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ON A COMMUTATIVITY OF AUTOMORPHISMS OF A BANACH ALGEBRA

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In this note we shall prove that if A is a semisimple commutative Banach algebra and S, T are automorphisms satisfying the equation $Sx+S^{-1}x=Tx+T^{-1}x$ for all $x \in A$, then S and T necessarily commute.

First we set our notations and some preliminary concepts. Following Rudin [1], let A^* denote the Banach space of all continuous linear functionals on A and S^* (resp. T^*) be the adjoint operators of S (resp. T). Let Δ be the set of all complex homomorphisms of A and $\hat{A}$ the set of all Gelfand transforms $\hat{x}$, $x \in A$. It is easy to see that $S^*(\Delta)=\Delta$, $T^*(\Delta)=\Delta$ and also $Sx+S^{-1}x=Tx+T^{-1}x$ if and only if $S^*x^*+S^{*^{-1}}x^*=T^*x^*+T^{*^{-1}}x^*$ for all $x \in A$ and $x^* \in A^*$.

Theorem. *Let A be a semisimple commutative Banach algebra and S, T be automorphisms satisfying the equation $Sx+S^{-1}x=Tx+T^{-1}x$, for all $x \in A$. Then S and T necessarily commute.*

Proof. Put $P_1=\{x^* \in A^* : S^*x^*=T^*x^*\}$, $P_2=\{x^* \in A^* : S^*x^*=T^{*^{-1}}x^*\}$. Obviously P_1 and P_2 are invariant under S^* and T^*. Put $L_1=P_1 \cap \Delta$, $L_2=P_2 \cap \Delta$. Then L_1 and L_2 are also invariant under S^* and T^*. Further, $L_1 \cup L_2 \subseteq \Delta$. We shall prove that, in fact, $L_1 \cup L_2=\Delta$.

Let $h \in \Delta$ and $x \in A$, then

$$(1) \qquad \hat{x}(S^*h)+\hat{x}(S^{*^{-1}}h)=\hat{x}(T^*h)+\hat{x}(T^{*^{-1}}h).$$

Now suppose on the contrary that $h \notin L_1 \cup L_2$, then (2) $S^*h \neq T^*h$ and $S^*h \neq T^{*^{-1}}h$ (or $S^{*^{-1}}h \neq T^*h$).

There are two possibilities for S^*h and $S^{*^{-1}}h$ in the sense that either a. $S^*h=S^{*^{-1}}h$, or b. $S^*h \neq S^{*^{-1}}h$. Since S^*h, $S^{*^{-1}}h$, T^*h, $T^{*^{-1}}h \in \Delta$, therefore there exists an element $\hat{x}_0 \in \hat{A}$ $(x_0 \in A)$ such that in case a., $\hat{x}_0(S^*h)=\hat{x}_0(S^{*^{-1}}h)=\lambda \neq 0$ (λ is a constant) and $\hat{x}_0(T^*h)=\hat{x}_0(T^{*^{-1}}h)=0$. This combined with (2) contradicts (1).

Similarly, $\hat{y}_0$ can be chosen in $\hat{A}$ such that in case b., $\hat{y}(S^*h)=\mu \neq 0$ and $\hat{y}_0(S^{*^{-1}}h)=\hat{y}_0(T^*h)=\hat{y}_0(T^{*^{-1}}h)=0$. Again this together with (2) contradicts (1).

Hence, in any case we must have $S^*h=T^*h$ or $S^*h=T^{*^{-1}}h$. This shows that $L_1 \cup L_2=\Delta$.

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It is immediate that S^* and T^* commute on Δ and, therefore, for any $h \in A$ and $x \in A$, it follows that $h(STx - TSx) = 0$. This implies that $STx - TSx \in \mathrm{Ker}(h)$ and hence $(STx - TSx) \in$ radical $(A) = \{0\}$. Thus $STx = TSx$ for all $x \in A$. This completes the proof of the theorem.

Remark that the theorem cannot be generalized to a Hilbert space. For example, let $0 < E$, $F < 1$ be projections on a Hilbert space H. Define linear transformations S, T by

$$T\xi = \lambda E\xi + \bar{\lambda}(1 - E)\xi, \qquad \lambda \text{ is a complex number, } |\lambda| = 1,$$

$$S\xi = \lambda F\xi + \bar{\lambda}(1 - F)\xi \qquad (\xi \in H).$$

S and T are invertible isometries on H with $S^{-1}\xi = \bar{\lambda}F\xi + \lambda(1 - F)\xi$ and also $S\xi + S^{-1}\xi = T\xi + T^{-1}\xi$. But S, T may not, in general, commute.

REFERENCE

1. W. Rudin. Functional analysis. New York, 1973.

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