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A LIMIT THEOREM FOR SEQUENCES OF WEAKLY DEPENDENT STOCHASTIC PROCESSES

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In this paper a functional central limit theorem for a sequence of stochastic processes is proved. Each value in these processes is a linear superposition of approximately independent random variables. Since the solution of boundary value problems and initial value problems depends linearly on the inhomogenous term our limit theorem may be applied to describe approximately the distribution of the solution of such problems.

In this paper we study sequences of stochastic processes $X_n(t)$, $0\leq t\leq 1$, having the property that for each $\varepsilon>0$ the random variables $X_n(t)$ and $X_n(s)$, $|t-s|>\varepsilon$ are approximately independent for large n. To give a more precise formulation of "approximately independent" we use mixing coefficients. Our aim is to study the asymptotical behavior of the sequence of distribution laws of the processes $W_n(t)=\int_0^t X_n(s)ds$. Since, roughly spoken, $W_n(t)$ is the sum of approximately independent random variables for large n we can show, under some technical assumptions, that the distribution law $\mathscr{L}(W_n)$ converges weakly to the distribution law of a continuous Gaussian process with independent increments (Theorem 1). Our results include the limit theorems for weakly correlated processes introduced in [3, 4]. Using theorem 1 we can derive a limit theorem for stochastic processes of the form $Y_n(t)=\int_0^1 K(t,s)X_n(s)ds$. The finite-dimensional distributions of $Y_n(t)$ converge to the corresponding distributions of a Gaussian process. Under some additional assumptions we can improve this result and deduce the weak convergence of the distribution laws. As solutions $Y(t)$ of boundary value problems and initial value problems of ordinary differential equations with stochastic inhomogenous term $X(s)$ may be written in the form $Y(t)=\int_0^1 K(t,s)X(s)ds$, we can apply our limit theorem to such problems of stochastic analysis.

1. Notations and results. By a general Wiener process $W_{a,b}(t)$, $0\leq t\leq 1$, we shall mean a separable almost sure continuous Gaussian process with independent increments and $W_{a,b}(0)=0$ a. s. where $a(t)$ and $b(t)$ are the mean and the variance of $W_{a,b}(t)$, respectively; a and b are continuous functions. Furthermore b is nondecreasing; $W_{a,b}(0)=0$ implies $a(0)=0$, $b(0)=0$.

Given a real random variable X we shall denote by $\langle X\rangle$ the mean and by D^2X the variance of X. For a stochastic process $(X(t))_{t\in T}$ on $[\Omega,\mathscr{F},\mathsf{P}]$ we denote by $\sigma((X(t))_{t\in T}$ the σ-algebra generated by the process $(X(t))_{t\in T}$.

Let now T be an interval $[a,b]$. We now introduce the coefficient of uniformly strong mixing

$$\varphi(\tau,X)=\sup\,[\mathsf{P}(A\cap B)-\mathsf{P}(A)\mathsf{P}(B)]/\mathsf{P}(A),$$

where the supremum is taken over all $A \in \sigma((X(t))_{a \leq t \leq s_1})$, $\mathsf{P}(A) > 0$, $B \in \sigma((X(t))_{s_2 \leq t \leq b})$ and all $a \leq s_1$, $s_2 \leq b$ with $|s_1 - s_2| > \tau$. To simplify the notation we often write $\varphi_n(\tau)$ instead of $\varphi(\tau, X_n)$ for a sequence X_n of processes.

Let C be the Banach space of all real continuous functions on $[0, 1]$ with the usual norm. $\mathfrak{C}$ denotes the σ-algebra of Borel sets of C.

Let $W(t)$, $0 \leq t \leq 1$, be a separable almost sure continuous stochastic process. It is clear that the mapping

$$F(\omega) = \begin{cases} W(t, 0) & \text{if} \quad W(. , \omega) \quad \text{is continuous,} \\ 0 & \text{otherwise} \end{cases}$$

is $\mathcal{F} - \mathfrak{C}$ measurable.

We define the distribution law $\mathcal{L}(W)$ of W by $\mathcal{L}(W) = \mathsf{P} \circ F^{-1}$ and write futher $\mu_n \Rightarrow_{n \to \infty} \mu$ for the weak convergence of the distribution laws on $[C, \mathfrak{C}]$. We now consider a real measurable stochastic process $X(t)$, $0 \leq t \leq 1$, satisfying the condition $\langle \int_0^1 X^2(t) dt \rangle < \infty$.

Using the notation $W(t) = \int_0^t X(s) ds$ we obtain

$$|W(t_2) - W(t_1)| \leq \left(\int_0^1 X^2(s) ds\right)^{1/2} |t_2 - t_1|^{1/2},$$

which means that $W(t)$ is a. s. continuous.

Our aim is to study the asymptotical behavior of $\mathcal{L}(W_n)$, $W_n(t)\ \int_0^t X_n(s) ds$, if $\varphi(\tau, X_n) \to_{n \to \infty} 0$ for each $\tau > 0$.

Theorem 1. *Let $X_n(t)$, $0 \leq t \leq 1$, be a sequence of real measurable stochastic processes satisfying the following conditions:*

(1) $$\sup_{0 \leq t \leq 1} \langle X_n(t)^4 \rangle = c_n < \infty, \; n = 1, 2, \ldots,$$

(2) $$\sup_n c_n \left(\int_0^1 \varphi_n^{1/8}(t) dt\right)^2 = c < \infty, \quad \lim_{n \to \infty} \varphi_n(\varepsilon) = 0, \quad \text{for each } \varepsilon > 0,$$

(3) *there exists a real continuous function $a(t)$, $0 \leq t \leq 1$, with $\sup_{0 \leq t \leq 1} |\langle W_n(t) \rangle - a(t)| \underset{n \to \infty}{\longrightarrow} 0$, where $W_n(t) = \int_0^t X_n(s) ds$,*

(4) *there is a nondecreasing continuous function $b(t)$, $0 \leq t \leq 1$, such that $D^2 W_n(t) \to_{n \to \infty} b(t)$ for each $t \in [0, 1]$,*

then $\mathcal{L}(W_n) \Rightarrow_{n \to \infty} \mathcal{L}(W_{a,b})$.

Remark 1. Suppose X_n is a sequence of processes satisfying the conditions (1)—(4), then $\lim_{n \to \infty} \langle W_n(t_1) \ldots W_n(t_k) \rangle = \langle W_{a,b}(t_1) \ldots W_{a,b}(t_k) \rangle$, $k = 1, 2, 3$, $0 \leq t_1, \ldots, t_k \leq 1$.

