}O((n^{-1}\log\log n)^{1/4}(\log n)^{1/2})$$

The relation (2.5) combined with Theorem 3 and theorems 3 and 5 of [4] enables one to prove a result similar to Theorem 5 of [4]. Let $C=C(0,1)$ be the space of continuous real valued functions endowed with the supremum norm. Let $K\subset C$ be the set of absolutely continuous functions $f(x)$ (with respect to the Lebesgue measure) for which $f(0)=f(1)=0$ and $\int_0^1(f'(y))^2dy\leq 1$.

Theorem 4. *Under conditions* (3.3), (3.4) *and* (3.6) *the set of limit points in C with respect to the supnorm of the sequence* $\{\frac{f(F^{-1}(y))q_n(y)}{(2\log\log n)^{1/2}}\}$ *is is equal to K a. s.*

**4. Linear quantile functions and hypotheses testing.** In this paragraph we consider the family $\mathcal{F}$ of distribution functions, defined through $\mathcal{F}=\{F|F(x)=F_0((x-\mu)/\sigma), -\infty<\mu<\infty, 0<\sigma<\infty\}$, where $F_0$ is a known absolutely continuous distribution function.

The hypothesis $H_0$ is $F\in\mathcal{F}$. Then $H_0$ is equivalent to

(4.1) $$F^{-1}(y)=\sigma F_0^{-1}(y)+\mu,$$

$$f(F^{-1}(y))=f_0(F_0^{-1}(y))/\sigma,$$

where $f(x)=F'(x)$, $f_0(x)=F_0'(x)$.

Let $X_1, X_2, \ldots, X_n$ be a sample taken from $F(x)$. Then under $H_0$ if $Z_i=(X_i-\mu)/\sigma$, $i=1,2,\ldots,n$ the set $Z_1, Z_2, \ldots, Z_n$ is a sample from $F_0(x)$ and the following relation holds

(4.2) $$Q_n^0(y)=[Q_n(y)-\mu]/\sigma,$$

where $Q_n(y)(Q_n^0(y))$ is the linear quantile function of $X_1, X_2, \dots, X_n(Z_1, Z_2, \dots, Z_n)$.

Using the differentiability of $Q_n(y)$ we construct now a random process for the purpose of testing $H_0$. Define $\widehat{U}_i=F_0[(X_i-\widehat{\mu})/\widehat{\sigma}]$, $i=1, 2, \dots, n$, where $\widehat{\mu}$ and $\widehat{\sigma}$ are some estimators of $\mu$ and $\sigma$. Denote $\widehat{D}_n(u)=F_0[(Q_n(u)-\widehat{\mu})/\widehat{\sigma}]$. We can consider $\widehat{D}_n(u)$ as the quantile function ot $\widehat{U}_1, \widehat{U}_2, \dots, \widehat{U}_n$. Differentiating $\widehat{D}_n(u)$ we get

(4.3) $$\widehat{d}_n(u)=f_0\,[(Q_n(u)-\widehat{\mu})/\widehat{\sigma}]Q_n'(u)(1/\widehat{\sigma}).$$

The function $\widehat{d}_n(u)$ depends also on $\widehat{\mu}$ and $\widehat{\sigma}$, but (4.3) suggests (see [8, p. 110]) to introduce a function $d_n(u)$ that does not depend on the estimates but nevertheless can be used for testing $H_0$. Define $d_n(u)=f_0(F_0^{-1}(u))Q_n'(u)(1/\sigma_n)$, where $\sigma_n$ is a norming constant, i. e. $\sigma_n=\int_0^1 f_0(F_0^{-1}(y))Q_n'(y)dy$.

Consider now the process $g_n(u)=n^{1/2}[\int_0^u d_n(y)dy-u]$, $0\leq u\leq 1$. It is $g_n(u)$ that we shall approximate by suitable two-parametric Gaussian process.

Denote

$$\varphi(y)=[f_0(F^{-1}(y))]'/f_0(F_0^{-1}(y))$$

(4.4) $$E_n(y)=f_0(F_0^{-1}(y))q_n^0(y)-n^{-1/2}K(y, n)$$

$$G_n(u)=n^{-1/2}K(u, n)-\int_0^u n^{-1/2}K(y, n)\varphi(y)dy+u\int_0^1 n^{-1/2}K(y, n)\varphi(y)dy,$$

where $q_n^0(y)=n^{1/2}[Q_n^0(y)-F_0^{-1}(y)]$ and $K(y, t)$ is a Kiefer process, corresponding to $q_n^0(y)$ like in the corollary following Theorem 3,

Theorem 5. *Under the hypothesis $H_0$,* (3.3), (3.4) *and* (3.6) *hold and if*

(4.5) *the function $|[f_0(F_0^{-1}(y))]'|$ is bounded in some neighbourhoods of* 0 *and* 1, *the functions $y^{1/r}|F_0^{-1}(y)|$ and $(1-y)^{1/r}|F_0^{-1}(y)|$ are bounded respectively in some neighbourhoods of* 0 *and* 1 *by a constant $\lambda$ where $r>1$,*

*then*

$$\sup_{0\leq u\leq 1}|g_n(u)-G_n(u)|=O(n^{-1/4}(\log\log n)^{1/4}(\log n)^{3/2})\ a.\ s.$$

Remark 1. From (3.4) it follows that $|\varphi(y)|\leq\gamma[y(1-y)]^{-1}$ for every $y\in(0, 1)$.

