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A CONVOLUTIONAL APPROACH TO THE MULTIPLIER PROBLEM CONNECTED WITH GENERALIZED EIGENVECTOR EXPANSIONS OF AN UNBOUNDED OPERATOR

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Some relations about the convolutions of a given operator with simple point spectrum in linear topological space, for their multipliers and for the coefficient multipliers and convolutions of its generalized eigenvector expansion are considered. Some applications on Dirichlet and Sturm-Liouville expansions are given.

0. Preliminarilies. This paper is devoted to some relations for the convolutions and their multipliers of a given linear operator L defined in a subspace X_L of a linear topological space X and having a simple point spectrum and for the coefficient multipliers and coefficient convolutions of its generalized eigenvector expansions. Some applications on real and complex Dirichlet expansions as on Sturm-Liouville expansions are given.

The basic tool in our approach is the notion convolution of a linear operator introduced from I. Dimovski in [1]. Since our purpose is to consider also operators which are not defined in the whole X we use a slight modification of Dimovski definition.

Definition 0.1. *Let $M: X_M \to X$, $X_M \subset X$ be a linear operator defined in a linear subspace X_M of a linear space X. A bilinear, commutative and associative operation $f*g$ in X is said to be a convolution of M in X iff 1) X_M is an ideal of X relative to $f*g$ and 2) the equality*

(0.1) *$M(f*g)=Mf*g$ holds for each $f \in X_M$ and each $g \in X$.*

Every operator M defined in the whole X satisfying (0.1) is said to be a *multiplier* of $f*g$ [2]. A nonzero element $f \in X$ is said to be an *annihilator* of $*$ iff $f*g=0$ for all $g \in X$. When $f*g$ is *without annihilators* we write for short: $*$ is w. a.

Throughout the paper we assume that the operator L is defined in a subspace X_L of X and that L has a nonempty point spectrum Λ of simple eigenvalues, i. e. the eigensubspace $\mathrm{Ker}(L-\lambda I)$ is one-dimensional for each $\lambda \in \Lambda$.

Definition 0.2. ([3, p. 171, 177]). *Let $\overset{\lambda}{u}_0$ be an eigenvector of L corresponding to $\lambda \in \Lambda$. Then $\overset{\lambda}{u}_0$ is said to be a generalized eigenvector of 0-th order. Inductively $\overset{\lambda}{u}_k$ is said to be a generalized eigenvector of k-th order corresponding to λ and associated with $\overset{\lambda}{u}_0$* iff $L\overset{\lambda}{u}_k=\lambda\overset{\lambda}{u}_k+\overset{\lambda}{u}_{k-1}$, *where $\overset{\lambda}{u}_{k-1}$ is of $(k-1)$-th order. By m_λ we denote the maximal integer such that $\overset{\lambda}{u}_{m_\lambda}$ exists.*

If such a number does not exist, then $m_\lambda=\infty$ *by definition. Let* H_λ *denote the root subspace corresponding to* $\lambda\in\Lambda$, *i. e.* $H_\lambda \overset{\text{def}}{=} \operatorname{Ker}(L-\lambda I)^{m_\lambda+1}$ *when* $m_\lambda<\infty$, *or* $H_\lambda \overset{\text{def}}{=} \bigcup_{k=0}^{\infty}\operatorname{Ker}(L-\lambda I)^k$ *when* $m_\lambda=\infty$. *It is clear that in every* H_λ *there is a linearly independent generalized eigenvector system* $S_\lambda=\{\overset{\lambda}{u}_k\}_{k=0}^{m_\lambda}$. *Let* $S\overset{\text{def}}{=}\bigcup_{\lambda\in\Lambda}S_\lambda$. *The system* S *is said to be complete iff its linear span* $\langle S\rangle$ *is dense in* X.

Definition 0.3. *Let* $\mathcal{P}=\{P_\alpha\}_{\alpha\in A}$ *be a family of projections in* X. *The system* $\mathcal{P}$ *is said to be orthogonal iff* $P_\alpha P_\beta=0$ *for all* $\alpha,\beta\in A$, $\alpha\neq\beta$. *The system* $\mathcal{P}$ *is said to be total in* X *iff* $P_\alpha f=0$ *for all* $\alpha\in A$ *implies* $f=0$. *The system* $\mathcal{P}$ *is said to be fundamental in* X *iff the linear span of* $\bigcup_{\alpha\in A}P_\alpha(X)$ *is dense in* X.

Definition 0.4. *Let* $\mathcal{P}=\{P_\alpha\}_{\alpha\in A}$ *be a total and orthogonal projection system in* X. *Then with each* $f\in X$ *one may associate its formal Fourier expansion* $f\sim\Sigma_{\alpha\in A}P_\alpha f$. *An operator* $M\colon X\to X$ *is said to be a coefficient multiplier of the system* $\mathcal{P}$, *or of formal Fourier expansion connected with* $\mathcal{P}$ *iff there is a family of scalars* $\{\mu_\alpha\}_{\alpha\in A}$ *such that:*

$$P_\alpha(Mf)=\mu_\alpha P_\alpha f \text{ for each } f\in X. \tag{0.2}$$

For details about the last definition, see e. g. [4, p. 10].

In section 1 a theorem about nonexistence of nontrivial continuous convolutions for a class of linear operators is proved. As consequence a result about nonexistence of nontrivial continuous convolutions of the differentiation operator d/dt in the space C^k is proved. Analogous result about nonexistence of continuous convolutions w. a. for the Cesaro operator $\frac{1}{t}\int_0^t$ in the space $C[0,T]$ is also proved. The main purpose in section 2 are two multiplier projection theorems (2.5 and 2.6). Section 3 is devoted to some relations about the multipliers of a convolution and the coefficient multipliers of a certain total and orthogonal multiplier projection system connected with L, as about the operators commuting with L and with all projections of this system. In section 4 some applications for Dirichlet and Sturm-Liouville expansions are made. In section 5 a connection between all convolutions of the operator L which are convolutions for the multiplier projections is considered. It is proved that all Mikusinski's rings defined by these convolutions are isomorphical. A theorem for existence of continuous convolutions w. a. for a class of closed operators in a Banach space is also proved.

We need an elementary proposition for generalized eigenvectors.

Lemma 0.1. *Let* $\lambda\in\Lambda$ *be fixed, let* $\{\overset{\lambda}{u}_k\}_{k=0}^{m_\lambda}$, $m_\lambda<\infty$ *be a generalized eigenvector system corresponding to* λ, *and let* $\{v_0,\ldots,v_{m_\lambda}\}$ *be an arbitrary solution of the system*

$$Lv_0=\lambda v_0,\ Lv_1=\lambda v_1+v_0,\ \ldots,\ Lv_{m_\lambda}=\lambda v_{m_\lambda}+v_{m_\lambda-1}. \tag{0.3}$$

Then there exists a scalar family $\{\overset{\lambda}{C}_0,\ldots,\overset{\lambda}{C}_{m_\lambda}\}$ *such that*

$$v_k=\overset{\lambda}{C}_k\overset{\lambda}{u}_0+\cdots+\overset{\lambda}{C}_0\overset{\lambda}{u}_k \text{ for } 0\le k\le m_\lambda. \tag{0.4}$$

If $v_0 \neq 0$, *i. e. if* $\overset{\lambda}{C}_0 \neq 0$, *then* $\{v_0, \ldots, v_{m_\lambda}\}$ *is a linearly independent system.*

1. Necessary conditions for existence of convolutions.

Lemma 1.1. *Let* $f*g$ *be an arbitrary convolution of* L *in* X.

a) *If* $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$ *then* $\overset{\lambda}{u}_k * \overset{\mu}{u}_p = 0$ *for* $0 \leq k \leq m_\lambda$, $0 \leq p \leq m_\mu$.

b) *If* $\lambda \in \Lambda$ *is fixed and if* $0 < m_\lambda \leq \infty$ *then* $\overset{\lambda}{u}_k * \overset{\lambda}{u}_p = 0$ *for all* $k, p \geq 0$ $k+p < m_\lambda$. *In particular* $\overset{\lambda}{u}_k * \overset{\lambda}{u}_p = 0$ *for all* $k, p \geq 0$ *if* $m_\lambda = \infty$.