We now apply Theorem 1 to a stochastic process defined on $[0, \infty)$.

Corollary 1. *Let $Z(t)$, $0 \leq t < \infty$, be a real mesurable stochastic process possessing the following properties*

(5) $$\langle Z(t)^4 \rangle \leq d < \infty, \; \langle Z(t) \rangle = 0 \; \textit{for every} \; 0 \leq t < \infty,$$

(6) $$\int_0^\infty \varphi^{1/8}(t, Z) dt < \infty,$$

(7) *there is a nondecreasing function b on $[0, \infty)$ with*

$$D^2 W_n(t) \to_{n \to \infty} b(t) \; \textit{for each} \; 0 \leq t < \infty, \; \textit{where} \; W_n(t) = n^{-1/2} \int_0^{nt} Z(s) ds,$$

then $\mathcal{L}(W_n) \to_{n \to \infty} \mathcal{L}(W_{0,b})$.

If $Z(t)$ is a stationary process the last assertation is the functional central limit theorem [1]. In this case the assumptions of Corollary 1. may be weakened [1]. We now consider a special class of sequences of stochastic process in [0,1].

Corollary 2. *Let $Z_n(t)$, $0 \leq t \leq 1$, be a sequence of measurable stochastic processes with*

(8) $$\langle Z_n^4(t)\rangle \leq d,\ \langle Z_n(t)\rangle = 0 \text{ for each } 0 \leq t \leq 1,$$

(9) $$\varphi(1/n, Z_n) = 0, \quad n = 1, 2, \dots,$$

(10) *there is a nondecreasing continuous function b on [0,1] with*

$$\lim_{n\to\infty} n \int_{-1/n}^{1/n} \langle Z_n(t) Z_n(t+s)\rangle ds = \bar{b}(t), \quad \text{for each } 0 \leq t \leq 1,$$

then $\mathscr{L}(W_n) \to_{n\to\infty} \mathscr{L}(W_{0,b})$, where $W_n(t) = \sqrt{n}\int_0^t Z_n(s)ds$ and $b(t) = \int_0^t \bar{b}(s)ds$.

We now give an example of a sequence of covariance functions $R_n(t, s)$ satisfying condition (10) of Corollary 2. We define R_n by

$$R_n(t, s) = \begin{cases} (1 - n\,|t-s|) & \text{for } |t-s| \leq 1/n, \\ 0 & \text{for } |t-s| > 1/n \end{cases}$$

and obtain $\lim_{n\to\infty} n \int_{-1/n}^{1/n} R_n(t, t+s)ds = 1$ and from this $b(t) = \bar{b}\,.\,t$.

In the above example R_n is the covariance function of a weakly stationary process. It is an interesting fact that in the case of weakly stationary processes the function $b(t)$ must be always linear.

Theorem 2. *Let $X_n(t)$, $0 \leq t \leq 1$, be a sequence of weakly stationary processes. Suppose:*
$\langle X_n(t)\rangle = 0$, $0 \leq t \leq 1$; $\lim_{n\to\infty} \varphi_n(\tau) = 0$ *for each* $\tau > 0$; *there is a continuous function* $b(t)$ *with* $\lim_{n\to\infty} \langle (\int_0^t X_n(s)ds)^2\rangle = b(t)$ *for each* $0 \leq t \leq 1$, *then* $b(t) = \sigma^2 \,.\, t$ *with a constant* $\sigma^2 \geq 0$.

For a sequence of real stochastic processes $Y, Y_1, Y_2, \dots$ we denote by $\mathfrak{E}(Y_n) \to_{n\to\infty} \mathfrak{E}(Y)$ the weak convergence of all finite-dimensional distributions.

Let T be a nonempty set and $K(t, s)$ a realvalued function on $T \times [0, 1]$.

We now formulate a limit theorem for the processes $Y_n(t) = \int_0^1 K(t, s) X_n(s)ds$.

Theorem 3. *Let $X_n(s)$, $0 \leq s \leq 1$, be a sequence of measurable stochastic processes satisfying the conditions* (1), (2), (4). *We assume* $\sup_{0\leq s\leq 1} |K(t, s)| < \infty$ *for each $t \in T$, and furthermore that $\int_0^1 K(t, s)\langle X_n(s)\rangle\, ds$ converges for each $t \in T$ and denote this value by $\alpha(t)$.*

Then $\mathfrak{E}(Y_n) \to_{n\to\infty} \mathfrak{E}(Y)$, where $Y(t)$ is a Gaussian process with mean $\alpha(t)$ and covariance function $R(t_1, t_2) = \int_0^1 (K(t_1, s) K(t_2, s) db(s)$.

We now consider the special case $T = [0, 1]$. We can improve the result of theorem 3 in this case if the kernel $K(t, s)$ has some additional properties. We suppose that $K(t, s)$ admits the representation

(11) $$K(t, s) = \int_0^s k(t, u)du,$$

where $k(t, u)$ is a realvalued measurable function on $[0, 1] \times [0, 1]$ with

$$\sup_{0\leq t\leq 1}\int_0^1 |k(t,u)|\,du<\infty,\ \lim_{t_1\to t_2}\int_0^1 |k(t_1,u)-k(t_2,u)|du=0\quad\text{for each}\quad 0\leq t_2\leq 1.$$

Theorem 4. *Suppose that* $X_n(t), 0\leq t\leq 1$, *is a sequence of stochastic processes satisfying the conditions* (1)—(4) *and* $K(t,s)$ *is a kernel which admits the representation* (11), *then* $\mathscr{L}(Y_n)\Rightarrow_{n\to\infty}\mathscr{L}(Y)$ *where* Y *is a Gaussian process with mean* $\alpha(t)=a(1)\int_0^1 k(t,u)du-\int_0^1 k(t,s)a(s)ds$ *and covariance function* $\int_0^1 K(t_1,s)K(t_2\, s), db(s)$.

Remark. If $a(t)$ in theorem 1 is a function of bounded variation it can be easily seen that the mean $\alpha(t)$ of $Y(t)$ may be written in the form $\alpha(t)=\int_0^1 K(t,s),\ da(s)$.

