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ON THE IMPOSSIBILITY OF CERTAIN DISTANCE-REGULAR GRAPHS

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1. In [10] Tutte considered the following question: *what is the least number of vertices N, say of a regular graph whose valency (k) and girth (γ) are given.*

He proved a lower bound for N, namely $N \geqq 1+k+k(k-1)+\cdots+k(k-1)^{(\gamma-3)/2}$ if γ is odd and $N \geqq 1+k+k(k-1)+\cdots+k(k-1)^{\gamma/2-2}+(k-1)^{\gamma/2-1}$ if γ is even.

A graph which attains this bound is called a Moore graph if γ is odd and a Generalized Polygon if γ is even. A lot of work has been done on the classification of such graphs. Generalized Polygons have been studied by Feit and Higman [7], Singleton [9], Benson [3] and Moore graphs by Hoffman and Singleton [8], Vijayan [11], Bannai and Ito [1] and Damerell [5].

In his book [4] Biggs considered both types of graphs as special cases of the distance-regular graph of diameter d whose intersection matrix is the $(d+1)\times(d+1)$ matrix

$$(1.1) \qquad B=\begin{bmatrix} 0 & 1 & & & & & \\ k & 0 & 1 & & & & \\ & k-1 & 0 & \cdot & & & \\ & & k-1 & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & 1 \\ & & & & & 0 & c \\ & & & & & k-1 & k-c \end{bmatrix}.$$

The case $c=k$ is a Generalized Polygon and the case $c=1$ is a Moore graph.

In this paper we investigate the feasibility of the intersection matrix B using the methods and formulae given by Biggs [4].

Supposing that a graph of this type does exist Biggs has derived a formula for the minimum polynomial of its adjacency matrix A; from this in theo-

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rem 1 we calculate the multiplicity of each eigenvalue. Using this result Bannai and Ito [2] have proved that the roots of the characteristic polynomial of B are all of degree ≤ 2 over the rationals.

Consequently, if we reduce that polynomial modulo 2 this must have all its roots in $GF(4)$. But in general this is not so.

Our main result is the following:

Provided that k is even, the graph does not exist for any $d>2$ when c is odd and for any $d\notin\{2, 4, 8, 3, 6, 12, 24, 5, 10, 20, 40\}$ when c is even.

2. The characteristic equation of B. We adopt the notation of Biggs [4] which is $h=k-1$, $q=+\sqrt{k-1}$, $\lambda=2q\cos\alpha$. Then the characteristic equation of B is given by the following result [4, lemma 23.3]:

The number λ is an eigenvalue of B if either $\lambda=k$ or $\lambda=2q\cos\alpha$ and

$$0=\frac{q\sin(d+1)\alpha}{\sin\alpha}+\frac{c\sin d\alpha}{\sin\alpha}+\frac{c-1}{q}\,\frac{\sin(d-1)\alpha}{\sin\alpha}\equiv F_d(\cos\alpha). \tag{2.1}$$

If we put $\theta=e^{i\alpha}$ and multiply by θ^d then (2.1) becomes

$$(q\theta+c+1)(q\theta+1)\theta^{2d}=(q+\theta)\{q+(c-1)\theta\}. \tag{2.2}$$

Lemma 2.1. *Let $G_d(\lambda)=q^{d-1}F_d(\cos\alpha)$. Then $(\lambda-k)G_d(\lambda)$ is the characteristic polynomial of B and*

$$G_1=\lambda+c,\quad G_2=\lambda^2+c\lambda+c-k, \tag{2.3}$$

$$G_d=\lambda G_{d-1}-hG_{d-2},\ d>2. \tag{2.4}$$

Proof. The Čebyšev polynomials of the second kind are of the form [6, (10, 11, 2)] $U_n(\cos\alpha)=\sin(n+1)\alpha/\sin\alpha$ and satisfy the following recurrence relation [5, (10. 11. 16)]:

$$U_{n+1}(\cos\alpha)=2\cos\alpha\, U_n(\cos\alpha)-U_{n-1}(\cos\alpha). \tag{2.5}$$

Hence from (2.1) we get

$$F_d(\cos\alpha)=qU_d(\cos\alpha)+cU_{d-1}(\cos\alpha)+\frac{d-1}{q}U_{d-2}(\cos\alpha). \tag{2.6}$$

From (2.5) we obtain $F_d(\cos\alpha)=2\cos\alpha\, F_{d-1}(\cos\alpha)-F_{d-2}(\cos\alpha)$ which gives (2.4).