Proof. a) The case $k=0, p=0$ is established by I. Dimovski [5]. Indeed now $\lambda \overset{\lambda}{u}_0 * \overset{\mu}{u}_0 = L\overset{\lambda}{u}_0 * \overset{\mu}{u}_0 = \overset{\lambda}{u}_0 * L\overset{\mu}{u}_0 = \mu \overset{\lambda}{u}_0 * \overset{\mu}{u}_0$. Hence $\overset{\lambda}{u}_0 * \overset{\mu}{u}_0 = 0$. The general case follows from the identity $\lambda \overset{\lambda}{u}_k * \overset{\mu}{u}_p + \overset{\lambda}{u}_{k-1} * \overset{\mu}{u}_p = L\overset{\lambda}{u}_k * \overset{\mu}{u}_p = \overset{\lambda}{u}_k * L\overset{\mu}{u}_p$ $= \mu \overset{\lambda}{u}_k * \overset{\mu}{u}_p + \overset{\lambda}{u}_k * \overset{\mu}{u}_{p-1}$ by induction relative to the integer $N \geq 0$ and the pairs (k, p) with $k+p=N$.

b) First let $0 \leq k < m_\lambda$ be fixed. Then there is $\overset{\lambda}{u}_{k+1}$ and $\lambda \overset{\lambda}{u}_{k+1} * \overset{\lambda}{u}_0 = \overset{\lambda}{u}_{k+1} * L\overset{\lambda}{u}_0$ $= L\overset{\lambda}{u}_{k+1} * \overset{\lambda}{u}_0 = \lambda \overset{\lambda}{u}_{k+1} * \overset{\lambda}{u}_0 + \overset{\lambda}{u}_k * \overset{\lambda}{u}_0$. Hence $\overset{\lambda}{u}_k * \overset{\lambda}{u}_0 = 0$ and the proposition is established for $p=0$. Let it be true for $p-1$, i. e. $\overset{\lambda}{u}_{k'} * \overset{\lambda}{u}_{p-1} = 0$ holds for all k', with $0 \leq k' < m_\lambda - p + 1$. Now if $0 \leq k \leq m_\lambda - p$ there exists $\overset{\lambda}{u}_{k+1}$ (since $p > 0$) and $\lambda \overset{\lambda}{u}_{k+1} * \overset{\lambda}{u}_p + \overset{\lambda}{u}_k * \overset{\lambda}{u}_p = L\overset{\lambda}{u}_{k+1} * \overset{\lambda}{u}_p = \overset{\lambda}{u}_{k+1} * L\overset{\lambda}{u}_p = \lambda \overset{\lambda}{u}_{k+1} * \overset{\lambda}{u}_p + \overset{\lambda}{u}_{k+1} * \overset{\lambda}{u}_{p-1}$. But $\overset{\lambda}{u}_{k+1} * \overset{\lambda}{u}_{p-1}$ $= 0$ since $k+1 < m_\lambda - p + 1$ and therefore, $\overset{\lambda}{u}_k * \overset{\lambda}{u}_p = 0$.

Theorem 1.2. *Let* L *has a complete generalized eigenvector system* S. *Then:*

a) *If* $\lambda \in \Lambda$ *and* $\dim H_\lambda = \infty$, *then every* $\overset{\lambda}{u}_k$, $k \geq 0$ *is an annihilator of every separately continuous convolution of* L, *i. e.* L *has not separately continuous convolutions w. a.*

b) *If* $\dim H_\lambda = \infty$ *for all* $\lambda \in \Lambda$, *then* L *has no nontrivial separately continuous convolutions.*

c) *If* L *has a separately continuous convolution w. a.* $f*g$, *then for each* $\lambda \in \Lambda$ *it follows that* $\dim H_\lambda < \infty$, $\overset{\lambda}{u}_0 * \overset{\lambda}{u}_{m_\lambda} \neq 0$ *and the system* $S \setminus \{\overset{\lambda}{u}_{m_\lambda}\}$ *is not complete.*

d) *If there is a* $\lambda \in \Lambda$ *with* $\dim H_\lambda < \infty$ *and if* $S \setminus \{\overset{\lambda}{u}_{m_\lambda}\}$ *is complete, then* L *has no separately continuous convolutions w. a.*

e) *If* $1 < \dim H_\lambda < \infty$ *for each* $\lambda \in \Lambda$ *and if* $\bigcup_{\lambda \in \Lambda} \{\overset{\lambda}{u}_0\}$ *is complete, then* L *has no nontrivial separately continuous convolutions.*

Proof. a) For an arbitrary convolution of L from Lemma 1.1 it follows that $\overset{\lambda}{u}_k * \overset{\mu}{u}_p = 0$ for fixed $\lambda \in \Lambda$, $0 \leq k$ and for all $\mu \in \Lambda$, $0 \leq p \leq m_\mu$. Now by approximation we obtain $\overset{\lambda}{u}_k * g = 0$ for each $g \in X$ when $*$ is separately continuous. b) The assertion follows in a similar way. c): It follows from a) that now $\dim H_\lambda < \infty$ for each $\lambda \in \Lambda$. If $\overset{\lambda}{u}_0 * \overset{\lambda}{u}_{m_\lambda} = 0$ or if $S \setminus \{\overset{\lambda}{u}_{m\lambda}\}$ is complete for some

$\lambda \in \Lambda$ then by completeness property it follows in a similar way that $\overset{\lambda}{u}_0$ is an annihilator. d) and e) follow easily.

Theorem 1.3. *The differentiation operator d/dt has not nontrivial continuous convolutions in the spaces: $C^k(a, b)$, $C^k[a, b]$ if it is considered with domain $C^{k+1}(a, b)$ or $C^{k+1}[a, b]$ respectively* ($0 \leq k \leq \infty$; $-\infty \leq a, b \leq +\infty$), *in the spaces $L^1(a, b)$, $L^1[a, b]$* ($-\infty < a < b < +\infty$) *and in the spaces $L^1_{\text{loc}}(a, b)$* ($-\infty \leq a < b \leq +\infty$) *if it is considered with domain $AC(a, b)$ or $AC[a, b]$.*

The proposition follows from the fact that now $\Lambda = \mathbf{C}$ and $\dim H_\lambda = \infty$ for all $\lambda \in \Lambda$, since $H_\lambda = \langle e^{\lambda t}, te^{\lambda t}, t^2 e^{\lambda t}, \ldots \rangle$ is dense in these spaces.

We note that d/dt has continuous convolutions if it is considered with smaller domain $C_\Phi^{k+1} = \{f \in C^{k+1} : \Phi(f) = 0\}$ or $AC_\Phi = \{f \in AC : \Phi(f) = 0\}$ (see section 4 and [13]).

Theorem 1.4. *The Cesaro operator $L = \frac{1}{t}\int_0^t$ has no continuous convolutions w. a. in the spaces $C[0, T]$ or $C[0, T)$, $0 < T \leq \infty$. Every continuous convolution of this operator has the form*

$$f * g = \alpha f(0) g(0) \quad \text{with} \quad \alpha \in \mathbf{C}. \tag{1.1}$$

Proof. Let $f * g$ be an arbitrary continuous convolution of L. Now every $\lambda = (n+1)^{-1}$, $n \geq 0$ is an eigenvalue with an eigenfunction t^n. For each $n > 0$ there is an associated generalized eigenfunction $-(n+1)^2 t^n \ln t \in C[0, T]$ (for $n = 0$ the associated function $-\ln t$ does not belong to $C[0, T]$). Then it follows that $t^n * t^k = 0$ for all integer $n \geq 0, k > 0$ and using an approximation we obtain that $f * g = 0$ for all $f, g \in C[0, T]$ when $f(0) = 0$ or $g(0) = 0$. Now let f, g are arbitrary functions from $C[0, T]$. Then $f = f(0) + \tilde{f}$, $g = g(0) + \tilde{g}$, where $\tilde{f}, \tilde{g} \in C[0, T]$, $\tilde{f}(0) = \tilde{g}(0) = 0$ and we have $f * g = f(0) g(0) 1 * 1 + \tilde{f} * g$, $+ f(0) * \tilde{g} = f(0) g(0) 1 * 1$. But $L(1 * 1) = L1 * 1 = 1 * 1$ since $L1 = 1$, i. e. $1 * 1$ is an eigenfunction corresponding to the simple eigenvalue $\lambda = 1$, hence $1 * 1 = \alpha \in \mathbf{C}$.

2. Convolutions and multiplier projections connected with a generalized eigenvector system. When $\dim H_\lambda = \infty$ Lemma 1.1 shows that every convolution of L is identically equal to 0 in the root subspace H_λ. Let us study now more precisely the act of an arbitrary convolution $f * g$ of L in H_λ when $\dim H_\lambda < \infty$. In this section we denote by $\{u_0, \ldots, u_m\}$ a generalized eigenvector basis in H_λ for a fixed $\lambda \in \Lambda$ satisfying the conditions $Lu_0 = \lambda u_0$, $Lu_1 = \lambda u_1 + u_0, \ldots, Lu_m = \lambda u_m + u_{m-1}$, i. e. $H_\lambda = \langle u_0, \ldots, u_m \rangle$ with $m = \dim H_\lambda - 1$.

Theorem 2.1. *There exist linear functionals $\{C_k(f)\}_{k=0}^m$ which do not depend on such kind choice of the basis $\{u_0, \ldots, u_m\}$ such that:*

$$f * u_k = C_k(f) u_0 + \cdots + C_0(f) u_k \quad \text{for } 0 \leq k \leq m \tag{2.1}$$

$$C_k(f * g) = C_k(f) C_0(g) + \cdots + C_0(f) C_k(g) \quad \text{for } 0 \leq k \leq m;\ f, g \in X. \tag{2.2}$$

The functionals $C_k(f)$ are continuous when $f * g$ is separately continuous If $f * g$ is w. a. in H_λ then $C_0(u_m) \neq 0$. In particular $C_0(u_m) \neq 0$ if S is complete and if $*$ is a separately continuous convolution w. a.

Proof. Let $v_k = f * u_k$, $0 \leq k \leq m$. Then it follows by (0.1) that $Lv_0 = \lambda v_0$, $Lv_k = \lambda v_k + v_{k-1}$, $1 \leq k \leq m$ and using Lemma 0.1 we obtain that there exist constants $C_k = C_k(f)$, $0 \leq k \leq m$ such that $v_k = C_k u_0 + \cdots + C_0 u_k$, $0 \leq k \leq m$. The independence of $C_k(f)$ can be obtained directly using Lemma 0.1 again.