Remark 2. The integrals in (4.4) exist with probability one. Indeed, for every $u\in[0, 1]$ we have a. s.

$$\left|\int_0^u n^{-1/2}K(y, n)\varphi(y)dy\right|\leq\int_0^1|n^{-1/2}K(y, n)|\gamma[y(1-y)]^{-1}dy$$

$$\leq 2\gamma\sup_{0<y<1}|K(y, n)|[4ny(1-y)\log\log\frac{n}{y(1-y)}]^{-1/2}\left|\int_0^1[\log\log\frac{n}{y(1-y)}]^{1/2}[y(1-y)]^{-1/2}dy\right.$$

$$\leq 2\gamma(1+o(1))2\int_0^{1/2}[\log\log\frac{n}{y(1-y)}]^{1/2}[y(1-y)]^{-1/2}dy<\infty.$$

The third inequality is implied by the following result of [2, p. 797]:

(4.6) $$\limsup_{n\to\infty} \sup_{0<y<1} |K(y,n)[4ny(1-y)\log\log \frac{n}{y(1-y)}]^{-1/2}|=1 \text{ a. s.}$$

Remark 3. The distribution function $F_0(x)=1-e^{-x}(x\geq 0)$ satisfies (3.3), (3.4), (3.6) and (4.5). In this case it can be shown in an elementary way that $EG_n(u)=0$, $EG_n(u_1)G_n(u_2)=u_1\wedge u_2-u_1u_2$, i. e. Theorem 5 implies a result of R. Barlow cited in [8, p. 110].

In order to prove the theorem we first show

Lemma 3. *If* $F_0$ *satisfies* (3.3), (3.4), (3.6) *and* (4.5), *then*

$$|\int_0^u E_n(y)\varphi(y)dy|=O(n^{-1/4}(\log\log n)^{1/4}(\log n)^{3/2}) \quad a.\ s.,$$

*where* $u\in[0,1]$.

Proof of Lemma 3. Chose $\beta$ such that

(4.7) $$\beta\geq \max(\frac{3}{4}\cdot\frac{r}{r-1};\ 2),$$

where $r$ is given by (4.5).

One has

$$|\int_0^u E_n(y)\varphi(y)dy|\leq\gamma\int_0^1|E_n(y)|[y(1-y)]^{-1}dy$$

$$=\int_0^{1/n^\beta}+\int_{1/n^\beta}^{1-1/n^\beta}+\int_{1-1/n^\beta}^{1}:=K_{n1}+K_{n2}+K_{n3}.$$

From (3.11) it follows that

(4.8) $$K_{n2}\leq \sup_{0<y<1}|E_n(y)|4\gamma\int_{1/n^\beta}^{1/2}y^{-1}dy$$
$$=O(n^{-1/4}(\log\log n)^{1/4}(\log n)^{1/2})\cdot\int_{1/n^\beta}^{1/2}y^{-1}dy=O(n^{-1/4}(\log\log n)^{1/4}(\log)^{3/2}) \text{ a. s.}$$

One has

(4.9) $$K_{n1}\leq\gamma\int_0^{1/n^\beta}n^{-1/2}|K(y,n)|[y(1-y]^{-1}dy$$
$$+\gamma n^{1/2}\int_0^{1/n^\beta}|[f_0(F_0^{-1}(y))]'|\cdot|Q_n^0(y)-F_0^{-1}(y)|dy:=k_{n1}+k_{n2}.$$

The relation (4.6) implies

(4.10) $$k_{n1}\leq 2\gamma\int_0^{1/n^\beta}[\log\log\frac{n}{y(1-y)}]^{1/2}[y(1-y)]^{-1/2}dy=O(n^{-1/4}),$$

and (1.1), (4.5) and (4.7), respectively,

$$k_{n2} \leq \gamma\lambda n^{1/2} \int_0^{1/n^\beta} |Z_{1:n} - [2n(\log n)^2]^{-1}|\, dy + \gamma\lambda n^{1/2} \int_0^{1/2^\beta} |F_0^{-1}(y)|\, dy$$

$$\leq \gamma\lambda n^{1/2} \int_0^{1/n^\beta} |F_0^{-1}(U_{1:n})|\, dy + \gamma\lambda n^{1/2} \int_0^{1/n^\beta} |F_0^{-1}(y)|\, dy + O(n^{-5/2})$$

(4.11)

$$\leq \gamma\lambda^2 n^{1/2} n^{-\beta} (U_{1:n})^{-1/r} + \frac{\gamma\lambda^2 n^{1/2}}{(1-1/r)} \int_0^{1/n^\beta} d(y^{1-1/r}) + O(n^{-5/2})$$

$$\leq \gamma\lambda^2 n^{1/2} n^{-\beta}(n(\log n)^2)^{1/r} + \frac{\gamma\lambda^2}{(1-1/r)} n^{-\beta(r-1)/r+1/2} + O(n^{-5/2}) = O(n^{-1/4}).$$

The last inequality uses the fact that $U_{1:n} > (n(\log n)^2)^{-1}$ a. s. As a consequence of (4.9), (4.10) and (4.11) we have $K_{n1} = O(n^{-1/4})$ a. s. and similarly $K_{n3} = O(n^{-1/4})$ a. s. Thus (4.8) proves the lemma.