2. Proofs. We first summarize without proofs several facts basic for the sequel.

Lemma 1 [1]. *Let* X, Y *be real random variables on* $[\Omega,\mathscr{F},\mathsf{P}]$ *with* $<|X|^p\rangle<\infty, \langle|Y|^q\rangle<\infty,\ 1/p+1/q=1, q>1$, *then*

$$|\langle XY\rangle-\langle X\rangle\langle Y\rangle|\leq 2\varphi^{1/p}\langle|X|^p\rangle^{1/p}\langle|Y|^q\rangle^{1/q}$$

where $\varphi=\sup_{A\in\sigma(X),\ B\in\sigma(Y)}|\mathsf{P}(A\cap B)-\mathsf{P}(A).\mathsf{P}(B)|/\mathsf{P}(A)$.

Lemma 2 [1]. *Let* $U_n(t), 0\leq t\leq 1$, *be a sequence of separable almost sure continuous processes. Assume that there are constants* $c\geq 0$, $\alpha>1$, $\gamma\geq 0$ *with* $\langle|U_n(t_2)-U_n(t_1)|^\gamma\rangle\leq c|t_2-t_1|^\alpha$ *for every* $0\leq t_1, t_2\leq 1$, *then* $\mathscr{L}(U_n)$ *is relatively compact.*

Lemma 3 [2]. *Every separable almost sure continuous stochastic process* $W(t), 0\leq t\leq 1$, *with independent increments and* $W(0)=0$ *is a general Wiener process.*

Proof of Theorem 1. The basic ideas of our proof are the same as in the proof of Theorem 19.2 in [1]. But the compactness, however, is proved directly without the criterion given there.

1) Denote

$$\bar{X}_n(t)=X_n(t)-\langle X_n(t)\rangle,\ \bar{V}_n(t_1,t_2)=\int_{t_1}^{t_2}\bar{X}_n(s)ds.$$

The inequality

$$\langle\bar{V}_n^4(t_1,t_2)\rangle\leq 3360\,c_n(t_2-t_1)^2\Big(\int_0^{t_2-t_1}\varphi_n^{1/8}(s)ds\Big)^2 \tag{12}$$

will play a fundamental role in the sequel.

In order to prove (12) we use the known inequality $\langle|U|^{p_1}\rangle^{1/p_1}\leq\langle|U|^{p_2}\rangle^{1/p_2}$, $1\leq p_1\leq p_2<\infty$. Using the notation in (2) we get

$$\langle\bar{X}_n^4(t)\rangle\leq\langle X_n^4(t)\rangle+4\langle|X_n(t)|^3\rangle\langle|X_n(t)|\rangle+6\langle X_n^2(t)\rangle\langle|X_n(t)|\rangle^2+3\langle|X_n(t)|\rangle\leq 14c_n.$$

Because of

$$\langle\bar{V}_n^4(t_1,t_2)\rangle\leq\int_{t_1}^{t_2}\int_{t_1}^{t_2}\int_{t_1}^{t_2}\int_{t_1}^{t_2}K\bar{X}_n(s_1)\bar{X}_n(s_2)\bar{X}_n(s_3)\bar{X}_n(s_4)\rangle|\,ds_1ds_2ds_3ds_4$$

we deal with $|\langle\bar{X}_n(s_1)\bar{X}_n(s_2)\bar{X}_n(s_3)\bar{X}_n(s_4)\rangle$ for $s_1\leq s_2\leq s_3\leq s_4$ and put $s_1=s$, $s_2=s+t$, $s_3=s+t+u$, $s_4=s+t+u+v$ with $t,u,v\geq 0$. Using Lemma 1 and $\langle\bar{X}_n(s)\rangle=0$ we obtain

$$|\langle \bar{X}_n(s)\bar{X}_n(s+t)\bar{X}_n(s+t+u)\bar{X}_n(s+t+u+v)\rangle|$$
$$\leq 2\varphi_n^{1/4}(t)\langle \bar{X}_n^4(s)\rangle^{1/4}\langle|\bar{X}_n(s+t)\bar{X}_n(s+t+u)\bar{X}_n(s+t+u+v)|^{4/3}\rangle^{3/4}$$

and by Hölder's inequality

$$\langle|\bar{X}_n(s+t)\bar{X}_n(s+t+u)\bar{X}_n(s+t+u+v)|^{4/3}\rangle$$
$$\leqq \langle \bar{X}_n^4(s+t)\rangle^{1/3}\langle \bar{X}_n^4(s+t+u)\rangle^{1/3}\langle \bar{X}_n^4(s+t+u+v)\rangle^{1/3}\leqq 14c_n.$$

Hence

$$|\langle \bar{X}_n(s)\bar{X}_n(s+t)\bar{X}_n(s+t+u)\bar{X}_n(s+t+u+v)\rangle|\leqq 28\varphi_n^{1/4}(t)\,.\,c_n. \tag{13}$$

A similar consideration yields

$$|\langle \bar{X}_n(s)\bar{X}_n(s+t)\bar{X}_n(s+t+u)\bar{X}_n(s+t+u+v)\rangle|\leqq 28\varphi_n^{1/4}(v)c_n. \tag{14}$$

We apply Lemma 1 again and obtain

$$|\langle \bar{X}_n(s)\bar{X}_n(s+t)\bar{X}_n(s+t+u)\bar{X}_n(s+t+u+v)\rangle|$$
$$\leq|\langle \bar{X}_n(s)\bar{X}_n(s+t)\rangle|\,|\langle \bar{X}_n(s+t+u)\bar{X}_n(s+t+u+v)\rangle|$$
$$+2\varphi_n^{1/2}(u)\langle \bar{X}_n^2(s)\bar{X}_n^2(s+t)\rangle^{1/2}\langle \bar{X}_n^2(s+t+u)\bar{X}_n^2(s+t+u+v)\rangle^{1/2}$$
$$\leqq|\langle \bar{X}_n(s)\bar{X}_n(s+t)\rangle\ \langle \bar{X}_n(s+t+u)\bar{X}_n(s+t+u+v)\rangle|+28\varphi_n^{1/2}(u)c_n$$

and

$$|\langle \bar{X}_n(s)\bar{X}_n(s+t)\rangle|\leqq 2\varphi_n^{1/2}(t)\langle \bar{X}_n^2(s)\rangle^{1/2}\langle \bar{X}_n^2(s+t)\rangle^{1/2}$$
$$\leqq 2\sqrt{14}\sqrt{c_n}\,\varphi_n^{1/2}(t),$$
$$|\langle \bar{X}_n(s+t+u)\bar{X}_n(s+t+u+v)\rangle|\leq 2\sqrt{14}\,\varphi_n^{1/2}(v)\sqrt{c_n}.$$