If $\{\tilde{u}_0, \dots, \tilde{u}_m\}$ is other generalized eigenvector basis of such kind, then $\tilde{u}_k = \Sigma_{j=0}^{k} \beta_{k-j} u_j$, $0 \leq k \leq m$ and

$$f * \tilde{u}_k = \sum_{l=0}^{k} \beta_{k-l} f * u_l = \sum_{l=0}^{k} \beta_{k-l} \sum_{i=0}^{l} C_i(f) u_{l-i}$$

$$= \sum_{i=0}^{k} C_i(f) \sum_{j=0}^{k-i} \beta_{k-i-j} u_j = \sum_{i=0}^{k} C_i(f) \tilde{u}_{k-i} = \sum_{i=0}^{k} C_{k-i}(f) \tilde{u}_i.$$

The formula (2.2) can be obtained directly but it is more convenient to use (2.7) and (2.8) we shall prove. At last if $f*g$ is w. a. in H_λ then $C_0(u_m)u_0 = u_0 * u_m \neq 0$, and we obtain the last proposition using Theorem 1.2c).

Lemma 2.2. *For an arbitrary basis of generalized eigenvectors of mentioned kind $\{u_0, \dots, u_m\}$ in H_λ one has:*

$$C_p(u_k) = \begin{cases} 0, & k+p<m \\ C_{k+p-m}(u_m), & k+p \geq m \end{cases} \tag{2.3}$$

$$u_k * u_p = \begin{cases} 0, & k+p<m \\ \sum_{i=0}^{k+p-m} c_{k+p-m-i} u_i, & k+p \geq m \end{cases} \quad \text{where } c_k \overset{\text{def}}{=} C_k(u_m). \tag{2.4}$$

Proof. Since $u_k * u_m = u_m * u_k$ using (2.1) and Lemma 1.1 we get (2.3). Now (2.4) can be obtained using (2.1) again.

I. Dimovski asserts (personal communication to the author) that (2.4) follows from the evident useful formula

$$u_k * u_p = (L - \lambda I)^{2m-k-p} u_m * u_m \quad \text{too.} \tag{2.5}$$

Theorem 2.3 *Let $\{u_0, \dots, u_m\}$ be a basis in H_λ such that $u_0 * u_m \neq 0$, and let $c_k \overset{\text{def}}{=} C_k(u_m)$ (i. e. $c_0 \neq 0$ now). Then there exists a multiplier projection $P\colon X \to H_\lambda$ of the form*

$$Pf = f * \{\alpha_0 u_m + \dots + \alpha_m u_0\}, \tag{2.6}$$

where $\alpha_0 = c_0^{-1}$, $\alpha_k = -c_0^{-1}(\alpha_{k-1} c_1 + \dots + \alpha_0 c_k)$, $k = 1, \dots, m$. P is the unique multiplier projection from X to H_λ when $*$ is w. a. in X.

Proof. It is clear from (2.1) that for arbitrary α_k the operator P defined by (2.6) maps X in H_λ. To be P a projection it is enough to determine α_k such that $Pu_k = u_k$, $0 \leq k \leq m$. By elementary calculations using (2.4) we obtain finally that it is enough α_k to be a solution of the triangle system $\alpha_0 c_0 = 1$, $\alpha_0 c_1 + \alpha_1 c_0 = 0, \dots, \alpha_0 c_m + \dots + \alpha_m c_0 = 0$. It is known [2] p. 20 that if $f*g$ is w. a. then all its multipliers commute. Now if P and Q are two multiplier projections on H_λ, then they commute and hence $P = Q$.

Corollary 2.4. *Under the same assumptions as in Theorem 2.3 there is a basis $\{\tilde{u}_0, \dots, \tilde{u}_m\}$ of generalized eigenvectors in H_λ such that $\tilde{u}_0 * \tilde{u}_m = \tilde{u}_0, \dots, \tilde{u}_k * \tilde{u}_m = \tilde{u}_k, \dots, \tilde{u}_m * \tilde{u}_m = \tilde{u}_m$. In this basis the convolution $f*g$ has the simplest form*

$$\tilde{u}_k * \tilde{u}_p = \begin{cases} 0, & k+p<m \\ \tilde{u}_{k+p-m}, & k+p \geq m \end{cases}$$

and the projection P has the representation

$$Pf = f * \tilde{u}_m = C_m(f)\tilde{u}_0 + \cdots + C_0(f)\tilde{u}_m. \tag{2.7}$$

The system $(\tilde{u}_0, C_m), \ldots, (\tilde{u}_m, C_0)$ *is biorthogonal.*

Proof. It can be easily verified that the vectors $\tilde{u}_k \overset{\text{def}}{=} \alpha_k u_0 + \cdots + \alpha_0 u_k$, $0 \leq k \leq m$ with α_k defined in Theorem 2.3 is the sought basis.

Remark 1. We note that other representation form of P in the primary basis $\{u_0, \ldots, u_m\}$ is:

$$Pf = \tilde{C}_m(f)u_0 + \cdots + \tilde{C}_0(f)u_m, \tag{2.7}$$

where $\tilde{C}_k(f) \overset{\text{def}}{=} \sum_{i=0}^{k} \alpha_{k-i} C_i(f)$, $0 \leq k \leq m$ and now the system $(u_0, \tilde{C}_m), \ldots, (u_m, \tilde{C}_0)$ is biorthogonal. The construction of the numbers α_k in Theorem 2.3 in essence is a construction of the biorthogonal basis of the basis $u_0, \ldots, u_m$. That means that the existence of a convolution w. a. ensures in general the existence of a biorthogonal system (more precisely see theorems 2.5, 2.6) but as can be seen in the following sections the convolution gives us much more information about the operator L.

Remark 2. It is clear also that the multiplers $Q_k f = f * u_k$, $0 \leq k \leq m$ map X in $\langle u_0, \ldots, u_k \rangle$ by Theorem 2.2 but Q_k is not a projection when $k < m$ and there is no multiplier projections mapping X onto $\langle u_0, \ldots, u_k \rangle$ when $k < m$ It is evident also that the multiplier projection P satisfies

$$P(f * g) = Pf * Pg \text{ for all } f, g \in X, \tag{2.8}$$

i. e. P is an homomorhism of the algebra $(X, +, \cdot, *)$ mapping X onto H_λ

Now let the root space H_λ be finite dimensional for all $\lambda \in \Lambda$ and let L have a separately continuous convolution and $*$, w. a. in each $H_\lambda = \langle \overset{\lambda}{u}_0, \ldots, \overset{\lambda}{u}_{m_\lambda} \rangle$, i. e $\overset{\lambda}{u}_0 * \overset{\lambda}{u}_{m_\lambda} \neq 0$ for each $\lambda \in \Lambda$ now. According corollary 2.4 without loss of generality we can assume that the basis $\{\overset{\lambda}{u}_0, \ldots, \overset{\lambda}{u}_{m_\lambda}\}$ is chosen such that $L\overset{\lambda}{u}_0 = \lambda \overset{\lambda}{u}_0$, $L\overset{\lambda}{u}_k = \lambda \overset{\lambda}{u}_k + \overset{\lambda}{u}_{k-1}$, $1 \leq k \leq m_\lambda$, and

$$\overset{\lambda}{u}_k * \overset{\lambda}{u}_{m_\lambda} = \overset{\lambda}{u}_k, \quad 0 \leq k \leq m_\lambda, \quad \lambda \in \Lambda. \tag{2.9}$$

Let us consider the system $\mathscr{P}_\Lambda = \{P_\lambda\}_{\lambda \in \Lambda}$ of continuous multiplier projections

$$P_\lambda f \overset{\text{def}}{=} f * \overset{\lambda}{u}_{m_\lambda}, \quad \lambda \in \Lambda. \tag{2.10}$$

Then the system $\mathscr{P}_\Lambda$ is orthogonal by Lemma 1.1, and it follows from corollary 2.4 that the system S is minimal, since there exists a biorthogonal system $\{(\overset{\lambda}{u}_k, \overset{\lambda}{C}_{m_\lambda - k}) : \lambda \in \Lambda, 0 \leq k \leq m_\lambda\}$.

Theorem 2.5. *Let the generalized eigenvector system S of L be complete, and let L have a separately continuous convolution w. a. $f * g$ in X. Then $\dim H_\lambda < \infty$ for all $\lambda \in \Lambda$. For each $\lambda \in \Lambda$ the multiplier projection P_λ defined by* (2.10) *is the unique continuous projection on H_λ commuting with L.*

$\mathscr{P}_\Lambda$ is a total, orthogonal and fundamental multiplier projection system. The system S is minimal, i. e. it has a biorthogonal system of continuous linear functionals.

P r o o f. By theorem 1.2 c) we have $\dim H_\lambda < \infty$ and $\overset{\lambda}{u}_0 * \overset{\lambda}{u}_{m_\lambda} \neq 0$ for all $\lambda \in \Lambda$, hence there is an orthogonal system $\mathscr{P}_\Lambda$ defined by (2.10). Obviously $\mathscr{P}_\Lambda$ is fundamental since $S \subset \bigcup_{\lambda \in \Lambda} \overset{\lambda}{P}(X)$. Let us prove the totality of $\mathscr{P}_\Lambda$. Indeed if $P_\lambda f = 0$ for all $\lambda \in \Lambda$, then $\overset{\lambda}{C}_k(f) = 0$ by (2.7) and from (2.1) we get that $f * \overset{\lambda}{u}_k = 0$ for all $\lambda \in \Lambda$, $0 \leq k \leq m_\lambda$. Now by the completeness of S it follows $f * g = 0$ for each $g \in X$, hence $f = 0$ since $f * g$ is w. a. For $\lambda \in \Lambda$ let now $Q_\lambda \colon X \to H_\lambda$ be other continuous projection commuting with L. Then from $Q_\lambda (L - \mu I)^{m_\mu + 1} = (L - \mu I)^{m_\mu + 1} Q_\lambda$ it follows $Q_\lambda (H_\mu) \subset H_\mu$ for each $\mu \in \Lambda$. Hence $P_\lambda Q_\lambda \overset{\mu}{u}_k = Q_\lambda P_\lambda \overset{\mu}{u}_k$ for $\mu \in \Lambda$, $0 \leq k \leq m_\mu$. This means $P_\lambda Q_\lambda = Q_\lambda P_\lambda$ in X, i. e. $P_\lambda = Q_\lambda$.