Lemma 4. *For $F_0(x)$ satisfying* (3.3), (3.4), (3.6) *and* (4.5) *we have for* $u \in [0, 1]$

(4.12) $$\left| \int_0^u f_0(F_0^{-1}(y))\, dq_n^0(y) \right| = O((\log\log n)^{1/2})\ a.\ s.$$

Proof of Lemma 4. We have

(4.13) $$\left| \int_0^u f_0(F_0^{-1}(y))\, dq_n^0(y) \right|$$

$$\leq |f_0(F_0^{-1}(u))q_n^0(u)| + |f_0(F_0^{-1}(y))q_n^0(y)| + \left| \int_0^u f_0(F_0^{-1}(y))q_n^0(y)\varphi(y)\, dy \right|.$$

Using Theorem 4 and following the lines of the proof in [6, p. 205-206] one gets that the set of limit points of the sequence $\sup_{0<y<1} \left| \frac{f_0(F_0^{-1}(y))\, q_n^0(y)}{(2\log\log n)^{1/2}} \right|$ is the interval [0, 1/2], i. e. the first and second summand of the right hand side of (4.13) are of the order $O((\log\log n)^{1/2})$. For the third summand one has

(4.14) $$\left| \int_0^u f_0(F_0^{-1}(y))q_n^0(y)\varphi(y)\, dy \right| = \left| \int_0^{\delta_n} + \int_{\delta_n}^{1-\delta_n} + \int_{1-\delta_n}^1 \right|,$$

where $\delta_n$ was defined in Theorem 3. This theorem and Lemma 2 yield the chain of inequalities

$$\left| \int_{\beta_n}^{1-\delta_n} f_0(F_0^{-1}(y))q_n^0(y)\,\varphi(y)\, dy \right|$$

$$\leq \left| \int_{\delta_n}^{1-\delta_n} [f_0(F_0^{-1}(y))q_n^0(y) - u_n^0(y)]\varphi(y)\, dy \right| + \left| \int_{\delta_n}^{1-\delta_n} u_n^0(y)\varphi(y)\, dy \right|$$

$$\leq \sup_{\delta_n<y<1-\delta_n} |f_0(F_0^{-1}(y|))q_n^0(y) - u_n^0(y)| \int_{\delta_n}^{1-\delta_n} |\varphi(y)|\, dy$$

(4.15)

$$+\sup_{\delta_n<y<1-\delta_n}|u_n^0(y)|[y(1-y)]^{-1/2}\int_{\delta_n}^{1-\delta_n}|\varphi(y)|[y(1-y)]^{1/2}dy$$

$$=O(n^{-1/2}\log\log n)\int_{\delta_n}^{1-\delta_n}|\varphi(y)|dy+O((\log\log n)^{1/2})\int_{\delta_n}^{1-\delta_n}[y(1-y)]^{-1/2}dy$$

$$=O(\log\log n)^{1/2}).$$

Further, Lemma 3 and (4.6) give us

$$(4.16)\quad |\int_0^{\delta_n}f_0(F_0^{-1}(y))q_n^0(y)\varphi(y)dy|\leqq|\int_0^{\delta_n}E_n(y)\varphi(y)dy|+|\int_0^{\delta_n}n^{-1/2}K(y,n)\varphi(y)dy|$$

$$\leqq O(n^{-1/4}(\log\log n)^{1/4}(\log n)^{3/2})+2\delta\int_0^{\delta_n}[\log\log\frac{n}{y(1-y)}]^{1/2}[y(1-y)]^{-1/2}dy$$

$$=O(n^{-1/4}(\log\log n)^{1/4}(\log n)^{3/2}),$$

where in the last equality $z=2n/y$. Similarly holds the relation

$$(4.17)\qquad |\int_{1-\delta_n}^{1}f_0(F_0^{-1}(y))q_n^0(y)\varphi(y)dy|=O(n^{-1/4}(\log\log n)^{1/4}(\log n)^{3/2}).$$

Now (4.14), (4.15), (4.16), and (4.17) imply (4.12) and thus prove Lemma 4.

Proof of Theorem 5. The relation (4.1) and (4.2) give

$$g_n(u)=n^{1/2}\{\frac{\int_0^u f_0(F_0^{-1}(y))dQ_n^0(y)}{\int_0^1 f_0(F_0^{-1}(y))dQ_n^0(y)}-u\}$$

$$(4.18)\qquad =\{\int_0^u f_0(F_0^{-1}(y))dq_n^0(y)-u\int_0^1 f_0(F_0^{-1}(y))dq_n^0(y)\}$$

$$+\frac{-n^{-1/2}\{\int_0^1 f_0(F_0^{-1}(y))dq_n^0(y)\}}{\int_0^1 f_0(F_0^{-1}(y))dQ_n^0(y)}\{\int_0^u f_0(F_0^{-1}(y))dq_n^0(y)-u\int_0^1 f_0(F_0^{-1}(y))dq_n^0(y)\}$$

$$:=g_{n1}(u)+E_{n1}(u).$$

By Lemma 3 for every $u\in[0,1]$

$$(4.19)\qquad |g_{n1}(u)-G_n(u)|=O(n^{-1/4}(\log\log n)^{1/4}(\log n)^{3/2})\ \text{a. s.}$$

By Lemma 4, again for every $u\in[0,1]$

$$(4.20)\qquad |E_{n1}(u)|=O(n^{-1/2}\log\log n).$$

The theorem is a consequence of (4.18), (4.19) and (4.20).

