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THE AXIOM OF COHOLOMORPHIC $(2n+1)$-SPHERES IN THE ALMOST HERMITIAN GEOMETRY

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In his book on Riemannian geometry [1] E. Cartan proved a characterization of a real-space-form, using the axiom of planes. There are many results in this direction also for a Kaehler manifold. B.-Y. Chen and K. Ogiue [4] have proved that a Kaehler manifold, which satisfies the axiom of coholomorphic 3-spheres is flat. In this paper we prove a generalization of this theorem for an almost Hermitian manifold.

1. Introduction. Let N be an n-dimensional submanifold of a $2m$-dimensional almost Hermitian manifold M with Riemannian metric g and almost complex structure J. Let $\tilde{\nabla}$ and ∇ be the covariant differentiations on M and N, respectively. It is well known, that the equation $\alpha(X, Y)=\tilde{\nabla}_X Y-\nabla_X Y$, where $X, Y\in\mathfrak{X}N$ defines a normal-bundle-valued symmetric tensor field, called the second fundamental form of the immersion. The submanifold N is said to be totally umbilical, if $\alpha(X, Y)=g(X, Y)H$ for all $X, Y\in\mathfrak{X}N$ where $H=(1/n)$ trace α is the mean curvature vector of N in M. In particular, if α vanishes identically, N is called a totally geodesic submanifold of M.

For $X\in\mathfrak{X}N$, $\xi\in\mathfrak{X}N^{\perp}$ we write $\tilde{\nabla}_X\xi=-A_{\xi}X+D_X\xi$, where $-A_{\xi}X$ (respectively $D_X\xi$) denotes the tangential (respectively, the normal) component of $\tilde{\nabla}_X\xi$. A normal vector field ξ is said to be parallel, if $D_X\xi=0$ for each $X\in\mathfrak{X}N$.

By an n-plane we mean an n-dimensional linear subspace of a tangent space. A $2n$-plane (respectively an n-plane) where $1\leq n\leq m$ is said to be holomorphic (respectively, antiholomorphic) if $J\alpha=\alpha$ (respectively $J\alpha\perp\alpha$). A $(2n+1)$-plane is called coholomorphic if it contains a holomorphic $2n$-plane.

An almost Hermitian manifold M is said to satisfy the axiom of holomorphic $2n$-planes (respectively $2n$-spheres) if for each point $p\in M$ and for any $2n$-dimensional holomorphic plane π in T_pM there exists a totally geodesic submanifold N (respectively a totally umbilical submanifold N with nonzero parallel mean curvature vector) containing p, such that $T_pN=\pi$, where n is a fixed integer, $1\leq n<m$.

An almost Hermitian manifold M is said to satisfy the axiom of antiholomorphic n-planes (respectively n-spheres) if for each point $p\in M$ and for any n-dimensional antiholomorphic plane π in T_pM there exists a totally geodesic submanifold N (respectively a totally umbilical submanifold N with nonzero parallel mean curvature vector) containing p, such that $T_pN=\pi$, where n is a fixed integer, $1<n\leq m$.

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An almost Hermitian manifold M is called an RK-manifold, if $R(X, Y, Z, U) = R(JX, JY, JZ, JU)$ for all $X, Y, Z, U \in T_pM$, $p \in M$.

We have proved in [5]:

Theorem A. *Let M be a $2m$-dimensional almost Hermitian manifold, $m \geq 2$. If M satisfies the axiom of holomorphic $2n$-planes or the axiom of holomorphic $2n$-spheres for some n, $1 \leq n < m$, then M is an RK-manifold with pointwise constant holomorphic sectional curvature.*

Theorem B. *Let M be a $2m$-dimensional almost Hermitian manifold, $m \geq 2$. If M satisfies the axiom of antiholomorphic n-planes or the axiom of antiholomorphic n-spheres for some n, $1 < n \leq m$, then M is an RK-manifold with pointwise constant holomorphic sectional curvature and with pointwise constant antiholomorphic sectional curvature. Consequently, if $m \geq 3$, then M is one of the following:*

1) a real-space-form, or

2) a complex- space-form.

These theorems generalize some results in [3, 6, 9].

It is not difficult to see that if $n > 1$ then the holomorphic analogue of Theorem B holds.

Following B.-Y. Chen and K. Ogiue [4], L. Vanhecke formulates the following axiom of coholomorphic $(2n+1)$-spheres [8]:

For each point $p \in M$ and for each coholomorphic $(2n+1)$-plane π in T_pM, there exists a $(2n+1)$-dimensional totally umbilical submanifold N of M containing p, such that $T_pN = \pi$, where n is a fixed integer, $1 \leq n < m$.

We shall prove the following theorem.

Theorem. *Let M be a $2m$-dimensional almost Hermitian manifold, $m \geq 2$. If M satisfies the axiom of coholomorphic $(2n+1)$-spheres for some n, then M is conformal flat.*

Hence, using [7] we have

Corollary 1. *Let M be a $2m$-dimensional connected Kaehler manifold, $m \geq 2$. If M satisfies the axiom of coholomorphic $(2n+1)$-spheres for some n, then either M is flat or M is locally a product of two 2-dimensional Keahler manifolds with constant curvature K and $-K$, respectively, $K > 0$.*

The case $m \geq 3$ in corollary 1 is treated in [4].