Therefore

$$|\langle \bar{X}_n(s)\bar{X}_n(s+t)\bar{X}_n(s+t+u)\bar{X}_n(s+t+u+v)\rangle|\leqq 28c_n[\varphi_n^{1/2}(u)+2\varphi_n^{1/2}(v)\varphi_n^{1/2}(t)]. \tag{15}$$

In order to estimate $\langle \bar{V}_n^4(t_1,t_2)\rangle$ we use the estimate (13) for $\{(t,u,v):u,v\leqq t\}$, the estimate (14) for $\{(t,u,v):u,t\leq v\}$ the estimate (15) for $\{(t,u,v):t,v\leqq u\}$ and obtain

$$\int\int\int\int_{s_1\leqq s_2\leqq s_3\leqq s_4}|\langle \bar{X}_n(s_1)\bar{X}_n(s_2)\bar{X}_n(s_3)\bar{X}_n(s_4)\rangle|ds_1ds_2ds_3ds_4$$

$$\begin{aligned}&\leq\int_{t_1}^{t_2}\Big(\int\int\int_{u,v\leqq t}28\varphi_n^{1/4}(t)c_n dt du dv\Big)ds+\int_{t_1}^{t_2}\Big(\int\int\int_{u,t\leqq v}28\varphi_n^{1/4}(v)c_n dt du dv\Big)ds\\ &+\int_{t_1}^{t_2}\Big(\int\int\int_{t,v\leqq u}28c_n[\varphi_n^{1/2}(u)+2\varphi_n^{1/2}(v)\varphi_n^{1/2}(t)]dt du dv\Big)ds\end{aligned} \tag{16}$$

$$\leq 28c_n(t_2-t_1)\Big[3\int\int\int_{u,v\leqq t}\varphi_n^{1/4}(t)dt du dv+2\int\int\int_{u,v\leqq t}\varphi_n^{1/2}(v)\varphi_n^{1/2}(u)dt du dv\Big],$$

It is $\varphi_n(u)\leq\varphi_n(v)\leq 1$ for $0\leq u\leq v\leq 1$ and therefore

(17) $$\int\int\int_{u,\, v\leqq t} \varphi_n^{1/2}(u)\varphi_n^{1/2}(v)\, dt du dv \leqq \int_0^{t_2-t_1}\int_0^{t_2-t_1}\int_0^{t_2-t_1} \varphi_n^{1/2}(u)\varphi_n^{1/2}(v) dt du dv$$

$$\leqq (t_2-t_1)\,(\int_0^{t_2-t_1} \varphi_n^{1/8}(u)du)^2$$

and

(18) $$\int\int\int_{u,\, v\leqq t} \varphi_n^{1/4}(t) dt du dv \leqq \int_0^{t_2-t_1} [\int_0^t\int_0^t \varphi_n^{1/4}(t) du dv] dt \leqq \int_0^{t_2-t_1} [\int_0^t\int_0^t \varphi_n^{1/8}(u)\varphi_n^{1/8}(v) du dv] dt$$

$$\leqq (t_2-t_1)\,(\int_0^{t_2-t_1} \varphi_n^{1/8}(u) du^2.$$

Applying (17) and (18) to (16) we get

$$\int\int\int\int_{s_1\leqq s_2\leqq s_3\leqq s_4} |\langle \bar{X}_n(s_1)\bar{X}_n(s_2)\bar{X}_n(s_3)\bar{X}_n(s_4)\rangle|\, ds_1 ds_2 ds_3 ds_4$$

$$\leqq 140 c_n (t_2-t_1)^2 (\int_0^{t_2-t_1} \varphi_n^{1/8}(t)dt)^2$$

and

$$\langle \bar{V}_n^4(t_1, t_2)\rangle \leqq 3360 c_n (t_2-t_1)^2 (\int_0^{t_2-t_1} \varphi_n^{1/8}(t)dt^2$$

which is the stated inequality.

According to inequality (12), Lemma 2 and assumption (2) $\mathscr{L}(\bar{W}_n)$ is relatively compact. Because of $\bar{W}_n(t) = W_n(t) - \langle W_n(t)\rangle$ and condition 3 $\mathscr{L}(W_n)$ is relatively compact, too.

2) A short calculation shows that the inequality in Lemma 1 is valid for complexvalued random variables ζ, η substituting the constant 2 by 8. Hence

(19) $$|\langle\zeta\eta\rangle - \langle\zeta\rangle\langle\eta\rangle| \leqq 8\varphi^{1/p}\langle|\zeta|^p\rangle^{1/p}\langle|\eta|^q\rangle^{1/q}$$

where $\langle|\zeta|^p\rangle < \infty$, $\langle|\eta^q|\rangle < \infty$, $1/p+1/q=1$, $p>1$. φ is defined analogously as we have done in the case of real random variables. Denote by $s_1, \ldots, s_r, t_1, \ldots, t_r$ real numbers with $0\leqq s_1 \leqq t_1 < s_2 \leqq t_2 < \cdots < s_r \leqq t_r \leqq 1$ and by

$$\zeta_{l,n} = \exp(ix_l(W_n(t_l) - W_n(s_l)),\quad \eta_{l,n} = \prod_{k=l+1}^{r} \exp(ix_k(W_n(t_k) - W_n(s_k)),$$

where $x_1, \ldots, x_r$ fixed real numbers. We obtain applying $(r-1)$-times (19)

$$|\langle \prod_{l=1}^{r} \zeta_{l,n}\rangle - \prod_{l=1}^{r}\langle\zeta_{l,n}\rangle| \leqq |\langle\zeta_{1,n}\eta_{1,n}\rangle - \langle\zeta_{1,n}\rangle\langle\eta_{1,n}\rangle| + |\langle\zeta_{1,n}\rangle|\,|\langle\eta_{1,n}\rangle - \prod_{l=2}^{r}\langle\zeta_{1,n}\rangle|$$

$$\leqq 8\,\varphi_n^{1/2}(s_2-t_1) + |\langle\eta_{1,n}\rangle - \prod_{l=2}^{r}\langle\zeta_{l,n}\rangle| \leqq 8(r-1)\varphi_n^{1/2}(\operatorname*{Min}_{l=2,\ldots,r}(s_l-t_{l-1})).$$

Assumption (2) implies

(20) $$\lim_{n\to\infty} |\langle \prod_{l=1}^{r} \zeta_{l,n}\rangle - \prod_{l=1}^{r}\langle\zeta_{l,n}\rangle| = 0,$$

3) Let μ be an accumulation point of $\mathscr{L}(W_n)$. The limit relation (20) implies

$$\int_C \prod_{l=1}^{r} \exp(ix_l(g(t_l)-g(s_l))\mu(dg) = \prod_{l=1}^{r} \int_C \exp(ix_l(g(t_l)-g(s_l))\mu(dg).$$

From $\mathbf{P}(W_n(0)=0)=1$, $n=1, 2, \ldots$, we get $\mu(\{f: f(0)=0\})=1$. Because of Lemma 3 μ is the distribution law of a general Wiener process. We denote by $a_\mu(t)$, $b_\mu(t)$ the mean and variance belonging to μ.