**5. Asymptotic distribution of the supremum of a normed linear uniform quantile process on subintervals on [0, 1].** Let $U_{1:n}<U_{2:n}<\dots<U_{n:n}$ be the order statistics of a uniform distribution on [0, 1] and $U_{0:n}=0$, $U_{n+1:n}=1$. Consider the linear quantile function $V_n(y)$ defined by

$$(5.1)\qquad \begin{array}{l}V_n(y)=U_{k:n}\ \text{for}\ y=k/(n+1),\ k=0,1,\dots,n+1\\ \text{linear in every subinterval}\ [(k-1)/(n+1),k/(n+1)],\ k=1,\dots,n+1.\end{array}$$

The process

(5.2) $$v_n(y)=(n+2)^{1/2}\{V_n(y)-y\},\quad 0\le y\le 1,$$

will be called a uniform linear quantile process. It has been studied among others by Penkov [9].

We introduce the notations

(5.3)
$$K_n(\varepsilon,\delta)=\sup_{\varepsilon<y<\delta} v_n(y)/\{y(1-y)\}^{1/2},$$
$$L_n(\varepsilon,\delta)=\sup_{\varepsilon<y<\delta} |v_n(y)|/\{y(1-y)\}^{1/2},$$
$$\check{K}_n(\varepsilon,\delta)=\sup_{\varepsilon<y<\delta} v_n(y)/\{V_n(y)[1-V_n(y)]\}^{1/2},$$
$$\tilde{L}_n(\varepsilon,\delta)=\sup_{\varepsilon<y<\delta} |v_n(y)|/\{V_n(y)[1-V_n(y)]\}^{1/2},\quad 0\le\varepsilon\le\delta\le 1,$$

and further those used by Jaeschke [7].

$a(x)=(2\log x)^{1/2}$,

$b(x)=2\log x+2^{-1}\log_2 x-2^{-1}\log\pi\ \ (x>e,\ \log_2 x=\log\log x)$,

$a_n=a(\log n)$,

$b_n=b(\log n)$,

$T_x(t)=(t+b(x))/a(x)$,

(5.4) $\mathsf{E}(t)=\exp\{-\exp(-t)\},\ t\in R$,

$\mu_n=(\log n)^3/(n+1),\ n\ge 3$,

$f_n(u)=(\mu_n\vee u)\wedge(1-\mu_n)\ (u\in[0,1],\ a\wedge b=\min(a,b),\ a\vee b=\max(a,b))$,

$\rho(\varepsilon,\delta)=2^{-1}\log\{\delta(1-\varepsilon)/\varepsilon(1-\delta)\},\ 0<\varepsilon\le\delta<1$,

$\rho_n=\rho_n(\varepsilon_n,\delta_n)=\rho(f_n(\varepsilon_n),f_n(\delta_n))$, where $\delta_n,\varepsilon_n\subset[0,1]$.

Theorem 6. *Put* $\lim_{n\to\infty}\rho_n/\log n=c$, *then*

(5.5) $$\lim_{n\to\infty}\mathsf{P}\{K_n(\varepsilon_n,\delta_n)<T_{\log n}(t)\}=\{\mathsf{E}(t)\}^c,\ t\in R$$

(5.6) $$\lim_{n\to\infty}\mathsf{P}\{L_n(\varepsilon_n,\delta_n)<T_{\log n}(t)\}=\{\mathsf{E}(t)\}^{2c},\ t\in R$$

(5.7) $$\lim_{n\to\infty}\mathsf{P}\{\check{K}_n(\varepsilon_n,\delta_n)<T_{\log n}(t)\}=\{\mathsf{E}(t)\}^c,\ t\in R$$

(5.8) $$\lim_{n\to\infty}\mathsf{P}\{\tilde{L}_n(\varepsilon_n,\delta_n)<T_{\log n}(t)\}=\{\mathsf{E}(t)\}^{2c},\ t\in R$$

Remarks: 1. From the definition of $\rho_n$ it follows that $c\in[0,1]$.