An almost Hermitian manifold M which satisfies $(\tilde{\nabla}_X J)X = 0$ for all $X \in \mathfrak{X}M$ is said to be an NK-manifold. Using the classification in [7] we have also

Corollary 2. *Let M be a $2m$-dimensional NK-manifold, $m \geq 2$. If M satisfies the axiom of coholomorphic $(2n+1)$-spheres for some n, then M is one of the following:*

1) a flat Kaehler manifold,

2) locally a product $M_1 \times M_2$, where M_1 (respectively M_2) is a 2-dimensional Kaehler manifold with constant curvature K (respectively $-K$),

3) a 6-dimensional manifold of constant curvature $K > 0$,

4) locally a product $M_3 \times M_2$, where M_3 is a 6-dimensional NK-manifold of constant curvature $K > 0$.

An almost Hermitian manifold M is said to be of pointwise constant type α, provided that for each point $p \in M$ and for each $X \in T_p(M)$ we have $\alpha(p)g(X, X) = \lambda(X, Y) = \lambda(X, Z)$ with $\lambda(X, Y) = R(X, Y, Y, X) - R(X, Y, JY, JX)$ whenever the planes defined by X, Y and X, Z are antiholomorphic and $g(Y, Y) = g(Z, Z) = 1$. If for $X, Y \in \mathfrak{X}(M)$ with $g(JX, Y) = g(X, Y) = 0$, $\lambda(X, Y)$ is a constant whenever $g(X, X) = g(Y, Y) = 1$ then M is said to have global constant type.

Corollary 3. *Let M be an almost Hermitian manifold with pointwise constant type* α. *If M satisfies the axiom of coholomorphic* $(2n+1)$*-spheres for some n and if* $\dim M \geqq 6$, *then M is a space of constant curvature* α *and M has global constant type.*

Corollary 3 is proved in [8] for an RK-manifold.

2. Preliminaries. Let M be a $2m$-dimensional almost Hermitian manifold with Riemannian metric g, almost complex structure J and covariant differentiation ∇. The curvature tensor R, associated with ∇ has the following properties:

1) $R(X, Y)Z = -R(Y, X)Z$

2) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$

3) $R(X, Y, Z, U) = -R(X, Y, U, Z)$

for all $X, Y, Z, U \in T_p(M)$, $p \in M$, where $R(X, Y, Z, U) = g(R(X, Y)Z, U)$. The Weyl conformal curvature tensor C is defined by

$$C(X, Y, Z, U) = R(X, Y, Z, U) - (1/(2m-2))\{g(X, U)S(Y, Z)$$
$$-g(X, Z)S(Y, U) + g(Y, Z)S(X, U) - g(Y, U)S(X, Z)\}$$
$$+(S(p)/((2m-1)(2m-2)))\{(X, U)g(Y, Z) - g(X, Z)g(Y, U)\},$$

where S and $S(p)$ are the Ricci tensor and the scalar curvature of M, respectively.

Now, let N be a submanifold of M, as in section 1. The normal component of $R(X, Y)Z$, where $X, Y, Z \in \mathfrak{X} N$ is given by

$$(R(X, Y)Z^{\perp} = (\bar{\nabla}_X\alpha)(Y, Z) - (\bar{\nabla}_Y\alpha)(X, Z), \tag{2.1}$$

where $(\bar{\nabla}_X\alpha)(Y, Z) = D_X\alpha(Y, Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)$ and if N is totally umbilical submanifold of M, (2.1) reduces to

$$(R(X, Y)Z^{\perp} = g(Y, Z)D_X H - g(X, Z)D_Y H. \tag{2.2}$$

3. Proof of the theorem. Let X, Y be arbitrary unit vectors in T_pM, $d \in M$, such that X is perpendicular to Y, JY. Applying the axiom of coholomorphic $(2n+1)$-spheres for a coholomorphic plane, which contains X, JX, JY and is perpendicular to Y and using (2.2) we obtain

$$R(X, JX, JY, Y) = 0, \tag{3.1}$$

$$R(JY, JX, X, Y) = 0, \tag{3.2}$$

$$R(X, JX, JX, Y) = g(D_X H, Y),$$

$$R(X, JY, JY, Y) = g(D_X H, Y).$$

Hence

$$R(X, JX, YX, JY) = R(X, JY, JY, Y). \tag{3.3}$$

From (3.2) we have $R(Y+JY, JX, X, Y-JY) = 0$ and consequently

$$R(X, Y, Y, JX) = R(X, JY, JY, JX). \tag{3.4}$$

If $m>2$, we take a unit vector Z, perpendicular to X, JX, Y, JY. Using again the axiom of coholomorphic $(2n+1)$-spheres and (2.2) we find

(3.5) $$R(X, JX, Y, Z)=R(X, Y, JY, Z)=0,$$

(3.6) $$R(X, JX, JX, Z)=R(X, Y, Y, Z),$$

(3.7) $$R(X, Y, Y, JX)=R(X, Z, Z, JX).$$

If $m\geq 4$, let U be a unit vector in T_pM, perpendicular to X, JX, Y, JY, Z, JZ From (3.6) we have $2R(X, JX, JX, U)=R(X, Y+Z, Y+Z, U)$, which gives $R(X, Y, Z, U)=-R(X, Z, Y, U)$.

Hence, by the properties of the curvature tensor R we obtain

(3.8) $$R(X, Y, Z, U)=0$$

Making use of (3.1)–(3.8) it is not difficult to prove that $R(X, Y, Z, U)=0$ for an arbitrary orthogonal quadriple $X, Y, Z, U\in T_pM$ According to a well known theorem of Schouten [2] the Weyl conformal curvature tensor M vanishes.

Remark. If a Riemannian manifold M of dimension $m>3$ is conformal flat, then there exists a totally umbilical submanifold N of dimension $n<m$ through every point of M and in every n-dimensional direction at that point (see [2]). Consequently, if M is a conformal flat $2m$-dimensional almost Hermitian manifold, $m\geq 2$, then M satisfies the axiom of coholomorphic $(2n+1)$-spheres for every n, $1\leq n<m$.

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