An important case for existence of multiplier projections without supposing completeness of S is the case when L is a closed operator in a Banach space X possessing a continuous convolution representing the resolvent R_λ of L by

$$R_\lambda f = r(\lambda) * f, \quad f \in X \tag{2.11}$$

for each λ of the resolvent set $\rho(L)$ and where $r(\lambda)$ is a continuous function on $\rho(L)$. Now $r(\lambda)$ is a nondivisor of 0 for each $\lambda \in \rho(L)$, i. e. $*$ is w. a.

T h e o r e m 2.6. *The function $r \colon \rho(L) \to X$ is a holomorphic function on the open set $\rho(L)$. If $\lambda_0 \in \Lambda$ is an isolated point of the spectrum which is a pole of the resolvent, then the Riesz projection $P_{\lambda_0} = -(2\pi i)^{-1} \int_{\Gamma_{\lambda_0}} R_\lambda d\lambda$ mapping X onto the invariant subspace H_{λ_0} is a multiplier projection of the form*

$$P_{\lambda_0} f = f * \varphi_{\lambda_0}, \text{ where } \varphi_{\lambda_0} = -\frac{1}{2\pi i} \int_{\Gamma_{\lambda_0}} r(\lambda) d\lambda \in H_{\lambda_0}, \tag{2.12}$$

$$\varphi_{\lambda_0} * \varphi_{\lambda_0} = \varphi_{\lambda_0}$$

and Γ_{λ_0} is a closed contour enclosing only λ_0 of the points of the spectrum. Let $\lambda' \neq \lambda''$ be isolated points of the spectrum. Then

$$\varphi_{\lambda'} * \varphi_{\lambda''} = 0. \tag{2.13}$$

P r o o f. Using Hilbert identity and that $*$ is w. a. we get easily $r(\lambda) * r(\lambda') = [r(\lambda) - r(\lambda')]/(\lambda - \lambda')$ and the analicity follows. Now by integration under convolution sign ($*$ is continuous) we get

$$P_{\lambda_0} f = -(2\pi i)^{-1} \int_{\Gamma_{\lambda_0}} R_\lambda f d\lambda = -(2\pi i)^{-1} \int_{\Gamma_{\lambda_0}} r(\lambda) * f d\lambda = \{-(2\pi i)^{-1} \int_{\Gamma_{\lambda_0}} r(\lambda) d\lambda\} * f$$

and (2.12) holds. From $P_{\lambda_0}^2 = P_{\lambda_0}$ it follows $(\varphi_{\lambda_0} * \varphi_{\lambda_0} - \varphi_{\lambda_0}) * f = 0$ for each $f \in X$, hence $\varphi_{\lambda_0} * \varphi_{\lambda_0} = \varphi_{\lambda_0}$. That means $\varphi_{\lambda_0} = P \varphi_{\lambda_0} \in H_{\lambda_0}$. Analogously $\varphi_{\lambda'} * \varphi_{\lambda''} = 0$ for $\lambda' \neq \lambda''$.

R e m a r k. Theorem 2.6 is also true, if $\lambda_0, \lambda', \lambda''$ are replaced by arbitrary s p e c t r a l s e t s $\sigma_0, \sigma', \sigma''$ of the spectrum, with $\sigma' \cap \sigma'' = \emptyset$ and H_{σ_0} is the invariant subspace $P_{\sigma_0}(X)$, where $P_{\sigma_0} = -(2\pi i)^{-1} \int_\Gamma R_\lambda d\lambda$ and Γ encloses σ_0.

3. Commuting operators, multipliers and coefficient multipliers connected with a total multiplier projection system. In this section we suppose that $L: X_L \to X$, $X_L \subset X$ has a separately continuous convolution w. a. $f * g$ in X, that $\dim H_\lambda < \infty$ for each $\lambda \in \Lambda$ and that there exists a total and orthogonal multiplier projection system $\mathscr{P}_\Lambda = \{P_\lambda\}_{\lambda \in \Lambda}$ of multiplier projections P_λ mapping X onto H_λ for each $\lambda \in \Lambda$.

It can be easily seen that $*$ has not annihilators if it is considered in H_λ for arbitrary fixed $\lambda \in \Lambda$. Then by Theorem 2.3 it follows that P_λ is represented by an element of H_λ hence it is continuous. Without loss of generality we may suppose that the generalized eigenvector basis $\{\overset{\lambda}{u}_0, \ldots, \overset{\lambda}{u}_{m_\lambda}\}$ in H_λ is chosen such that $L\overset{\lambda}{u}_0 = \lambda \overset{\lambda}{u}_0$, $L\overset{\lambda}{u}_k = \lambda \overset{\lambda}{u}_k + \overset{\lambda}{u}_{k-1}$, $1 \leq k \leq m_\lambda$ and $\overset{\lambda}{u}_k * \overset{\lambda}{u}_p = 0$ for $k+p < m_\lambda$; $\overset{\lambda}{u}_k * \overset{\lambda}{u}_p = \overset{\lambda}{u}_{k+p-m_\lambda}$ for $k+p \geqq m_\lambda$ and $P_\lambda f = f * \overset{\lambda}{u}_{m_\lambda}$, $f \in X$ ($\overset{\lambda}{u}_{m_\lambda} * \overset{\lambda}{u}_{m_\lambda} = \overset{\lambda}{u}_{m_\lambda}$).

Let us note that there exist operators possessing continuous convolutions w. a. and total and orthogonal but not fundamental system of continuous multiplier projections, and their generalized eigenvector system S is not complete (e. g. see section 4, examples 1,2 — the case of Dirichlet expansions).

In what follows we give only these parts of the proofs using essentially existence of the total multiplier projection system.

Definition 3.1. *An operator $M: X \to X$ is said to be commuting with the system $\mathscr{P}_\Lambda$ iff $MP_\lambda = P_\lambda M$ in X for all $\lambda \in \Lambda$.*

We note also that if the operator $L: X_L \to X$, $X_L \subset X$ has nonempty resolvent set $\rho(L)$, then its resolvent R_λ is a multiplier of $f * g$ for these $\lambda \in \rho(L)$ for which R_λ is defined in the whole X. This is certainly true for each $\lambda \in \rho(L)$ if L is a closed operator in a Banach space (see [7] ch. VIII, 1).

Theorem 3.1 *Let either the operator L be defined in the whole X or its resolvent set $\rho(L) \neq \emptyset$ and the resolvent R_ν be defined in the whole X at least for one $\nu \in \rho(L)$. Let $M: X \to X$ be a linear operator. Then* a), b), c) *and* d) *are equivalent.*

a) *$M(X_L) \subset X_L$ and M commutes with L in X_L and with $\mathscr{P}_\Lambda$ in X.*

b) *$M(H_\lambda) \subset H_\lambda$ and M commutes with L in H_λ for each $\lambda \in \Lambda$ and with $\mathscr{P}_\Lambda$ in X.*

c) *There is a scalar family $\{\overset{\lambda}{\mu}_k : 0 \leqq k \leqq m_\lambda, \lambda \in \Lambda\}$ such that*

$$M\overset{\lambda}{u}_k = \overset{\lambda}{\mu}_k \overset{\lambda}{u}_0 + \cdots + \overset{\lambda}{\mu}_0 \overset{\lambda}{u}_k, \quad 0 \leqq k \leqq m_\lambda \tag{3.1}$$

$$\overset{\lambda}{C}_k(Mf) = \overset{\lambda}{\mu}_k \overset{\lambda}{C}_0(f) + \cdots + \overset{\lambda}{\mu}_0 \overset{\lambda}{C}_k(f), \quad 0 \leqq k \leqq m_\lambda \text{ hold for each } \lambda \in \Lambda. \tag{3.2}$$

d) *M is a multiplier of $f * g$.*

If the closed graph theorem holds in X, then a), b), c) *or* d) *imply that M is a continuous operator.*

Proof. a) $\Rightarrow$ b) $\Leftrightarrow$ c) are trivial. b) $\Rightarrow$ d). Using (2.5) and (2.9) it can be proved easily that $M(f * g) = Mf * g$ for $f, g \in H_\lambda$, $\lambda \in \Lambda$. Now let $h = M(f * g) - Mf * g$ for arbitrary $f, g \in X$. Then by (2.8) we get $P_\lambda h = M(P_\lambda f * P_\lambda g) - MP_\lambda f * P_\lambda g = 0$ hence $h = 0$. d) $\Rightarrow$ a). Now if L is defined in the whole X the proposition follows from the fact that all multipliers of a convolution w. a.

commute ([2] p. 20). In the other case R_ν, $\nu \in \rho(L)$ is a multiplier, hence R_ν and M commute in X. Let $f \in X_L$, then $f = R_\nu g$, $g \in X$ $(X_L = R_\nu(X))$ hence $Mf = MR_\nu g = R_\nu M g \in X_L$, i. e. $M(X_L) \subset X_L$ and $(LM - ML)f = (L_\nu M - ML_\nu)f = L_\nu M R_\nu f - ML_\nu R_\nu f = L_\nu R_\nu M f - ML_\nu R_\nu f = Mf - Mf = 0$, where $L_\nu = L - \nu I$.