2. It is easy to see that for $c>0$ the relations (5.5)—(5.8) are equivalent to

(5.9) $$\lim_{n\to\infty}\mathsf{P}\{K_n(\varepsilon_n,\delta_n)<T_{\rho_n}(t)\}=\mathsf{E}(t),$$

$$(5.10)\qquad \lim_{n\to\infty} \mathsf{P}\{L_n(\varepsilon_n, \delta_n) < T\rho_n(t)\} = \mathsf{E}^2(t),$$

$$\lim_{n\to\infty} \mathsf{P}\{\hat{K}_n(\varepsilon_n, \delta_n) < T\rho_n(t)\} = \mathsf{E}(t),$$

$$\lim_{n\to\infty} \mathsf{P}\{\hat{L}_n(\varepsilon_n, \delta_n) < T\rho_n(t)\} = \mathsf{E}^2(t).$$

3. In the case $\varepsilon_n = 0$, $\delta_n = 1$ (5.10) implies a result of Eicker [5, Theorem 3, (1.9)].

Proof of Theorem 6. We use the following

Lemma 5 [7]. *Let $0 < \varepsilon_n \le \delta_n < 1$, $\tau_n = \rho(\varepsilon_n, \delta_n) \to \infty$ and $\{B(y), 0 \le y \le 1\}$ s a Brownian bridge. Then for $\forall t \in R$*

$$\lim_{n\to\infty} \mathsf{P}\{\sup_{\varepsilon_n < y < \delta_n} B(y)\{y(1-y)\}^{-1/2} < T\tau_n(t)\} = \mathsf{E}(t),$$

$$\lim_{n\to\infty} \mathsf{P}\{\sup_{\varepsilon_n < y < \delta_n} |B(y)|\{y(1-y)\}^{-1/2} < T\tau_n(t)\} = \mathsf{E}^2(t).$$

Lemma 6. *Let $\varepsilon_n \wedge (1-\delta_n) \ge \mu_n = (\log n)^3/(n+1)$. Then there exists a sequence of Brownian bridges $\{B_n(y), 0 \le y \le 1\}$, so that*

$$K_n(\varepsilon_n, \delta_n) - \sup_{\varepsilon_n < y < \delta_n} B(y)\{y(1-y)\}^{-1/2} = o((\log_2 n)^{-1/2}) \ a.\ s.$$

Proof of Lemma 6. It can be proved that Theorem 1 remains valid, if $u_n(y)$ is replaced by $v_n(y)$, i. e. there exists a sequence of Brownian bridges $\{B_n(y), 0 \le y \le 1\}$ satisfying the relation

$$\sup_{0 \le y \le 1} |v_n(y) - B_n(y)| = O(n^{-1/2} \log n).$$

It follows that

$$\sup_{\varepsilon_n \le y \le \delta_n} \left| \frac{v_n(y)}{\{y(1-y)\}^{1/2}} - \frac{B_n(y)}{\{y(1-y)\}^{1/2}} \right| = O(n^{-1/2} \log n)/(\mu_n)^{1/2} = O((\log n)^{-1/2})$$

$$= o((\log_2 n)^{-1/2}),$$

which prove Lemma 6.

Lemma 7. *Let $\mu_n = (\log n)^3/(n+1)$. The relation*

$$a_n\{L_n(0, \mu_n) \vee L_n(1-\mu_n, 1)\} - b_n \to -\infty$$

*holds.*

Proof of Lemma 7. Because of the symmetry of the intervals $(0, \mu_n)$ and $(1-\mu_n, 1)$ with respect to the point $(1/2)$ it is sufficient to prove

$$(5.11)\qquad a_n L_n(0, \mu_n) - b_n \to_{\mathsf{P}} -\infty.$$

By (5.4) to show that (5.11) holds it is enough to veryfy that $\lim_{n\to\infty} \mathsf{P}\{L_n(0, \mu_n) \ge a_n\} = 0$. Actually we shall prove

$$(5.12)\qquad \lim_{n\to\infty} \mathsf{P}\{L_n(0, \mu_n) \ge a_n^{2/h}\} = 0,\ h = 2, 3, \ldots$$

We have indeed

$$\mathsf{P}\{L_n(0, \mu_n) \ge a_n^{2/h}\} \le \mathsf{P}\{L_n(0, \tfrac{1}{n+1}) \ge \tfrac{1}{2} a_n^{2/h}\} + \mathsf{P}\{L_n(\tfrac{1}{n+1}, \mu_n) \ge \tfrac{1}{2} a_n^{2/h}\} := \mathsf{P}_{1n} + \mathsf{P}_{2n}$$

and from (5.1)

$$
\begin{aligned}
\mathsf{P}_{1n} &= \mathsf{P}\{\sup_{0<y\le 1/(n+1)} \frac{|v_n(y)|}{\{y(1-y)\}^{1/2}} \ge \frac{1}{2} a_n^{2/h}\} \\
&\le \mathsf{P}\{|U_{1:n}-\frac{1}{n+1}| \ge \frac{a_n^{2/h}}{4(n+1)}\} \le \frac{\mathsf{E}|U_{1:n}-\mathsf{E}U_{1:n}|}{(a_n^{2/h}/4(n+1))} \le \frac{4\,.\,11\,.\,(n+1)}{n(2\log_2 n)^{1/h}} \to 0,
\end{aligned} \tag{5.13}
$$

where the last inequality follows from lemma 2 of Wellner [10]. Again from (5.1) we have

$$
\mathsf{P}_{2n} \le \mathsf{P}\{\max_{1\le k\le [\log^3 n]+1} \frac{2(n+2)^{1/2}\,|U_{k:n}-\mathsf{E}U_{k:n}|}{(k/(n+1))^{1/2}} \ge \frac{1}{2} a_n^{2/h}\}
$$

$$
\le \mathsf{P}\{\max_{1\le k\le [\log^3 n]+1} \{\frac{4(n+2)(n-k+1)}{k^{1/2}\, a_n^{2/h}}\} \{\frac{|U_{k:n}-\mathsf{E}U_{k:n}|}{n\,|\,k+1}\} \ge 1\}.
$$