The inequality (12) implies

$$\sup_n \langle W_n^4(t)\rangle < \infty. \tag{21}$$

It is known that in this case the sequences $W_n(t)$, $(W_n(t)-\langle W_n(t)\rangle)^2$ are uniformly integrable. Therefore we have by assumptions (3) and (4) $a_\mu(t)=a(t)$, $b_\mu(t)=b(t)$.

Since the distribution law of a general Wiener process is uniquely determined by its mean and variance there exists at most one accumulation point of $\mathscr{L}(W_n)$. According to the relative compactness, which has been proved in part 2, we get $\mathscr{L}(W_n) \Rightarrow_{n\to\infty} \mathscr{L}(W_{a,b})$. The proof of the assertion in Remark 1 follows from the last limit relation and the fact that the sequence $W_n(t_1)\ldots W_n(t_k)$, $n=1, 2, \ldots$, is uniformly integrable according to (21).

P r o o f o f C o r o l l a r y 1. We apply Theorem 1 to $X_n(s)=\sqrt{n}\,Z(sn)$. In order to deduce the relations (1)—(4) we first note, that $\varphi(t, X_n) \leq \varphi(nt, Z)$. Let $c_n = n^2 d$, then

$$c_n\Big(\int_0^1 \varphi^{1/8}(t, X_n)dt\Big)^2 \leq d\Big(\int_0^n \varphi^{1/8}(s, Z)ds\Big)^2,$$

$n=1, 2, \ldots$, and $\lim_{n\to\infty} \varphi(\varepsilon, X_n) \leq \lim_{n\to\infty} \varphi(\varepsilon \circ n, Z)=0$. The assumptions $\langle Z(t)\rangle=0$ and (7) imply (3) and (4).

P r o o f o f C o r o l l a r y 2. We first deal with $\langle(\sqrt{n}\int_0^t Z_n(s)ds)^2\rangle$. Using the assumptions (9), (10) we get

$$\lim_{n\to\infty} \langle(\sqrt{n}\int_0^t Z_n(s)ds)^2\rangle = \lim_{n\to\infty} n \int_0^t \int_0^t \langle Z_n(s_1)Z_n(s_2)\rangle ds ds_2$$

$$= \lim_{n\to\infty} n \int_{1/n}^{t-1/n} \int_{-1/n}^{1/n} \langle Z_n(t)Z_n(t+s)\rangle ds = \int_0^t \bar{b}(s)ds.$$

Put $c_n = n^2 d$, then according to (9) $c_n(\int_0^1 \varphi_n^{1/8}(t)dt^2 \leq n^2 d 1/n^2 = d$.

P r o o f o f T h e o r e m 2. Put $I_n(s_1, s_2)=\int_{s_1}^{s_2} X_n(s)ds$, then

$$\langle(I_n(p, q))^2\rangle = \int_p^q \int_p^q R_n(s-t)dsdt = \int_0^{q-p} \int_0^{q-p} R_n(s-t)dsdt = \langle(I_n(0, q-p))^2\rangle \underset{n\to\infty}{\to} b(q-p).$$

For any real numbers $s_1 < s_2 < s_3 < s_4$ we obtain by Lemma 1

$$\lim_{n\to\infty} |\langle \int_{s_1}^{s_2} X_n(s)ds \int_{s_3}^{s_4} X_n(s)ds\rangle| \tag{22}$$

$$\leq 2 \lim_{n\to\infty} \varphi_n^{1/2}(s_3-s_2)\langle(\int_{s_1}^{s_2} X_n(s)ds)^2\rangle^{1/2} \langle(\int_{s_3}^{s_4} X_n(s)ds)^2\rangle^{1/2}$$

$$\leq 2\,(b(s_2-s_1)\,b\,(s_4-s_3))^{1/2}\lim_{n\to\infty}\varphi_n^{1/2}\,(s_3-s_2)=0.$$

Therefore $\lim_{n\to\infty}\langle(I_n(s_1,s_2)+I_n(s_3,s_4))^2\rangle=b(s_2-s_1)+b\,(s_4-s_3)$.

Let X, Y be random variables with $\langle X^2\rangle<\infty$, $\langle Y^2\rangle<\infty$. A simple calculation shows that $|\langle X^2\rangle-\langle(X+Y)^2\rangle|\leq\langle Y^2\rangle+2\langle X^2\rangle^{\frac{1}{2}}\langle Y^2\rangle^{1/2}$. Applying this inequality we get

$$|\langle(I_n(s_1,s_2)+I_n(s_3,s_4))^2\rangle-\langle I_n^2(s_1,s_4)\rangle|\leq\langle I_n^2(s_2,s_3)\rangle$$

$$+2\langle I_n^2(s_2,s_3)\rangle^{1/2}\,[\langle I_n^2(s_1,s_2)\rangle^{1/2}+\langle I_n^2(s_3,s_4)\rangle^{1/2}].$$

The last inequality implies as $n\to\infty$ $|\,b(s_2-s_1)+b(s_4-s_3)-b(s_4-s_1)\,|\leq b(s_3-s_2)+2b^{1/2}(s_3-s_2)\,[b^{1/2}(s_2-s_1)+b^{1/2}(s_4-s_3)]$. If s_3 tends to s_2 we obtain $b\,(s_4-s_2)+b(s_2-s_1)=b(s_4-s_1)$ as b is a continuous function.

This functional equation proves, together with the continuity of b, is our assertion.