Definition 3.2. *Let $\mathcal{M}$ be a set of operators M: $X_M \to X$, $X_M \subset X$. An element $a \in X$ is said to be a cyclic element of $\mathcal{M}$ in X iff the set $\{M_1^{k_1} \cdots M_n^{k_n} a$: $M_i \in \mathcal{M}$, $k_i = 0, 1, 2, \ldots\}$ exists and its span is dense in X.*

Theorem 3.2. *Let $f * g$ be a separately continuous bilinear commutative and associative operation w. a. in X. Then:*

a) *If $\mathcal{M}$ is a set of multipliers of $f * g$ with a cyclic element in X, then every continuous linear operator M: $X \to X$ commuting with all operators of $\mathcal{M}$ is a multiplier of $f * g$.*

b) *If L: $X_L \to X$, $X_L \subset X$ has a cyclic element, and if $f * g$ is a convolution of L, then every continuous linear operator M: $X \to X$, $M(X_L) \subset X_L$, commuting with L in X_L is a multiplier of $f * g$.*

c) *If L: $X_L \to X$, $X_L \subset X$ has a multiplier resolvent of the form* (2.11), *and the span of the set $\{r(\lambda)$: $\lambda \in \rho(L)\}$ is dense in X then every continuous linear operator M: $X \to X$, $M(X_L) \subset X_L$ commuting with L in X_L is a multiplier of $f * g$.*

d) *If L: $X_L \to X$, $X_L \subset X$ has a complete generalized eigenvector system S and if $f * g$ is a convolution of L, then every continuous linear operator M: $X \to X$, $M(X_L) \subset X_L$ commuting with L in X_L is a multiplier of $f * g$.*

Proof. a) and b) follows using I. Dimovski's idea [6]. Let a be the cyclic element of $\mathcal{M}$. From $Ma * a = a * Ma$ we get $MM_1^{k_1} \cdots M_s^{k_s} a * N_1^{p_1} \cdots N_n^{p_n} a = M_1^{k_1} \cdots M_s^{k_s} a * MN_1^{p_1} \cdots N_n^{p_n} a$ for arbitrary M_i, $N_j \in \mathcal{M}$, k_i, $p_j = 0, 1, 2, \ldots$ and from density of the span it follows that $Mf * g = f * Mg$ for all $f, g \in X$. b) and c) follow in the same way. To prove d) we note that from $ML = LM$ in X_L it follows as in Theorem 3.1 that $M(H_\lambda) \subset H_\lambda$ for $\lambda \in \Lambda$. Then it is clear that $M\overset{\lambda}{u}_{m_\lambda} * \overset{\mu}{u}_{m_\mu} = \overset{\lambda}{u}_{m_\lambda} * M\overset{\mu}{u}_{m_\mu}$ for arbitrary $\lambda, \mu \in \Lambda$ since if $\lambda \neq \mu$ both sides of this equality vanish. But $\overset{\lambda}{u}_k = (L - \lambda I)^{m_\lambda - k} \overset{\lambda}{u}_{m_\lambda}$ hence $Mf * g = f * Mg$ for all $f, g \in H_\lambda$ and by approximation we obtain the validity of this equality in X.

Remark 3.1 It can be proved easily that every cyclic element is a nontrivial nondivisor of 0 for $f * g$.

Remark 3.2 Let Λ be countable and let S be complete. If there are $\alpha_\lambda \in \mathbf{C}$, $\alpha_\lambda \neq 0$ such that $\Sigma_{\lambda \in \Lambda} \alpha_\lambda \overset{\lambda}{u}_{m_\lambda}$ is convergent, then this series defines a cyclic element relative to the operator set $\mathcal{M} = \{L\} \cup \mathcal{P}_\Lambda$ and this element is not divisor of 0. Especially such elements always exist if X is a Banach space.

Theorem 3.3. *Let M: $X \to X$ be a linear operator. Then* a), b) *and* c) *are equivalent.*

a) *M is a coefficient multiplier of $\mathcal{P}_\Lambda$.*

b) *The equalities $\overset{\lambda}{C}_k(Mf) = \overset{\lambda}{\mu}\overset{\lambda}{C}_k(f)$ hold for all $0 \leq k \leq m_\lambda$, $\lambda \in \Lambda$.*

c) *For all $\lambda \in \Lambda$, $P_\lambda M = MP_\lambda$ and $M\overset{\lambda}{u}_k = \overset{\lambda}{\mu}\overset{\lambda}{u}_k$, $0 \leq k \leq m_\lambda$.*

If M is continuous and if S is complete, then $M\overset{\lambda}{u}_k = \overset{\lambda}{\mu}\overset{\lambda}{u}_k$, $\lambda \in \Lambda$, $0 \leq k \leq m_\lambda$ implies that M is a coefficient multiplier of $\mathcal{P}_\Lambda$.

P r o o f. a) $\Rightarrow$ b) follows from (2.7). a) $\Rightarrow$ c). Since every coefficient multiplier is a multiplier of $f*g$ we have $MP_\lambda = P_\lambda M$. Now using (2.7) again we get $M\overset{\lambda}{u}_k = \overset{\lambda}{\mu}\overset{\lambda}{u}_k$. The case c) $\Rightarrow$ a) is similar.

The above considerations show that the representation problem for the coefficient multipliers can be reduced to the problem for finding of a certain multiplier set of a convolution w. a. $f*g$ for L.

The multiplier problem for an arbitrary bilinear, commutative and associative operation $f*g$ in X is trivial if $*$ has a unit element $e \in X$. Now every its multiplier M can be represented by the formula $Mf = n*f$, where $n = Me$. A more general case is considered by I. D i m o v s k i [6]. He proved that if there exists an operator of the form $Rf = r*f$, $f \in X$, $r \in X$ is fixed, which is a right inverse of a given operator $D: X_D \to X$, $X_D \subset X$, then every multiplier M of $*$ can be represented by the formula

$$Mf = D(n*f) \quad \text{with} \quad n = Mr. \tag{3.3}$$

This is a universal formula which does not solve the multiplier problem in general because the open problem for characterizing of the representation elements n arises: *to describe the set of* $n \in X$ *for which* $n*f \in X_D$ *for each* $f \in X$. This problem must be solved separately for each concrete case, as it is made for instance in section 4 for differentiation and for Sturm-Liouville operator.

Another nontrivial problem arising in our approach is the question for finding of a convolution for the operator $L: X_L \to X$, $X_L \subset X$ in X. In section 4 are given examples of convolutions for the differentiation d/dt and for the Sturm-Liouville operator $d^2/dt^2 - q(t)$ considered with suitable domains, such that their resolvents are represented by these convolutions too. In section 5 sufficient conditions for existence of convolutions for a certain class of closed operators in a Banach space are given.

We note that if the operator L has a nonempty resolvent set $\rho(L)$ and if the resolvent R_ν is defined in the whole X for any $\nu \in \rho(L)$ then R_ν is a right inverse of $L_\nu \overset{\text{def}}{=} L - \nu I$ and if $R_\nu = r_\nu * f$, $f \in X$, then (3.3) in the form

$$Mf = (L - \nu I)(n*f), \quad \text{where } n = Mr_\nu \tag{3.3'}$$

can be used.

T h e o r e m 3.4. a) *Let* $\dim H_\lambda = 1$ *for all* $\lambda \in \Lambda$. *Then a linear operator* $M: X \to X$ *is a coefficient multiplier of the system* $\mathcal{P}_\Lambda$ *iff* M *is a multiplier of* $f*g$.

b) *A multiplier of the form* $Mf = n*f$, $f \in X$ *is a coefficient multiplier of* $\mathcal{P}_\Lambda$ *iff* n *satisfies the conditions* $\overset{\lambda}{C}_k(n) = 0$ *for* $k = 1, 2, \ldots, m_\lambda$, *for these* $\lambda \in \Lambda$, with $\dim H_\lambda > 1$ *i. e. iff* $n \sim \Sigma_{\lambda \in \Lambda} \mu_\lambda \overset{\lambda}{u}_{m_\lambda}$ *with* $\mu_\lambda \in \mathbf{C}$.

c) *Let* M *be a multiplier of the form* $Mf = (L - \nu I)(n*f)$, $f \in X$, *for* $\nu \in \rho(L)$ *and* $R_\nu f = r_\nu * f$. *Then* M *is a coefficient multiplier of* $\mathcal{P}_\Lambda$ *iff* n *satisfies the conditions* $\overset{\lambda}{C}_1(n) = (\lambda - \nu)\overset{\lambda}{C}_0(n)$, $\overset{\lambda}{C}_k(n) = 0$ *for* $k = 2, 3, \ldots, m_\lambda$, for these $\lambda \in \Lambda$, *with* $\dim H_\lambda > 1$ *i. e. iff* $n \sim \Sigma_{\lambda \in \Lambda} \mu_\lambda R_\nu \overset{\lambda}{u}_{m_\lambda}$ *or equivalently iff* $n \sim \Sigma_{\lambda \in \Lambda} \mu_\lambda P_\lambda r_\nu$.

P r o o f. a) follows from theorems 3.1 and 3.3. b) follows from the same theorems and the formula (2.2). The first part of the proof of b) follows by the equalities $P_\lambda n = P_\lambda M r_\nu = \mu_\lambda P_\lambda r_\nu = \mu_\lambda r_\nu * \overset{\lambda}{u}_{m_\lambda} = \mu_\lambda R_\nu \overset{\lambda}{u}_{m_\lambda}$. The conversation of b) follows by the equalities $P_\lambda M f = P_\lambda L_\nu(n * f) = L_\nu P_\lambda(n * f) = L_\nu(P_\lambda n) * f = L_\nu [\mu_\lambda R_\nu \overset{\lambda}{u}_{m_\lambda} * f] = \mu_\lambda L_\nu R_\nu(\overset{\lambda}{u}_{m_\lambda} * f) = \mu_\lambda P_\lambda f$ since P_λ commutes with L_ν and R_ν and $L_\nu = L - \nu I$.