Now putting $d=1, 2$ into (2.1) we obtain (2.3). The polynomial $(\lambda-k)G_d$ is monic and of degree $d+1$ having as its roots those given by [4, lemma 23.3]. Therfore $(\lambda-k)G_d$ is the characteristic polynomial of B.

Lemma 2.2. *Let x be any real number $\neq\pm 2q$ and φ, ψ be the roots of*

$$y^2-xy+h=0 \tag{2.7}$$

Then for any $d\geq 0$

$$G_d(x)=(k-c)\frac{\varphi^{d-1}-\psi^{d-1}}{\psi-\varphi}+(x+c)\frac{\psi^d-\varphi^d}{\psi-\varphi}. \tag{2.8}$$

Proof. Since $h=q^2$ and $x=2q\cos\alpha$, the roots of (2.7) are $\varphi,\psi=qe^{\pm i\alpha}$. Hence for integer r: $(\varphi^r-\psi^r)=2iq^r\sin r\alpha$. Take (2.6), substitute for U_d from (2.5), get $F_d(\cos\alpha)=(x+c)U_{d-1}(\cos\alpha)+\frac{c-k}{q}U_{d-2}(\cos\alpha)$. Multiplying by q^{d-1} and expressing in terms of $\sin\alpha$ we get (2.8).

3. The multiplicity of λ as an eigenvalue of A. In the following Theorem we express the multiplicity $m(\lambda)$ of an eigenvalue λ of $A(\Gamma)$ as a function of λ. If the graph exists $m(\lambda)$ must be an integer, this imposes restrictions on λ.

Theorem 1. *Let λ be an eigenvalue of the matrix B of* (1.1) *and $m(\lambda)$ its multiplicity as an eigenvalue of A. If $\lambda \neq -2q$ then*

$$(3.1)\qquad \frac{N}{m(\lambda)}=\frac{\lambda-h-1}{(h+1)(\lambda^2-4h)}\left[\frac{(h+1-c)(\lambda c+2h+2c-2)}{\lambda(c-1)+h+(c-1)^2}+2d(\lambda+h+1)\right]\equiv S(\lambda)$$

(*say*) *where N= the number of vertices of the supposed graph.*

Proof. To simplify the algebra we adopt the convention that if W is a, rational function of θ, the $\widehat{W}$ is the function got by replacing θ by θ^{-1} throughout. By [4, p. 158] the miltiplicity is given by $N/m(\lambda)=\Sigma(\lambda)=\Sigma_{i=0}^{d} k_i u_i^2$ where $k_0=1$, $k_i=kh^{i-1}(1\leq i<d)$, $k_d=c^{-1}kh^{d-1}$ and $\boldsymbol{u}=(u_0, u_1, \ldots, u_d)$ is a left eigenvector of B corresponding to the eigenvalue λ.

Since the first d columns of B are the same as in the matrix B of [5] the formulae derived there for $\boldsymbol{u}$ in terms of any eigenvalue λ hold $u_i=C(\theta/q)^i+D(\theta^{-1}/q)^i$, where

$$(3.2)\qquad C=(h\theta-\theta^{-1})/k(\theta-\theta^{-1}),\quad D=\widehat{C},\quad \theta=e^{i\alpha},\quad \theta+\theta^{-1}=\lambda/q.$$

