

Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

The attached copy is furnished for non-commercial research and education use only.
Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.

Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on
Serdica Bulgaricae Mathematicae Publicationes
and its new series Serdica Mathematical Journal
visit the website of the journal <http://www.math.bas.bg/~serdica>
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

ON A COMMUTATIVITY OF AUTOMORPHISMS OF A BANACH ALGEBRA

A. B. THAHEEM

In this note we shall prove that if A is a semisimple commutative Banach algebra and S, T are automorphisms satisfying the equation $Sx + S^{-1}x = Tx + T^{-1}x$ for all $x \in A$, then S and T necessarily commute.

First we set our notations and some preliminary concepts. Following Rudin [1], let A^* denote the Banach space of all continuous linear functionals on A and S^* (resp. T^*) be the adjoint operators of S (resp. T). Let Δ be the set of all complex homomorphisms of A and \hat{A} the set of all Gelfand transforms $\hat{x}, x \in A$. It is easy to see that $S^*(\Delta) = \Delta$, $T^*(\Delta) = \Delta$ and also $Sx + S^{-1}x = Tx + T^{-1}x$ if and only if $S^*x^* + S^{*-1}x^* = T^*x^* + T^{*-1}x^*$ for all $x \in A$ and $x^* \in A^*$.

Theorem. *Let A be a semisimple commutative Banach algebra and S, T be automorphisms satisfying the equation $Sx + S^{-1}x = Tx + T^{-1}x$, for all $x \in A$. Then S and T necessarily commute.*

Proof. Put $P_1 = \{x^* \in A^* : S^*x^* = T^*x^*\}$, $P_2 = \{x^* \in A^* : S^*x^* = T^{*-1}x^*\}$. Obviously P_1 and P_2 are invariant under S^* and T^* . Put $L_1 = P_1 \cap \Delta$, $L_2 = P_2 \cap \Delta$. Then L_1 and L_2 are also invariant under S^* and T^* . Further, $L_1 \cup L_2 \subseteq \Delta$. We shall prove that, in fact, $L_1 \cup L_2 = \Delta$.

Let $h \in \Delta$ and $x \in A$, then

$$(1) \quad \hat{x}(S^*h) + \hat{x}(S^{*-1}h) = \hat{x}(T^*h) + \hat{x}(T^{*-1}h).$$

Now suppose on the contrary that $h \notin L_1 \cup L_2$, then (2) $S^*h \neq T^*h$ and $S^*h \neq T^{*-1}h$ (or $S^{*-1}h \neq T^*h$).

There are two possibilities for S^*h and $S^{*-1}h$ in the sense that either a. $S^*h = S^{*-1}h$, or b. $S^*h \neq S^{*-1}h$. Since $S^*h, S^{*-1}h, T^*h, T^{*-1}h \in \Delta$, therefore there exists an element $\hat{x}_0 \in \hat{A}$ ($x_0 \in A$) such that in case a., $\hat{x}_0(S^*h) = \hat{x}_0(S^{*-1}h) = \lambda \neq 0$ (λ is a constant) and $\hat{x}_0(T^*h) = \hat{x}_0(T^{*-1}h) = 0$. This combined with (2) contradicts (1).

Similarly, \hat{y}_0 can be chosen in \hat{A} such that in case b., $\hat{y}_0(S^*h) = \mu \neq 0$ and $\hat{y}_0(S^{*-1}h) = \hat{y}_0(T^*h) = \hat{y}_0(T^{*-1}h) = 0$. Again this together with (2) contradicts (1).

Hence, in any case we must have $S^*h = T^*h$ or $S^*h = T^{*-1}h$. This shows that $L_1 \cup L_2 = \Delta$.

It is immediate that S^* and T^* commute on Δ and, therefore, for any $h \in A$ and $x \in A$, it follows that $h(STx - TSx) = 0$. This implies that $STx - TSx \in \text{Ker}(h)$ and hence $(STx - TSx) \in \text{radical}(A) = \{0\}$. Thus $STx = TSx$ for all $x \in A$. This completes the proof of the theorem.

Remark that the theorem cannot be generalized to a Hilbert space. For example, let $0 < E, F < 1$ be projections on a Hilbert space H . Define linear transformations S, T by

$$T\xi = \lambda E\xi + \bar{\lambda}(1-E)\xi, \quad \lambda \text{ is a complex number, } |\lambda| = 1,$$

$$S\xi = \lambda F\xi + \bar{\lambda}(1-F)\xi \quad (\xi \in H).$$

S and T are invertible isometries on H with $S^{-1}\xi = \bar{\lambda}F\xi + \lambda(1-F)\xi$ and also $S\xi + S^{-1}\xi = T\xi + T^{-1}\xi$. But S, T may not, in general, commute.

REFERENCE

1. W. Rudin. Functional analysis. New York, 1973.

Department of Mathematics
Garyounis University, P. O. Box 9480
Benghazi *Libya*

Received 17. 1. 1980