Using an inequality of Birnbaum and Marshall [1] and the fact that $\{(U_{k:n}-\mathsf{E}U_{k:n})/(n-k+1),\ 1\le k\le n\}$ is a martingale we get

$$
\mathsf{P}_{2n} \le \sum_{k=1}^{[\log^3 n]+1} (q_k^{2r}-q_{k+1}^{2r})\mathsf{E}\{\frac{|U_{k:n}-\mathsf{E}U_{k:n}|}{n-k+1}\}^{2r},
$$

where $q_k = 4(n+2)(n-k+1)/k^{1/2}a_n^{2/h}$, $k=1,\dots,[\log^3 n]+1$ and $r$ is an arbitrary integer. Some calculations yield

$$
(q_k^{2r}-q_{k+1}^{2r}) \le D_r(n+1)^{2r}(n-k)^{2r}(2\log_2 n)^{2r/h}k,
$$

where $D_r$ depends only on $r$. Again by Lemma 2 of Wellner [10] we have

$$
\mathsf{E}\{\frac{|U_{k:n}-\mathsf{E}U_{k:n}|}{(n-k+1)}\}^{2r} \le \frac{C_r(k/n^2)^r}{(n-k+1)^{2r}},
$$

where $C_r = 1+2.5^{2r}\,\Gamma(2r+1)$.

The last three inequalities and the common relation $\sum_{k=1}^{m} 1/k = \log m + C + O(1/m)$, $C=$ constant, lead to

$$
\mathsf{P}_{2n} \le \frac{D_r C_r\,.\,(n+1)^{3r}}{n^{2r}} \frac{O(\log_2 n)}{(2n\log_2 n)^{2r/h}} \to 0, \tag{5.14}
$$

if $r$ is chosen to satisfy $2r/h>1$. The relations (5.13) and (5.14) imply (5.12) which proves Lemma 7.

Now let us prove (5.5). It suffices to consider only the case $c>0$, where (5.5) is equivalent to (5.9), see Remark 2 after Theorem 6. For $n$ large enough and $c>0$ one has $\mu_n<\delta_n$, $\varepsilon_n<1-\mu_n$, i. e.

$$
K_n(\varepsilon_n,\delta_n) = K_n(\varepsilon_n,\mu_n\vee\varepsilon_n)\vee K_n(\mu_n\vee\varepsilon_n,\delta_n\wedge(1-\mu_n))\vee K_n(\delta_n\wedge(1-\mu_n)>\delta_n). \tag{5.15}
$$

Lemma 7 says that we have to discuss only the third term on the right hand side of (5.15), for which by Lemma 6 we get the asymptotic distribution (5.9). The proof of (5.6) is quite similar.

In order to prove (5.7) and (5.8) we need the following two lemmas.

Lemma 8. *If* $\varepsilon_n \wedge (1-\delta_n) \geq \lambda_n = \log n/(n+1)$, *then*

$$\tilde{K}_n(\varepsilon_n, \delta_n) - K_n(\varepsilon_n, \delta_n) = o((\log_2 n)^{-1/2}) \ a.\ s. \tag{5.16}$$

Proof of Lemma 8. One can show that (3.1) remains valid, if $u_n(y)$ is replaced by $v_n(y)$, i. e.

$$\limsup_{n\to\infty} \sup_{(0.25\log_2 n)/n \leq y \leq 1-(25\log_2 n)/n} \{y(1-y)\log_2 n\}^{-1/2} |v_n(y)| \leq 4 \tag{5.17}$$

Here from

$$\sup_{\lambda_n \leq y \leq 1-\lambda_n} \frac{y}{V_n(y)} = 1 + O((\log_2 n/\log n)^{1/2}) \ \text{a. s.}$$
$$\sup_{\lambda_n \leq y \leq 1-\lambda_n} \frac{1-y}{1-V_n(y)} = 1 + O((\log_2 n/\log n)^{1/2}) \ \text{a. s.} \tag{5.18}$$

and the relations (5.17) and (5.18) give

$$\tilde{K}_n(\varepsilon_n, \delta_n) \leq K_n(\varepsilon_n, \delta_n) \sup_{\varepsilon_n < y < \delta_n} \left\{\frac{y(1-y)}{V_n(y)[1-V_n(y)]}\right\}^{1/2} \leq K_n(\varepsilon_n, \delta_n)[1 + O((\log_2 n/\log n)^{1/2})]$$

$$= K_n(\varepsilon_n, \delta_n) + O((\log_2 n)^{1/2} O((\log_2 n/\log n)^{1/2}) = K_n(\varepsilon_n, \delta_n) + o((\log_2 n)^{-1/2}),$$

which proves (5.16).