Proof of Theorem 3. 1) Let $Z_n(t)$, $t\in T$, be a sequence of stochastic processes and $Z(t)$, $t\in T$, be a Gaussian process with mean zero and covariance function $R(t,s)$. Using characteristic functions it is easy to see that $\mathfrak{G}(Z_n)\to_{n\to\infty}\mathfrak{G}(Z)$ if for every $t_1,\ldots,t_k\in T$, $x_1,\ldots,x_k\in(-\infty,+\infty)$ the distribution law of $x_1Z_n(t_1)+\cdots+x_kZ_n(t_k)$ converges weakly to a normal distribution with mean zero and variance $\Sigma_{i,j=1}^k x_ix_jR(t_i,t_j)$.

Put $\bar{Y}_n(t)=\int_0^1K(t,s)(X_n(s)-\langle X_n(s)\rangle)ds$. Let $\bar{X}(t)$, $t\in T$, be a Gaussian process with mean zero and covariance function $R(t_1,t_2)=\int_0^1K(t_1,s)K(t_2,s)db(s)$. In order to prove $\mathfrak{G}(Y_n)\to_{n\to\infty}\mathfrak{G}(\mathrm{Y})$, according to the above remark, it is enough to show that for every measurable bounded f the distribution of $\int_0^1 f(s)\bar{X}_n(s)ds$ converges to a normal distribution with mean zero and variance $\int_0^1 f^2(s)db(s)$.

According to Theorem 1 this statement is valid for step functions $f(s)=\alpha_1I_{[0,a_1)}+\cdots+\alpha_{n-1}I_{[a_{n-1},a_n)}+\alpha_nI_{[a_n,1]}$, where I_A stands for the indicator function of the set A.

Let now f be a bounded measurable function on $[0,1]$ and f_m be a sequence of step functions with

$$\int_0^1(f(s)-f_m(s))^2ds\underset{m\to\infty}{\to}0,$$

$$\int_0^1 f_m^2(s)\,d\,b\,(s)\underset{m\to\infty}{\to}\int_0^1 f^2(s)d\,b(s). \tag{24}$$

Using inequality (12) we get

$$\sup_n\langle|\int_0^1(f(s)-f_m(s))\bar{X}_n(s)ds\,|\rangle\leq[\int_0^1(f(s)-f_m(s))^2ds]^{1/2}\sup_n[\int_0^1\bar{X}_n^2(s)ds]^{1/2} \tag{25}$$

$$\leq[\int_0^1(f(s)-f_m(s))^2ds]^{1/2}\,(3360\,.\,c)^{1/4}$$

with c from (2). Setting $\sigma^2=\int_0^1 f^2(s)d\,b(s)$, $\sigma_m^2=\int_0^1 f_m^2(s)d\ b(s)$ we obtain for a real z

$$|\langle\exp(iz\int_0^1 f(s)\bar{X}_n(s)ds)\rangle-\exp(-\tfrac{1}{2}z^2\sigma^2)|\leq z\langle|\int_0^1(f(s)-f_m(s))\,\bar{X}_n(s)ds\,|\rangle$$

$$+|\langle\exp(iz\int_0^1 f_m(s)\bar{X}_n(s)ds)\rangle-\exp(-\tfrac{1}{2}z^2\sigma_m^2)|+|\exp(-\tfrac{1}{2}z^2\sigma_m^2)-\exp(-\tfrac{1}{2}z^2\sigma^2)|.$$

The first term tends to zero as $m\to\infty$ uniformly in n according to (25).

Because of theorem 1 the second term converges to zero as $n\to\infty$ for fixed m since f_m is a step function.

The third term is small for large m because of (24). Thus we proved that the distribution of $\int_0^1 f(s)\bar{X}_n(s)ds$ tends to a normal distribution with mean zero and variance $\int_0^1 f^2(s)d\,b(s)$.

To finish the proof we remark that $\mathfrak{G}(\bar{Y}_n)\to_{n\to\infty}\mathfrak{G}(\bar{Y})$ and

$$\langle Y_n(t)\rangle=\int_0^1 K(t,s)\langle X_n(s)\rangle ds \underset{n\to\infty}{\to} \alpha(t)=\langle Y(t)\rangle$$

imply $\mathfrak{G}(Y_n)\to_{n\to\infty}\mathfrak{G}(Y)$.

Proof of Theorem 4. We have by Fubini's theorem

$$Y_n(t)=\int_0^1 K(t,s)X_n(s)ds=\int_0^1\int_0^s k(t,u)X_n(s)duds \tag{26}$$

$$=\int_0^1\left(\int_u^1 X_n(s)ds\right)k(t,u)du=W_n(1)\int_0^1 k(t,u)du-\int_0^1 W_n(u)k(t,u)du \text{ and therefore}$$

$$Y_n(t)-\langle Y_n(t)\rangle=[W_n(1)-\langle W_n(1)\rangle]\int_0^1 k(t,u)du-\int_0^1(W_n(u)-\langle W_n(u)\rangle)k(t,u)du.$$

We now consider the mapping T which is defined by

$$(Tf)(t)=f(1)\int_0^1 k(t,u)du-\int_0^1 f(u)k(t,u)du.$$

Using the assumed properties of k it is easy to see that T is a continuous mapping from C into C. In the proof of Theorem 1 it was shown that $\mathscr{L}(W_n(t)-\langle W_n(t)\rangle)$ is relatively compact. The continuity of T implies that $\mathscr{L}(Y_n(t)-\langle Y_n(t)\rangle)$ is relatively compact, too.

Theorem 3 yields $\mathfrak{G}(Y_n(t)-\langle Y_n(t)\rangle)\to_{n\to\infty}\mathfrak{G}(\bar{Y}(t))$ where $\bar{Y}(t)$ is a Gaussian process with mean zero and covariance function $\int_0^1 K(t,s)K(t_2,s)d\,b(s)$. Thus we obtain

$$\mathscr{L}(Y_n(t)-\langle Y_n(t)\rangle)\underset{n\to\infty}{\Rightarrow}\mathscr{L}(\bar{Y}(t)). \tag{27}$$

Using the definition of $a(t)$ in Theorem 1 we get by (26)

$$|\langle Y_n(t)\rangle-a(1)\int_0^1 k(t,u)du+\int_0^1 a(u)k(t,u)du|$$

$$\leq|\langle W_n(1)\rangle-a(1)|\int_0^1|k(t,u)|du+\int_0^1|\langle W_n(u)\rangle-a(u)|\,|k(t,u)|du.$$

According to the assumption about $k(t,u)$ the mapping $t\frown k(t,u)$ is a continuous one from $[0,1]$ into $L_1([0,1])$. We have therefore $\sup_{0\leq t\leq 1}\int_0^1|k(t,u)|du<\infty$ and by condition (3)

(28) $$\sup_{0\leq t\leq 1} |\langle Y_n(t)\rangle + a(t)\int_0^1 k(t,u)du - \int_0^1 a(u)k(t,u)du| \underset{n\to\infty}{\to} 0.$$

(27) and (28) imply $\mathscr{L}(Y_n) \underset{n\to\infty}{\Rightarrow} \mathscr{L}(Y)$.