Thus $k_0u_0^2=1=h^{-1}+kh^{-1}(C^2+2C\widehat{C}+\widehat{C}^2)$, $k_iu_i^2=kh^{-1}(C^2\theta^{2i}+2C\widehat{C}+\widehat{C}^2\theta^{-2i})$, $1\leq i<d$, and $k_du_d^2=c^{-1}kh^{-1}(C^2\theta^{2d}+2C\widehat{C}+\widehat{C}^2\theta^{-2d})$. Hence

$$(3.3)\qquad \Sigma(\lambda)=\sum_{i=0}^{d} k_iu_i^2=h^{-1}+kh^{-1}c^{-1}\{Z+Y+\widehat{Z}\},$$

where

$$(3.4)\qquad Y=2C\widehat{C}(dc+1)$$

and $Z=C^2\{c\Sigma_{i=0}^{d}\theta^{2i}+(1-c)\theta^{2d}\}$. Consider Z. Sum the series and multiply by $(q\theta+c-1)k^2(\theta-1)^3$ then

$$k^2(\theta^2-1)^3(q\theta+c-1)Z=(h\theta^2-1)(q\theta-1)\{(q\theta+c-1)(q\theta+1)\theta^{2d}(\theta^2+c-1)$$
$$-c(q\theta+c-1)(q\theta+1)\},$$

using (2.2) to eliminate θ^{2d} we obtain

$$k^2(\theta^2-1)^2(q\theta+c-1)Z=(h\theta^2-1)(q\theta-1)\{(c-1)\theta^2+qc\theta+(c-1)(c-q^2)\}.$$

Multiplying by $\theta^{-2}(q\theta^{-1}+c-1)$ we have

$$AZ=\theta^3hq(c-1)^2+\theta^2\{h^2(c^2-1)-h(c-1)^2\}+\theta\{h^2(3qc-qc^2-q)+hq(c-1)^3$$
$$-q(c-1)^2\}-h^3(c-1)+h^2(2c^2-4c+1)+h(-c^3+c^2-c+1)+(c-1)^2$$
$$+\theta^{-1}(c-1)\{h^2q-hq-qc(c-1)+q(c+1)\}+\theta^{-2}\{h^2(c-1)-hc(c-2)$$
$$-(h-c)(c-1)^2\}+\theta^{-3}q(c-1)(c-h),$$

where $A=k^2(\theta^2-1)^2(q\theta+c-1)\theta^{-2}(q\theta^{-1}+c-1)=\widehat{A}=k^2(\lambda^2-4h)\{(c-1)\lambda+(c-1)^2+h\}$. Hence by (3.2)

$$A(Z+\hat{Z})=(\theta^3+\theta^{-3})q(c-1)\{h(c-2)+c\}$$
$$+(\theta^2+\theta^{-2})\{h^2(c^2+c-2)-h(3c^2-6c+2)+c(c-1)^2\}$$
$$+(\theta+\theta^{-1})q\{h^2(4c-c^2-2)+h(c-1)(c^2-2c-2)-hc$$
$$+(c^2-1)(2-c)\}-2h^3(c-1)+2h^2(2c^2-4c+1)+2h(-c^3+c^2-c$$
$$+1)+2(c-1)^2=\lambda^3(c-1)\{h(c-2)+c\}+\lambda^2\{h^2(c^2+c-2)-h(3c^2 \tag{3.4}$$
$$-6c+2)+c(c-1)^2\}+\lambda\{h^3(-c^2+4c-2)+h^2(c-1)(c^2-5c$$
$$+4)-h^2c+h(c-1)(-c^2-2c+2)\}$$
$$-2h^4(c-1)+2h^3(c^2-5c+3)+2h^2(-c^3+4c^2-7c+3)-2h(c-1)^3.$$

Substituting (3.5) and (3.4) into (3.3) we get (3.1).

Proposition 3.1. *$\lambda=2q$ is never an eigenvalue.*

Proof. Let q be irrational. Then if $-2q$ is an eigenvalue so is $+2q$. But $\lambda=+2q$ implies $\alpha=0$ (since $\lambda=2q\cos\alpha$) and in that case (2.1) gives, after using L'Hopital's rule, that $q(d+1)+cd+(c-1)(d-1)/q=0$, which is impossible since this is strictly positive. Therefore q has to be rational and integral. Now if the integer $\lambda=-2q$ is an eigenvalue then $\alpha=\pi$ and again by L'Hopital's rule from (2.1) we get

$$(-1)^{d-1}\{q(d+1)-cd+(c-1)(d-1)/q\}=0,$$

therefore $c-1=\frac{-dq^2+dq-q^2}{-dq+d-1}=q+\frac{q^2-q}{dq-d+1}$, hence $\frac{q^2-q}{dq-d+1}$ has to be an integer, therefore $\frac{q}{dq-d+1}=q-\frac{d(q^2-q)}{dq-d+1}$ must be an integer, hence $dq-d+1\leqq q$, which is not true when $q>1$.