Lemma 9. *Put* $\lambda_n = (\log n)/(n+1)$. *The relation* $a_n[\tilde{L}_n(0, \lambda_n) \vee \tilde{L}_n(1-\lambda_n, 1)] - b_n \to_P -\infty$ *holds.*

Proof of Lemma 9. From Lemma 4 of Jaeschke [7] and $|\tilde{E}_n(y) - V_n^{-1}(y)| \leq 1/n$ for $y \in [0, 1]$ ($\tilde{E}_n(y)$ being defined in paragraph 2) we can show that

$$a_n \sup_{U_{1:n} < y < \mu_n} \frac{(n+2)^{1/2} |V_n^{-1}(y) - y|}{\{y(1-y)\}^{1/2}} - b_n \to_P - \to \infty,$$

where $\mu_n = (\log n)^3/(n+1)$. Taking into account that $U_{[\log n]+1:n} < \mu_n$ a. s., it follows

$$a_n \sup_{U_{1:n} < y < U_{[\log n]+1:n}} \frac{(n+2)^{1/2} |V_n^{-1}(y) - y|}{\{y(1-y)\}^{1/2}} - b_n \to_P -\infty$$

and

$$a_n \tilde{L}_n(1/(n+1), \lambda_n) - b_n \to_P -\infty. \tag{5.19}$$

We prove now that

$$a_n \tilde{L}_n(0, 1/(n+1)) - b_n \to_P -\infty. \tag{5.20}$$

Indeed, we have

$$P\{\sup_{0 < y \leq 1/(n+1)} 4y/V_n(y) \geq a_n\} = P\{4/U_{1:n} \geq (n+1)a_n\}$$

$$= \frac{1}{B(1, n)} \int_0^{4/(n+1)(\log_2 n)^{1/2}} (1-x)^{n-1} dx \to 0,$$

which with (5.12) implies

$$\mathsf{P}\{\tilde{L}_n(0, \frac{1}{n+1}) \geq a_n\} \leq \mathsf{P}\{L_n(0, \frac{1}{n+1})\, 2 \sup_{0<y\leq 1/(n+1)} (\frac{y}{V_n(y)})^{1/2} \geq a_n\}$$

$$\leq \mathsf{P}\{L_n(0, \frac{1}{n+1}) \geq a_n^{1/2}\} + \mathsf{P}\{4 \sup_{0<y\leq 1/(n+1)} \frac{y}{V_n(y)} \geq a_n\} \to 0, \; n \to \infty,$$

which prove (5.20) and with (5.19) also $a_n\tilde{L}_n(0, \lambda_n) - b_n \to_{\mathsf{P}} -\infty$. Similarly one has $a_n\tilde{L}(1-\lambda_n, 1) - b_n \to_{\mathsf{P}} -\infty$. Thus Lemma 9 is proved completely.

Now (5.7) follows from (5.5) and the last two lemmas. Similarly, (5.8) follows.

Now we proceed to some generelizations of (5.5) and (5.6). Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the order statistics of a sample from an absolutely continuous distribution $F(x)$ on $[0, 1]$. Put $X_{0:n}=0$, $X_{n+1:n}=1$. As an analogue of (5.1) (5.2) and (5.3) we introduce

$$W_n(y) = X_{k:n} \text{ for } y = k/(n+1),\ k=0, 1, \ldots, n+1, \text{ linear in every}$$
$$\text{subinterval } [(k-1)/(n+1), k/(n+1)],\ k=1, \ldots, n+1,$$
$$w_n(y) = (n+2)^{1/2}\{W_n(y) - F^{-1}(y)\},\ 0 \leq y \leq 1,$$
$$M_n(\varepsilon, \delta) = \sup_{\varepsilon<y<\delta} f(F^{-1}(y))w_n(y)/\{y(1-y)\}^{1/2},$$
$$N_n(\varepsilon, \delta) = \sup_{\varepsilon<y<\delta} f(F^{-1}(y))\,|w_n(y)|/\{y(1-y)\}^{1/2},\ 0 \leq \varepsilon \leq \delta \leq 1, \text{ where } f(x) = F'(x).$$

Theorem 7. *Suppose that $F(x)$ satisfies the following:*
*$F(x)$ is twice differentiable on $[0, 1]$ and $F'(x) = f(x) \neq 0$ on $(0, 1)$,*

$$\sup_{0<x<1} F(x)(1-F(x))\,|f'(x)|/f^2(x) \leq \gamma < 1,$$

(5.21) *$f(x)$ is nondecreasing (nonincreasing) on an interval to the right of 0 (to the left of 1).*

*Then if* $\lim_{n\to\infty} \rho_n(\varepsilon_n, \delta_n)/\log n = c$,

$$\lim_{n\to\infty} \mathsf{P}\{M_n(\varepsilon_n, \delta_n) < T_{\log n}(t)\} = \{\mathsf{E}(t)\}^c,$$

$$\lim_{n\to\infty} \mathsf{P}\{N_n(\varepsilon_n, \delta_n) < T_{\log n}(t)\} = \{\mathsf{E}(t)\}^{2c}.$$

Proof of Theorem 7. As in the proof of Theorem 6 it is enough to show, that for $c>0$

$$\lim_{n\to\infty} \mathsf{P}\{M_n(\varepsilon_n, \delta_n) < T_{\rho_n}(t)\} = \mathsf{E}(t)$$

$$\lim_{n\to\infty} \mathsf{P}\{N_n(\varepsilon_n, \delta_n) < T_{\rho_n}(t)\} = \mathsf{E}^2(t).$$

Let us prove first two lemmas.