Remark 3.3. In the case when $\{H_\lambda\}_{\lambda \in \Lambda}$ is a direct sum decomposition of X (see definition 5.2) then $n = \Sigma_{\lambda \in \Lambda} \mu_\lambda \overset{\lambda}{u}_{m_\lambda}$ or $n = \Sigma_{\lambda \in \Lambda} \mu_\lambda R_\nu \overset{\lambda}{u}_{m_\lambda} = \Sigma_{\lambda \in \Lambda} \mu_\lambda P_\lambda r_\nu$.

4. Examples of certain convolutions and multipliers connected with linear differential operators of first and of second order.

1. Convolutions and multipliers connected with the problem $y' = \lambda y$, $\Phi(y) = 0$ with a functional Φ. a) Let D be a finite convex domain in **C**, let $\bar{D}$ denotes the closure of D, and let $0 \in \bar{D}$. With $H(\bar{D})$ is denoted the space of all holomorphic functions $f(z)$ in $\bar{D}$ endowed with the usual inductive topology (see [8] p. 378-381). Let Φ be an arbitrary continuous linear functional in $H(\bar{D})$. Then it has the form (see [8] p. 378-381) $\Phi f = \frac{1}{2\pi i} \int_\Gamma f(\zeta)\gamma(\zeta)d\zeta$ with a holomorphic function $\gamma(\zeta)$ on the complement of $\bar{D}$ with $\gamma(\infty) = 0$. Let us consider the spectral problem $y' = \lambda y$, $\Phi(y) = 0$, where $y \in H(\bar{D})$. It determines a generalized eigenfuntion system $S = \{e^{\lambda_n z}, ze^{\lambda_n z}, \ldots, z^{m_n - 1} e^{\lambda_n z}\}_{n=1}^{\infty}$, where $\{\lambda_n\}_{n=1}^{\infty}$ are the zeros of the entire function $E(\lambda) \overset{\text{def}}{=} \Phi_z(e^{\lambda z})$ and where $m_n < \infty$ are theirs multiplicities. Now $\Lambda \overset{\text{def}}{=} \{\lambda_n\}_{n=1}^{\infty}$ is the spectrum of the operator $L = d/dz$ considered in the space $H_\Phi = \{f \in H(\bar{D}) : \Phi(f) = 0\}$. The root subspace corresponding to a λ_n is $H_{\lambda_n} = \langle e^{\lambda_n z}, ze^{\lambda_n z}, \ldots, z^{m_n - 1} e^{\lambda_n z}\rangle$. The operator d/dz has a convolution in $H(\bar{D})$:

$$f * g = \Phi_\zeta \{\int_\zeta^z f(z + \zeta - \tau) g(\tau) d\tau\} \tag{4.1}$$

representing the resolvent R_λ (defined by the problem $y' - \lambda y = f$, $\Phi(y) = 0$) by the formula $R_\lambda f = \{e^{\lambda z}/E(\lambda)\} * f$. The problem for expanding of the functions in Dirichlet series of the form $\Sigma_{n=0}^{\infty} P_n f$, where $P_n f = \frac{1}{2\pi i} \int_{\Gamma_n} \Phi_\zeta\{\int_0^\zeta f(\zeta - x) e^{\tau x} dx\} \times \frac{e^{\tau z}}{E(\tau)} d\tau = \Sigma_{k=0}^{m_n - 1} \overset{n}{C}_k(f) z^k e^{\lambda_k z}$ (Γ_n is a contour containing only λ_n in its inside) is considered by A. F. Leontiev [9, 222-336]. Now the convolutional approach can be applied to determine the coefficient multipliers of the complex Dirichlet expansions since it happens that these projections are of the form $P_n = -\frac{1}{2\pi i} \int_{\Gamma_n} R_\lambda d\lambda$ and they are multiplier projections of the form

$$P_n f = f * \varphi_n, \quad \text{where } \varphi_n(z) = -\frac{1}{2\pi i} \int_{\Gamma_n} \frac{e^{\tau z}}{E(\tau)} d\tau \in H_{\lambda_n}. \tag{4.2}$$

In [20] I. Dimovski has proved this representation and that $\varphi_n * \varphi_m = 0$ for $n \neq m$ and that $\varphi_n * \varphi_n = \varphi_n$. The system $\mathcal{P}_\Lambda = \{P_n\}_{n=1}^{\infty}$ is a total system in $H(\bar{D})$ according to a theorem of A. F. Leontiev [9], but the system S is not complete in $H(\bar{D})$, since ([11], [12]) every function $f \in H(\bar{D})$ from the closure of the linear span must satisfy the equation of the mean periodic functions: $\Phi_\zeta\{f(\zeta+z)\}=0$ for $|z|$ less than a certain $\delta>0$.

Now from theorems 3.1, 3.2 and 3.3 it follows:

Theorem 4.1. (I. Dimovski [20]) *Let λ be an arbitrary fixed complex number which is not a zero of $E(\lambda)$. Then a continuous linear operator $M: H(\bar{D})\to H(\bar{D})$, $M(H_\Phi)\subset H_\Phi$ commutes with d/dz in H_Φ iff M is a multiplier of $f*g$ in $H(\bar{D})$ or equivalently iff M can be represented in the form*

$$Mf=(d/dz-\lambda)(m*f) \text{ with } m\in H(\bar{D}), \tag{4.3}$$

or in the equivalent form

$$Mf=\alpha f+n*f \text{ with } n\in H(\bar{D}),\ \alpha\in\mathbf{C}. \tag{4.3'}$$

Corollary 4.2. *An operator M in $H(\bar{D})$ is a coefficient multiplier of the complex Dirichlet expansion defined by the projections* (4.2) *iff M admits a representation of the form* (4.3′) *with $n\sim\Sigma_{n=1}^{\infty}\mu_n\varphi_n$, or equivalently iff M admits a representation of the form* (4.3) *with $m\sim\Sigma_{n=1}^{\infty}\mu_n R_\lambda\varphi_n$, i. e. $m\sim\Sigma_{n=1}^{\infty}\mu_n P_n\{e^{\lambda z}/E(\lambda)\}$. If the entire function $E(\lambda)$ has simple zeros only, then the set of coefficient multipliers coincides with the set of operators represented by* (4.3) *or* (4.3′) *vith arbitrary $m\in H(\bar{D})$ or $n\in H(\bar{D})$ respectively.*

b) The proposed approach can be applied to the real exponential Dirichlet expansions too which are generalized eigenvector expansions of the problem $y'=\lambda y$, $\Phi(y)=0$ with an arbitrary continuous linear functional Φ in $C[0, T]$ Now $E(\lambda)\overset{\text{def}}{=}\Phi_t(e^{\lambda t})$, the resolvent R_λ and the projections P_n have the same form

$$P_n f=f*\varphi_n, \text{ where } \varphi_n(t)\overset{\text{def}}{=}-\frac{1}{2\pi i}\int_{\Gamma_n}\frac{e^{\tau t}}{E(\tau)}d\tau \tag{4.4}$$

and Berg-Dimovski convolution

$$f*g=\Phi_\xi\{\int_\xi^t f(t+\xi-\tau)g(\tau)d\tau\} \tag{4.5}$$

can be extended in the space $L^1[0, T]$ (see [13]). Many cases can be considered, e. g. $X=L^1[0, T]$, $C[0, T]$ or $BV[0, T]$ (see [14], [25]). We shall formulate the main result when $X=C[0, T]$. Now a complete description of the multipliers of $f*g$ can be obtained when has the form

$$\Phi(f)=kf(t_0)+\int_0^T K(t)f(t)dt \tag{4.6}$$

with $t_0\in[0, T]$, $K\in BV[0, T]$, $k\in\mathbf{C}$ or when it has the form

$$\Phi(f)=k_1 f(0)+k_2 f(T)+\int_0^T K(t)f(t)dt \tag{4.6'}$$

with $K\in BV[0, T]$, $k_1, k_2\in\mathbf{C}$. For details see [14]. We note that from a Leontiev's theorem ([12] theorem 7) it follows that the system $\mathcal{P}_\Lambda=\{P_n\}_{n=1}^{\infty}$ is total in $C[0, T]$. For a functional Φ of the form (4.6) or (4.6′) the result is similar one, but it is not identical to the complex case:

Theorem 4.3. *Let* λ *be an arbitrary complex number with* $E(\lambda)\neq 0$. *Then a continuous linear operator* M *in* $C[0, T]$ *has for an invariant subspace* $C_\Phi^1 \overset{\text{def}}{=} \{f \in C^1[0, T]:\ \Phi(f)=0\}$ *and commutes with* d/dt *in* C_Φ^1 *iff* M *is a multiplier of* $f*g$, *or equivalently iff* M *can be represented in the form*

$$Mf=(d/dt-\lambda)(m*f) \tag{4.7}$$

with a $m \in BV \cap C$ *when* $k\neq 0$ or when $|k_1|+|k_2|\neq 0$, *or with a* $m \in C[0, T]$ *when* $k=0$, *or when* $k_1=k_2=0$, *i. e, when* $\Phi(f)=\int_0^T K(t)f(t)dt$ *with* $K \in BV[0, T]$.