Now if $q=1$ then $k=2$ (since $q=+\sqrt{k-1}$) in which case our graph is a polygon with $2d+1$ edges.

4. Using Therorem 1, Bannai and Ito have proved the following.

Theorem 2. *If Γ is a distance-regular graph with intersection matrix B and valency $k>2$, then the roots of the characteristic polynomial of B are all of degree $\leqq 2$ over the rationals.*

For proof see [2, (Theorem A)].

In this section and the next we apply this to prove our main result by considering the polynomial $G_d(\lambda)$ reduced modulo 2. We denote reduction mod 2 by an asterisk.

Theorem 3. *If h and c are both odd and $d>2$, the graph does not exist.*

Proof. From lemma 2.1 we get

$$G_d^*=\lambda G_{d-1}^*+G_{d-2}^*, \tag{4.1}$$

$$G_1^*=\lambda+1,\quad G_2^*=\lambda^2+\lambda+1. \tag{4.2}$$

The solution to the recurrence (4.1) is

$$G_d^*=K\rho^d+L\sigma^d, \tag{4.3}$$

where ρ, σ are the roots of the auxiliary equation $y^2+\lambda y+1=0$. Substituting

(4.3) into (4.2) we find $K=(1+\sigma)/(\rho+\sigma)$ and $L=1+(1+\rho)/(\rho+\sigma)$. So $G_d^*=(\rho^{2d+1}+1)/\rho^d(\rho+1)$. Thus

(4.4) $$G^*(\lambda)=0 \text{ iff } \rho^{2d+1}+1=0.$$

We call the elements of $GF(4)$ 0, 1, ω, ω^2 where ω, ω^2 denote cube roots of 1. Now $\lambda=0$ is impossible since from (4.1) we have $G_d^*(0)=1$ for all d. $\lambda=1$ corresponds to $\rho=\omega$ or ω^2. $\lambda=\omega$ or ω^2 corresponds to $\rho=a$ root of the equation $\rho^2+\omega\rho+1=0$ or $\rho^2+\omega^2\rho+1=0$. These do not have roots in $GF(4)$, but

$$(x^2+\omega x+1)(x^2+\omega^2 x+1)=x^4+x^3+x^2+x+1=(x^5+1)/(x+1).$$

So $\lambda=\omega$ or ω^2 corresponds to $\rho=a$ primitive 5th root of 1. Therefore $G^*(\lambda)$ has roots of degree ≤ 2 if and only if the equation (4.4) has roots in the set $R=\{\omega, \omega^2, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4\}$, where $\varepsilon^5=1$.

Equation (4.4) has distinct roots in its splitting field and they form a cyclic group under multiplication. If η is a generator then its order equals $2d+1$. Therefore if $2d+1>5$ i. e. if $d>2$ the equation (4.4) has at least one root $\eta:\eta\notin R$ i. e. the graph does not exist. Thus we have proved Theorem 3.

5. Theorem 4. *If c is even and h is odd and d has an odd factor $f>5$ the graph does not exist.*

Proof. Again reducting G_d modulo 2 from Lemma 2.1 we obtain

(5.1) $$G_d^*=\lambda G_{d-1}^*+G_{d-2}^*,\ G_0^*=0,\ G_1^*=\lambda.$$

As in p. 4 the solution to (5.1) is $G_d^*=K\rho^d+L\sigma^d$ but this time with $K=L=1$. Thus $G_d^*=\rho^d+\rho^{-d}$. Let $d=2^t\times f$ (f being odd) then $G_d^*=(\rho^f+\rho^{-f})^{2^t}$ and as in 4 if $f>5$ the equation $\rho^f+1=0$ has a root not in the set R. So $G_d^*=0$ has a root in $GF(4)$. Hence if d has an odd factor $f>5$ the graph does not exist.