Lemma 10. *Under the conditions of Theorem 7 let $V_n(y)$ and $v_n(y)$ be defined by* (5.1) *and* (5.2), *where $U_{k:n} = F(X_{k:n})$, $k=0, 1, \ldots, n+1$, $K_n(\varepsilon, \delta)$ and $L_n(\varepsilon, \delta)$ being defined by* (5.3). *If $\varepsilon_n \wedge (1-\delta_n) \geq \theta_n = (\log_2 n)^4/(n+1)$, then*

$$M_n(\varepsilon_n, \delta_n) - K_n(\varepsilon_n, \delta_n) = o((\log^2 n^{-1/2}) \text{ a. s.}$$

Proof of Lemma 10. As in Theorem 3 (3.7) we have

$$\sup_{0<y<1} |f(F^{-1}(y))w_n(y) - v_n(y)| = O(n^{-1/2} \log_2 n) \text{ a. s.,}$$

where from

$$M_n(\varepsilon_n, \delta_n) - K_n(\varepsilon_n, \delta_n) = O(n^{-1/2} \log_2 n)/\theta_n^{1/2} = O((\log_2 n)^{-1}) = o((\log_2 n)^{-1/2}),$$

which proves the lemma.

Lemma 11. *Under the conditions of Theorem 7*

$$a_n[N_n(0, \theta_n) \vee N_n(1-\theta_n, 1)] - b_n \to_{\mathbf{P}} -\infty$$

*holds, where $a_n$ and $b_n$ are defined by* (5.4)

Proof of Lemma 11. Again by symmetry of $(0, \theta_n)$ and $(1-\theta_n, 1)$ with respect to $1/2$ it is sufficient to show that

$$a_n N_n(0, \theta_n) - b_n \to_{\mathbf{P}} -\infty \tag{5.22}$$

Let $\tilde{V}_n(y) = U_{k:n} (= F(X_{k:n}))$ for $(k-1)/(n+1) < y \leq k/(n+1)$, $k = 1, \ldots, n+1$, $\tilde{v}_n(y) = (n+2)^{1/2}\{\tilde{V}_n(y) - y\}$. First we prove the following inequality: for every $y \in [2/(n+1), \theta_n]$ holds

$$|f(F^{-1}(y))w_n(y)| \leq |v_n(y)| + A(\gamma)\{|\tilde{v}_n(y')| + (n+2)^{-1/2}\}, \tag{5.23}$$

where $A(\gamma)$ is a constant depending only on $\gamma$ and $y' = y - 1/(n+1)$. Indeed, let $y \in ((k-1)/(n+1), k/(n+1)]$.

If $W_n(y) \geq F^{-1}(y)$, the definition of $W_n(y)$ and the convexity of $F^{-1}(y)$ imply

$$\begin{aligned} |f(F^{-1}(y))w_n(y)| &= (n+2)^{1/2} f(F^{-1}(y))\{W_n(y) - F^{-1}(y)\} \\ &\leq (n+2)^{1/2} f(F^{-1}(y))\{F^{-1}(V_n(y)) - F^{-1}(y)\} = (n+2)^{1/2} \int_y^{V_n(y)} \frac{f(F^{-1}(y))}{f(F^{-1}(u))} du \leq |v_n(y)|. \end{aligned} \tag{5.24}$$

The last of these inequalities follows from (5.21). If $W_n(y) < F^{-1}(y)$ then

$$\begin{aligned} |f(F^{-1}(y))w_n(y)| &= (n+2)^{1/2} f(F^{-1}(y))\{F^{-1}(y) - W_n(y)\} \\ &\leq (n+2)^{1/2} f(F^{-1}(y))\{F^{-1}(y) - F^{-1}(\tilde{V}_n(y'))\} \\ &= (n+2)^{1/2}\left[\frac{f(F^{-1}(y))}{f(F^{-1}(y))}\right]\left[f(F^{-1}(y'))(F^{-1}(y) - F^{-1}(y') + F^{-1}(y) - F^{-1}(\tilde{V}_n(y')))\right]. \end{aligned} \tag{5.25}$$

Using Lemma 1 of [4] and (5.21), one can prove that

$$f(F^{-1}(y))/f(F^{-1}(y')) \leq 2^\gamma, \tag{5.26}$$

$$f(F^{-1}(y))(F^{-1}(y) - F^{-1}(y')) \leq y - y' = 1/(n+1), \tag{5.26}$$

$$|F^{-1}(y') - F^{-1}(\tilde{V}_n(y'))| \leq 2^\gamma |y' - \tilde{V}_n(y')|/(1-\gamma).$$

The relations (5.24), (5.25) and (5.26) imply (5.23).

As in Lemma 7 we have

$$a_n \sup_{2/(n+1)<y<\theta_n} A(\gamma)\frac{|\tilde{v}_n(y)|}{\{y(1-y)\}^{1/2}} - b_n \to_P -\infty,$$

which with (5.23) shows that

(5.27) $$a_n N_n(2/(n+1), \theta_n) - b_n \to_P -\infty.$$

On the other hand,

(5.28) $$a_n N_n(0, 2/(n+1)) - b_n \to_P -\infty.$$

Now (5.22) is implied by (5.27) and (5.28), which proves Lemma 11.

Theorem 7 itself is a consequence of (5.5), (5.6) and Lemmas 10 and 11.

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