3. Applications to stochastic differential equation problems. 3.1 We consider the stochastic boundary value problem

(29) $$L[u]=X(t); \quad U_i[u]=0, \ i=1,\ldots,m, \ 0\leq t\leq 1.$$

L is a deterministic differential operator of order m, $L[u]=\Sigma_{k=0}^m g_k(t)u^{(k)}$, $g_m(t)\neq 0$. $U_i[u]=0, i=1,\ldots,m$, are deterministic boundary conditions

$$U_i[u]=\sum_{k=0}^{m-1}(\alpha_{ik}u^{(k)}(0)+\beta_{ik}u^{(k)}(1)).$$

$X(t)$ is a separable a. s. continuous stochastic process. Further on we assume that Green's function $G(x,y)$ of the differential operator $L[u]$ and the above boundary conditions $U_i[u]$ exists. Under these assumptions the solution $u(t)$ of the boundary value problem (29) exists almost sure and may be written in the form $u(t)=\int_0^1 G(t,s)X(s)ds$. We substitute X by a sequence X_n of processes satisfying the assumptions of theorem 3. Then we obtain $\mathfrak{E}(u_n)\to_{n\to\infty}\mathfrak{E}(\zeta)$ where $\zeta(t), 0\leq t\leq 1$, is a Gaussian process with mean $\langle\zeta(t)\rangle=a(t)$ and covariance function $R(t_1,t_2)=\int_0^1 G(t_1,s)G(t_2,s)d\,b(s)$.

We now suppose that $L[u]$ is a selfadjoint and positively definite operator. Then Green's function $G(x,y)$ is symmetrical $G(x,y)=G(y,x)$ and there are at most countable many eigenvalues $0\langle\lambda_1\leq\lambda_2\leq\ldots$ of the eigenvalue problem $L[u]=\lambda u$, $U_i[u]=0$ $i=1,2,\ldots,m$. The eigenfunctions corresponding to the eigenvalues have the property $\int_0^1\varphi_i(t)\varphi_j(t)dt=\delta_{ij}$. By Mercer's theorem one gets $G(x,y)=\Sigma_{i=1}^\infty\frac{1}{\lambda}\varphi_i(x)\varphi_i(y)$, where the series converges uniformly. Therefore the covariance function may be expressed by the eigenfuctions

$$R(t_1,t_2)=\sum_{i,j=1}^\infty\frac{1}{\lambda_i\lambda_j}\varphi_i(t_1)\varphi_j(t_2)\int_0^1\varphi_i(s)\varphi_j(s)db(s).$$

We obtain in the special case $b(t)=\sigma^2 t$

$$R(t_1,t_2)=\sigma^2\sum_{i=1}^\infty\frac{1}{\lambda_i^2}\varphi_i(t_1)\varphi_i(t_2).$$

Example. $u^{\text{IV}}=X$, $u(0)=u''(0)=u(1)=u''(1)=0$.

Substituting X by a sequence X_n of processes with mean zero satisfying the assumptions in Theorem 3 we obtain $\mathfrak{E}(u_n)\to_{n\to\infty}\mathfrak{E}(\zeta)$, where ζ is a Gaussian process with mean zero and covariance function

$$R(t_1,t_2)=\int_0^1 G(t_1,s)G(t_2,s)d\,b(s),$$

where

$$G(x,y)=\begin{cases} x(y-1)(x^2+y^2-2y)/6, & 0\leq x\leq y\leq 1,\\ y(x-1)(y^2+x^2-2x)/6, & 0\leq y\leq x\leq 1.\end{cases}$$

A simple calculation shows that $\lambda_k=(k\pi)^4$, $\varphi_k(t)=\sqrt{2}\sin(k\pi t)$. Therefore we have for a sequence weakly of stationary processes X_n satisfying the conditions in Theorem 3

$$\lim_{n\to\infty}\langle u_n(t_1)\,u_n(t_2)\rangle=\frac{2\sigma^2}{8\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^8}(\sin k\pi t_1)(\sin k\pi t_2),$$

because $b(t)=\sigma^2 t$.

3.2. In this section we deal with initial value problems for linear differential equations having a stochastic inhomogenous term $X(t)$

$$(30)\qquad \sum_{k=0}^{m} a_k(t)\,u^{(k)}(t)=X(t)\quad 0\leq t\leq 1\,;\quad u^{(p)}(0)=u_{p,0},\ p=0,1,\dots,m-1.$$

$X(t)$ is a separable a. s. continuous stochastic process. We assume that $a_k(t)$ are continuous functions. Such initial value problems have unique solutions.

If $u_k(t)$, $k=1,\dots,m$, denotes a fundamental systems of solutions of the homogenous differential equation the solution of problem (30) may be written as

$$u(t)=\sum_{k=1}^{m}c_k u_k(t)+\sum_{k=1}^{m}u_k(t)\int_0^t\frac{W_k(s)}{W(s)}ds.$$

$W(t)$ denotes Wronski's determinant $W(t)=\det(u_i^{(j)}(t),\ 1\leq i\leq m,\ 0\leq j\leq m-1)$. $W_k(t)$ is the determinant which is obtained from Wranski's determinant substituting the k-th column by $(0,\dots,0,X(t))$, $k=1,\dots,m$. The constants $c_1,\dots,c_m$ are determined by the initial conditions $\Sigma_{k=1}^{m}c_k u_k^{(p)}(0)=u_{p,0}$.