Now if $f\varepsilon\{1, 3, 5\}$ the question remains open. In order to tackle this part we prove the following two lemmas.

Lemma 5.1. *If $d=2^t\times f$ where f is odd, then $2^{2^{t-1}}$ divides $c-k$ and $G_d(\pm 2)$.*

Proof. $G_d^*=\rho^d+\rho^{-d}=(\rho^f+\rho^{-f})^{2^t}=(H_f(\rho))^{2^t}$ (say) then for all odd f, $\lambda=\rho+\rho^{-1}$ divides H_f. Thus λ^{2^t} divides $G^*(\lambda)$.

By Theorem 2 $G_d(\lambda)$ is a product of at least $f\times 2^{t-1}$ quadratic factors over the rationals. Reduced modulo 2 this product is divisible by λ^{2^t} therefore at least 2^{t-1} factors of G_d^* have constant term zero. Hence at least 2^{t-1} factors of G_d have an even constant term. Thus $2^{2^{t-1}}$ divides the constant term of G_d which is $(-h)^{(d-2)/2}(c-k)$ by lemma 2.1. Therefore $2^{2^{t-1}}$ divides $c-k$ since h is odd. By the same argument $2^{2^{t-1}}$ divides the constant term of $G_d(\lambda\pm 2)$ which equals $G_d(\pm 2)$.

Lemma 5.2. *Let $d=2^t\times f$ where $f=1$ or 3 or 5. Let φ and ψ be the roots of $y^2\pm 2y+h=0$. Then*: (i) $\varphi^d+\psi^d\equiv 2 \bmod 4$ *and* (ii) $(\varphi^d-\psi^d)/(\varphi^f-\psi^f)$ *is divisible by 2^t but not by 2^{t+1}.*

Proof. First suppose $t=0$, so $d=1$ or 3 or 5. Then $\varphi+\psi=2$, $\varphi^3+\psi^3=(\varphi+\psi)^3-3\varphi\psi(\varphi+\psi)\equiv 2 \bmod 4$, $\varphi^5+\psi^5=(\varphi^3+\psi^3)(\varphi^2+\psi^2)-\varphi^2\psi^2(\psi+\varphi)\equiv 2 \bmod 4$ So (i) holds for $t=0$.

Suppose now the result holds for given t. Then $\varphi^{2d}+\psi^{2d}=(\varphi^d+\psi^d)^2-2(\varphi\psi)^d\equiv 2h^d\equiv 2 \bmod 4$. Hence (i) holds by induction for every t.

Now $(\varphi^d-\psi^d)/(\varphi^f-\psi^f)=(\varphi^{d/2}+\psi^{d/2})(\varphi^{d/4}+\psi^{d/4})\cdots(\varphi^f+\psi^f)$. By part (i) each factor $\equiv 2 \bmod 4$, hence this product is divisible by 2^t and not by 2^{t+1}.

Theorem 5. *If c is even and h is odd and* $d\notin\{2, 4, 8, 3, 6, 12, 24, 5, 10, 20, 40\}$ *the graph does not exist.*

Proof. From Lemma 2.2 we have

$$C_d(\pm 2)=(k-c)\,\frac{\psi^{d-1}-\varphi^{d-1}}{\varphi-\psi}+(\pm 2+c)\,\frac{\varphi^d-\psi^d}{\varphi-\psi}\,. \tag{5.2}$$

Now $(\psi^{d-1}-\varphi^{d-1})/(\varphi-\psi)$ is an integer and by Lemma 5.1 $2^{2^{t-1}}$ divides $k-c$. By Lemma 5.2 the second term of (5.2) is $=(\pm 2+c)\times 2^t\times$odd number-$T$ say, and since c is even, either 4 divides c which implies $\pm 2+c\equiv 2 \bmod 4$ or 4 does not divide c which implies $\pm 2+c\equiv 4 \bmod 8$, hence the largest power of 2 dividing T is $\leq 2^{t+1}$, therefore $2^{2^{t-1}}\leq 2^{t+2}$, or $t\leq 3$. Thus the only possible graphs are those with diameter

$$d\in\{2, 4, 8, 3, 6, 12, 24, 5, 10, 20, 40\}.$$

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