Corollary 4.4 *An operator* M *in* $C[0, T]$ *is a coefficient multiplier of real Dirichlet expansion defined by the projections* (4.4) *iff* M *admits a representation of the form* (4.7) *with* $m \in BV \cap C$, $m \sim \sum_{n=1}^\infty \mu_n R_\lambda \varphi_n$ $(m \sim \sum_{n=1}^\infty \mu_n P_n \{e^{\lambda t}/E(\lambda)\})$ *in the cases* $k\neq 0$ *or* $|k_1|+|k_2|\neq 0$ *or with* $m \in C[0, T]$, $m \sim \sum_{n=1}^\infty \mu_n R_\lambda \varphi_n$, $(m \sim \sum_{n=1}^\infty \widetilde{\mu_n} P_n\{e^{\lambda t}/E(\lambda)\})$ *in the case* $\Phi(f)=\int_0^T K(t)f(t)dt$, $K \in BV[0, T]$. *Let now* $E(\lambda)$ *have simple zeros only. If* $k\neq 0$ *or if* $|k_1|+|k_2|\neq 0$, *then the set of coefficient multipliers coincides with the set of operators represented by* (4.7) *with arbitrary* $m \in BV \cap C$. *If* $\Phi(f)=\int_0^T K(t)f(t)dt$, $K \in BV[0, T]$ *then the set of coefficient multipliers coincides with the set of operators represented by* (4.7) *with* $m \in C[0, T]$.

c) A similar approach can be applied to the multiple dimensional complex Dirichlet expansions [10; 16; 24] but now they are generalized eigenvector expansions relative to a system of linear operators with simple point spectrum, and the approach must be applied with certain modifications.

2. Convolutions and multipliers connected with Sturm—Liouville expansions. Let us consider the Sturm-Liouville operator $L=d^2/dt^2-q(t)$ with a complex valued $q \in L^1[0, T]$. Using a convolution $f*g$ found from the author and I. Dimovski ([17, 18]) we can apply the present approach to the spectral problem $Ly=\lambda y$, $\alpha y(0)+\beta y'(0)=0$, $\Phi(y)=0$; $\alpha, \beta \in \mathbf{C}$. Here Φ is a continuous linear operator in $C^1[0, T]$. For details we refer [15; 19; 26]. There is a complete description of the multipliers of $f*g$, of the operators commuting with L in the space of continuously differentiable functions f with absolutely continuous derivative f' satisfying $\alpha f(0)+\beta f'(0)=0$, $\Phi(f)=0$, and of the coefficient multipliers of the generalized Sturm-Liouville expansons in some spaces, e. g. when the functional Φ has the form $\Phi(f)=f'(T)+\Psi(f)$ with $\Psi \in C[0, T]^*$, or if $\Phi(f)=\int_0^T K(t)f(t)dt$ with $K \in BV[0, T]$.

Remark (added in proof). While the paper was being printed the author managed to generalize the results mentioned in Section 4, 1b) and 2) for more general classes of functionals Φ. For details see [26], [27].

5. Convolutions connected with generalized eigenvector expansions. Existence. By the same assumptions as in section 3 we shall find a connection between the convolution $f*g$ and other convolutions of $L: X_L \to X$, $X_L \subset X$ and P_λ, $\lambda \in \Lambda$.

Definition 5.1. *An operation* $\tilde{*}: X \times X \to X$ *is said to be a convolution of a set of operators* $\mathcal{M}$ *if it is a convolution for each operator of* $\mathcal{M}$.

Theorem 5.1. *Let either the operator* L *be defined in the whole* X *or its resolvent set* $\rho(L)\neq\emptyset$ *and the resolvent* R_ν *be defined in the whole* X *at least for one* $\nu \in \rho(L)$. *Let* $\tilde{*}: X \times X \to X$ *be an operation in* X. *Then the conditions* a), b), c), d) *are equivalent.*

a) $\tilde{*}$ *is a convolution for* L *and* $\mathcal{P}_\Lambda$ *in* X.

b) $\tilde{*}$ *is a convolution for* L *in* H_λ *for all* $\lambda \in \Lambda$ *and for* $\mathcal{P}_\Lambda$ *in* X.

c) *The mixed "generalized associative" relations:*

$$(5.1)\qquad (f \tilde{*} g) * h = f * (g \tilde{*} h) = f \tilde{*} (g * h) \text{ hold for } f, g, h \in X.$$

d) *The relations*

$$(5.2)\qquad \overset{\lambda}{C}_k(f \tilde{*} g) = \sum_{i=0}^{k} \overset{\lambda}{\gamma}_{k-i} \overset{\lambda}{C}_i(f * g) = \sum_{i=0}^{k} \overset{\lambda}{\gamma}_{k-i} \sum_{j=0}^{i} \overset{\lambda}{C}_{i-j}(f) C_j(g)$$

hold for $0 \leq k \leq m_\lambda$, $\lambda \in \Lambda$ *and all* $f, g \in X$. *Here* $\{\overset{\lambda}{\gamma}_k : 0 \leq k \leq m_\lambda,\ \lambda \in \Lambda\}$ *is a scalar system determined by* $\overset{\lambda}{u}_{m_\lambda} \tilde{*} \overset{\lambda}{u}_{m_\lambda} = \overset{\lambda}{\gamma}_{m_\lambda} \overset{\lambda}{u}_0 + \cdots + \overset{\lambda}{\gamma}_0 \overset{\lambda}{u}_{m_\lambda}$.

The convolution $\tilde{*}$ *is w. a. in* X *iff* $\overset{\lambda}{\gamma}_0 \neq 0$ *for all* $\lambda \in \Lambda$.

If the closed graph theorem holds in X *then every operation* $\tilde{*}$ *satisfying* (5.1) *is separately continuous, i. e. every convolution of* L *and* $\mathcal{P}_\Lambda$ *is separately continuous.*

P r o o f. Obviously a) $\Rightarrow$ b). b) $\Rightarrow$ c) Using (2.5) applying on the convolutions $*$ and $\tilde{*}$ it follows easily that (5.1) holds in H_λ. Now by totality (as in Theorem 3.1) we obtain that (5.1) holds in X.

c) $\Rightarrow$ a). Since $*$ is w. a. it can be easily proved that every operation $\tilde{*}$ satisfying (5.1) is bilinear, commutative and associative, and it is separately continuous if the closed graph theorem holds in X. We shall prove that every multiplier of $*$ is a multiplier of $\tilde{*}$. Indeed if $\omega = M(f \tilde{*} g) - Mf \tilde{*} g$ then from (5.1) it follows $\omega * h = 0$ for each $h \in X$ hence $h = 0$ since $*$ is w. a. Hence $\tilde{*}$ is a convolution of P_λ, $\lambda \in \Lambda$ and of L if L is defined in the whole X. The case if L is not defined in the whole X can be considered as in theorems 3.1, 5.5 to prove that X_L is an ideal of X with respect $\tilde{*}$ and that $\tilde{*}$ is its convolution.

a) $\Rightarrow$ d). From (2.4) applied on the convolutions $*$ and $\tilde{*}$ it follows that (5.2) holds in H_λ and using the evident relation $\overset{\lambda}{C}_k(f) = \overset{\lambda}{C}_k(P_\lambda f)$ for $f \in X$ it can be proved that (5.2) holds in X.

d) $\Rightarrow$ c) Now let $\omega = (f \tilde{*} g) * h - f \tilde{*} (g * h)$. Then from (5.2) and (2.2) it follows that $\overset{\lambda}{C}_k(\omega) = 0$ for $0 \leq k \leq m_\lambda$, $\lambda \in \Lambda$, hence $\omega = 0$. It is clear that $\overset{\lambda}{\gamma}_0 \neq 0$ for all $\lambda \in \Lambda$ iff $\tilde{*}$ is w. a. in H_λ for all $\lambda \in \Lambda$. Now it is not difficult to prove that $\tilde{*}$ is w. a. in X iff $\tilde{*}$ is w. a. in H_λ for all $\lambda \in \Lambda$.

We shall study more precisely the properties of the operations $\tilde{*}: X \times X \to X$ satisfying the mixed "generalized associative" relations (5.1).

L e m m a 5.2. *Let* X *be a linear space, and let* $f * g$ *be a bilinear, commutative and associative operation in* X *and let its set* H_* *of nontrivial nondivisors of* 0 *be nonempty. Then:*

a) (I. D i m o v s k i [6]) *An element* $r \in X$ *belongs to* H_* *iff the operator* $Rf = r * f$, $f \in X$ *is right invertible. If a linear operator* $D: X_D \to X$, $X_D \subset X$ *satisfies* $DR = I$ in X, *then every operation* $\tilde{*}: X \times X \to X$ *satisfying* (5.1) *can be represented in the form*

(5.3) $$f \tilde{*} g = D^2(n * f * g), \text{ where } n = r \tilde{*} r \in X.$$

b) *A nontrivial operation* $\tilde{*}: X \times X \to X$ *satisfying* (1.5) *is w. a. in* X *iff* $r \tilde{*} r \in H_*$ *for each* $r \in H_*$. *If there exists* $r_0 \in H_*$ *with* $r_0 \tilde{*} r_0 \in H_*$ *then* $r \tilde{*} r \in H_*$ *for all* $r \in H_*$ *and* $\tilde{*}$ *is w. a.*

c) *If* $f * g$ *has not divisors of* 0 *then every operation* $\tilde{*}: X \times X \to X$ *satisfying* (5.1) *has not divisors of* 0.

The proposition can be proved by elementary algebraic considerations.