Let $A_{k,l}(t)$ denote the determinant obtained from $W(t)$ by omitting the k-th line and the l-th column. Then we have for $u(t)$

$$u(t)=\sum_{k=1}^{m}c_k u_k(t)+\int_0^t\frac{1}{W(s)}\Big[\sum_{k=1}^{m}(-1)^{m+k}A_{m,k}(s)u_k(t)\Big]X(s)ds.$$

We put

$$K(t,s)=\frac{1}{W(s)}\sum_{k=1}^{m}(-1)^{m+k}A_{m,k}(s)u_k(t)\,I_{[0,t]}(s).$$

Let $X_n(s)$ be a sequence of processes satisfying the conditions in Theorem 3. Denote by u_n the sequence of solutions of the initial value problem belonging to the sequence of inhomogenous terms X_n. We obtain by Theorem 3 $\mathfrak{G}(u_n)\to_{n\to\infty}\mathfrak{G}(u)$, where u is a Gaussian process with mean

$$\langle u(t)\rangle=\sum_{k=1}^{m}c_k u_k(t)+\lim_{n\to\infty}\int_0^1 K(t,s)\langle X_n(s)\rangle ds$$

and covariance function $R(t_1,t_2)=\int_0^1 K(t_1,s)K(t_2,s)d\,b(s)$. Denote

$$u_{1,n}(t)=u_n(t)=\sum_{k=1}^{m}c_k u_k(t)+\int_0^1 K_1(t,s)X_n(s)ds,$$

$$u_{2,n}(t)=u_n'(t)=\sum_{k=1}^{m}c_k u_k'(t)+\int_0^1 K_2(t,s)X_n(s)ds,$$

where $K_1(t, s)=K(t, s)$, $K_2(t, s)=\frac{\partial K}{\partial t}(t, s)$. We assume additional that $\lim_{n\to\infty}\int_0^1 K_2(t, s)\langle X_n(s)\rangle ds$ exists. Denote

$$Y_{i,n}(t)=\int_0^1 K_i(t, s)X_n(s)ds,\ K(t, s)=x_1K_1(t, s)+x_2K_2(t, s),$$

where x_i are fixed real numbers, then we obtain from Theorem 3 $\mathfrak{G}(x_1 u_{1,n} + x_2 u_{2,n})\to_{n\to\infty}\mathfrak{G}(\zeta)$, where ζ is a Gaussian process with mean $\lim_{n\to\infty}\int_0^1(x_1K_1(t, s) + x_2 K(t, s)\langle X_n(s)\rangle ds$ and covariance function

$$R(t_1, t_2)=\sum_{i,j=1}^{2} x_i x_j\int_0^1 K_i(t_1, s)K_j(t_2, s)d\,b(s).$$

Using similar arguments as in the proof of Theorem 3 we obtain the convergence of all finite-dimensional distributions of the vector process $(u_{1,n}(t), u_{2,n}(t))$ to the corresponding distributions of a Gaussian vector process $(\zeta_1(t), \zeta_2(t))$ having mean

$$\langle\zeta_1(t)\rangle=\sum_{k=1}^{m} c_k u_k(t)+\lim_{n\to\infty}\int_0^1 K_1(t, s)\langle X_n(s)\rangle ds,$$

$$\langle\zeta_2(t)\rangle=\sum_{k=1}^{m} c_k u_k'(t)+\lim_{n\to\infty}\int_0^1 K_2(t, s)\langle X_n(s)\rangle ds$$

and covariance function $R_{ij}(t_1, t_2)=\langle\zeta_i(t_1)\zeta_j(t_2)\rangle-\langle\zeta_i(t_1)\rangle\langle\zeta_j(t_2)\rangle$, where

$$R_{ij}(t_1, t_2)=\int_0^1 K_i(t_1, s)K_j(t_2, s)d\,b(s).$$

The limit theorem for the vector processes $(u_{1,n}(t), u_{2,n}(t))$ may be applied to calculate approximately the expectation of the number of level crossings.

If we deal with initial value problems with constant coefficients $a_k(t)=a_k$ the roots of the associated algebraic equation $\sum_{k=0}^{m} a_k\lambda^k=0$ possess in many cases negative real parts. Thus we have for large t approximately $u_k(t)\approx 0$.

Example. We consider a one mass system described by the initial value problem $u''+2\beta u'+w_0^2u=X(t)$, $u(0)=u_{00}$, $u'(0)=u_{10}$ with stochastic exitation $X(t)$. We assume $w_0^2-\beta^2>0$, $\beta>0$. These conditions are fulfilled, for instance, for vibration problems. $u_1(t)=e^{-\beta t}\cos(wt)$, $u_2(t)=e^{-\beta t}\sin(wt)$ form a fundamental systems of solutions. Here we have put $w=(w_0^2-\beta^2)^{1/2}$. The initial conditions imply $c_1=u_{00}$, $c_2=(u_{10}+u_{00})/w$.

The solution of our initial value problem may be written as

$$u(t)=e^{-\beta t}(c_1\cos(wt)+c_2\sin(wt))+\frac{1}{w}\int_0^t X(s)\,e^{-\beta(t-s)}\sin(w(t-s))ds.$$

For a sequence $X_n(s)$ satisfying the conditions in Theorem 3 with

$$K(t, s)=\frac{1}{w}e^{-\beta(t-s)}\sin(w(t-s))\,I_{[0,t]}(s)$$

we obtain $\mathfrak{G}(u_n)\to_{n\to\infty}\mathfrak{G}(u)$, where u is a Gaussian process with mean

$$\lim_{n\to\infty} \frac{1}{w} \int_0^t \langle X_n(s)\rangle e^{-\beta(t-s)} \sin(w(t-s))ds + e^{-\beta t}(c_1 \cos(wt) + c_2 \sin(wt))$$

and covariance function

$$R(t_1, t_2) = \frac{1}{w^2} e^{-\beta(t_1+t_2)} \int_0^{\mathrm{Min}\,(t_1,t_2)} e^{2\beta s}(\sin(w(t_1-s))\sin(w(t_2-s))d\, b(s).$$

Provided that the X_n are weakly stationary we have $b(t)=\sigma^2 t$ and

$$R(t_1, t_2) = \frac{\sigma^2}{4w_0^2} e^{-\beta|t_1-t_2|} (\frac{1}{\beta} \cos(w\,|\,t_1-t_2\,|) + \frac{1}{w} \sin(w\,|\,t_1-t_2\,|))$$

$$+\frac{\sigma^2}{4w^2w_0^2} e^{-\beta(t_1+t_2)} (\beta \cos(w(t_1+t_2)) - w \sin(w(t_1+t_2)) + \frac{1}{\beta} w_0^2 \cos(w\,|\,t_1-t_2\,|)).$$

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