Theorem 5.3. *Let* $f * g$ *be a separately continuous, bilinear commutative and associative operation w. a. in* X. *Then:*

a) *If* $\mathcal{M}$ *is a set of multipliers of* $f * g$ *and if* $\mathcal{M}$ *has a cyclic element in* X, *then every separately continuous convolution* $\tilde{*}: X \times X \to X$ *for the set of operators* $\mathcal{M}$ *satisfies the mixed "generalized associative" relations* (5.1).

b) (I. Dimovski [6]). *If the operator* $L: X_L \to X, X_L \subset X$ *has a cyclic element in* X *and if* $f * g$ *is a convolution of* L *in* X *then every separately continuous convolution* $\tilde{*}$ *of* L *in* X *satisfies* (5.1).

c) *If the operator* $L: X_L \to X, X_L \subset X$ *has multiplier resolvent of the form* (2.11) *and if the span of the set* $\{r(\lambda): \lambda \in \rho(L)\}$ *is dense in* X *and* $\nu \in \rho(L)$ *is fixed, then every separately continuous convolution* $\tilde{*}$ *of* L *in* X *satisfies* (5.1) *and can be represented in the form:*

(5.3′) $$f \tilde{*} g = (L - \nu I)^2 (n * f * g), \text{ where } n = r(\nu) \tilde{*} r(\nu) \in X.$$

d) *If the operator* $L: X_L \to X, X_L \subset X$ *has complete generalized eigenvector system and if* $f * g$ *is a convolution of* L *in* X *then every separately continuous convolution* $\tilde{*}$ *of* L *in* X *satisfies* (5.1).

Proof. Since the operator $M_g f \overset{\text{def}}{=} f \tilde{*} g$ ($g \in X$ is fixed) commutes with the corresponding class of operators, then M_g is a muliplier of $f * g$ by Theorem 3.2. Hence (5.1) follows.

Remark. I. Dimovski [20] has shown that a general operational calculus of Mikusinski's type for an operator $L: X \to X$ can be constructed in the form of a ring of multiplier quotients $\boldsymbol{M}_* = \{M/N: M \in \mathcal{M}_*, N \in \boldsymbol{H}_*\}$ where $\mathcal{M}_*$ is the set of multipliers of a convolution w. a. $f * g$ for L in X and $\boldsymbol{H}_*$ is its multiplicative subset of all nontrivial nondivisors of 0 in $\mathcal{M}_*$.

Now by assumptions of Theorem 5.1 it follows:

Theorem 5.4. *All convolutions w. a. for* L *and* $\mathcal{P}_\Lambda$ *in* X *have one and the same ring of multiplier quotients.*

Proof. Now $*$ and $\tilde{*}$ have one and the same set of multipliers.

This proposition shows that the Mikusinski's operational calculi for L and $\mathcal{P}_\Lambda$ are equivalent in some sense to the "transform" calculus defined by the formal Fourier expansion $f \sim \sum_{\lambda \in \Lambda} P_\lambda f$.

Definition 5.2 [21, p. 86]. *A sequence of closed subspaces* $\{H_i\}_{i=0}^{\infty}$ *in a Banach space* X *is said to be a direct sum decomposition (or Schauder decomposition) of* X *iff for each* $f \in X$ *there exists a unique sequence* $\{f_i\}_{i=0}^{\infty}$ *such that* $f_i \in H_i$ *for all* $i = 0, 1, 2, \ldots$, *and* $f = \Sigma_{i=0}^{\infty} f_i$ *in the topology of* X. *It is known* [21, p. 88] *that there exists an orthogonal sequence of continuous projections* $P_i: X \to H_i$ *such that* $f = \Sigma_{i=0}^{\infty} P_i f$ *for each* $f \in X$.

Theorem 5.5. *Let $L: X_L \to X$, $X_L \subset X$ be a closed operator in a Banach space X with countable simple point spectrum Λ, let its generalized eigen subspaces H_λ be finite-dimensional for all $\lambda \in \Lambda$ and let $\{H_\lambda\}_{\lambda \in \Lambda}$ form a direct sum decomposition of X. Then:*

a) *The operator L and the projection system $\mathcal{P}_\Lambda = \{P_\lambda\}_{\lambda \in \Lambda}$ corresponding to the direct sum decomposition have a continuous convolution w. a. and X is a Banach algebra with respect to it.*

b) *The Banach algebra X has nontrivial nondivisors of* 0.

Proof. First, let us suppose that L is a continuous operator defined in the whole X.

Let $\{\overset{\lambda}{u}_0, \dots, \overset{\lambda}{u}_{m_\lambda}\}$ be a basis in H_λ satisfying the conditions $L\overset{\lambda}{u}_0 = \lambda \overset{\lambda}{u}_0$, $L\overset{\lambda}{u}_k = \lambda \overset{\lambda}{u}_k + \overset{\lambda}{u}_{k-1}$, $1 \leq k \leq m_\lambda$ for each $\lambda \in \Lambda$. Let us introduce the functionals $\overset{\lambda}{C}_k(f)$: $0 \leq k \leq m_\lambda$, $\lambda \in \Lambda$ by formula

$$P_\lambda f = \sum_{k=0}^{m_\lambda} \overset{\lambda}{C}_{m_\lambda - k}(f) \overset{\lambda}{u}_k. \tag{5.4}$$

It is clear that $\overset{\mu}{C}_{m_\mu - q}(\overset{\lambda}{u}_p) = 0$ for $(\lambda, p) \neq (\mu, q)$ and $\overset{\mu}{C}_{m_\mu - q}(\overset{\lambda}{u}_p) = 1$ for $\lambda = \mu$, $p = q$.

Let now $f, g \in X$. Theorem 5.1 shows how an operation in H_λ must be introduced to be a convolution for L and P_λ in X. Let

$$P_\lambda f * P_\lambda g = \gamma_\lambda \sum_{k=0}^{m_\lambda} \overset{\lambda}{u}_{m_\lambda - k} \sum_{j=0}^{k} \overset{\lambda}{C}_{k-j}(f) \overset{\lambda}{C}_j(g), \tag{5.5}$$

where the numbers $\gamma_\lambda \in \mathbf{C}$ will be chosen later. Since all $\overset{\lambda}{C}_k(f)$ are continuous there exists $\delta_\lambda \in \mathbf{C}$, $\lambda \in \Lambda$ such that $\|P_\lambda f * P_\lambda g\| \leq \gamma_\lambda \delta_\lambda \|f\| \, \|g\|$ for all $f, g \in X$. Let us chose γ_λ such that the series $\Sigma_{\lambda \in \Lambda} \gamma_\lambda \delta_\lambda$ to be convergent with sum less than 1. Then it is clear that the series $\Sigma_{\lambda \in \Lambda} P_\lambda f * P_\lambda g$ is convergent in X for all $f, g \in X$ and the operation $f * g \overset{\text{def}}{=} \Sigma_{\lambda \in \Lambda} P_\lambda f * P_\lambda g$ is a continuous bilinear operation satisfying the inequality $\|f * g\| \leq \|f\| \, \|g\|$, $f, g \in X$.

We shall prove that $*$ is a convolution for L and P_λ, $\lambda \in \Lambda$ in X. Indeed it follows easily from (5.4) that $\overset{\lambda}{u}_p * \overset{\mu}{u}_q = 0$ for $\lambda \neq \mu$; $\overset{\lambda}{u}_p * \overset{\mu}{u}_q = 0$ for $\lambda = \mu$, $p + q < m_\lambda$ and $\overset{\lambda}{u}_p * \overset{\mu}{u}_q = \gamma_\lambda \overset{\lambda}{u}_{p+q-m_\lambda}$ for $\lambda = \mu$, $p + q \geq m_\lambda$. Now it is not difficult to verify the relations $f * g = g * f$, $f * (g * h) = (f * g) * h$, $P_{\lambda_0}(f * g) = P_{\lambda_0} f * g$ for arbitrary fixed $\lambda_0 \in \Lambda$ and $L(f * g) = Lf * g$, for arbitrary $f = \overset{\lambda}{u}_p$, $g = \overset{\mu}{u}_q$ $_\lambda$and $h = \overset{\nu}{u}_r$. Hence by the completeness of the generalized eigenvector system $S = \{u_k : 0 \leq k \leq m_\lambda, \lambda \in \Lambda\}$ and the continuity of P_{λ_0}, L and $*$ the validity of these relations follows in X.

Now let us suppose that $L: X_L \to X$, $X_L \subset X$ is a closed operator, and let $\nu \in \rho(L)$ be fixed. It is known ([7] ch. VIII, 1) that now the resolvent R_ν is a continuous operator defined in the whole X. It is clear that every generalized eigenspace H_λ of L is a generalized eigenspace of R_ν too corresponding to the eigenvalue $(\lambda - \nu)^{-1}$ and conversely, i. e. $\{H_\lambda\}_{\lambda \in \Lambda}$ is a direct sum decomposition for the bounded operator R_ν too. According to the first part of the proof there exists a convolution $f * g$ for R_ν and P_λ, $\lambda \in \Lambda$ in X. We shall prove that $f * g$ is a convolution for L. Indeed, since $X_L = R_\nu(X)$ if $f \in X_L$, $g \in X$, then $f = R_\nu h$, $h \in X$, hence $f * g = (R_\nu h) * g = R_\nu(h * g) \in X_L$, i. e. X_L is an ideal of X with respect to $f * g$. Let now $k = L(f * g) - (Lf) * g$ for $f \in X_L$, $g \in X$. Then

$k=L_\nu(f*g)-(L_\nu f)*g$, where $L_\nu=L-\nu I$ and $R_\nu k=R_\nu L_\nu(f*g)-(R_\nu L_\nu f)*g=f*g-f*g=0$. Therefore, $k=L_\nu R_\nu k=0$.

The proposition b) follows from remark 3.2.

Some results of this paper when the operator L is defined in the whole X are announced in [22; 23].

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