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PERFECTION OF IDEALS GENERATED BY THE PFAFFIANS OF AN ALTERNATING MATRIX, 1

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Let R be a Noetherian commutative ring, X be an alternating $s\times s$ matrix whose above-diagonal entries are algebraically independent over R, and let $Pf_{2t}(X)$ denote the ideal in $R[X_{ij}]$ generated by the pfaffians of all the alternating $2t\times 2t$ submatrices of X. It is proved that $Pf_{2t}(X)$ is perfect of depth $(s-2t+1)(s-2t+2)/2$, and that it is prime when R is a domain. The theorem is a consequence of a more general result, involving ideals generated by both pfaffians and determinants of submatrices of X.

In the first part we prove that the ideals of this class are prime or radical, when R is a domain.

An important invariant for the ideal I in a commutative ring is its depth (depth I = the length of the longest R-sequence contained in I). In view of the inequality depth $I\leq \mathrm{pd}_R R/I$ holding for ideals in a Noetherian ring R, where $\mathrm{pd}_R R/I$ is the projective (homological) dimension of R/I as an R-module, the perfect ideals (for which depth $I=\mathrm{pd}_R R/I$) are of special interest: cf. e. g. [6].

An important class of perfect ideals was constructed by Hochster and Eagon ([6]) who proved that the ideal generated by $t\times t$. minors of an $r\times s$ matrix M with entries in R is perfect, if its depth reaches the maximal possible value. Here, as in the rest of this paper, all rings are commutative with identity.

In [11] Kutz proved an analogous statement supposing M is a symmetric matrix. Buchsbaum and Eisenbud aroused interest in the case of an alternating $s\times s$ matrix M (i. e. $M^T=-M$ and diagonal entries are 0) in their paper [4]. They showed that in this case the pfaffians are the fundamental invariants and not the minors (see Lemma 3) and proved some important statements for the ideals generated by pfaffians of $2t\times 2t$ alternating submatrices $Pf_{2t}(M)$ (see 1). Moreover, it can be deduced from their results, that $Pf_{2t}(M)$ is perfect, when $s=2t+1$ and its depth is 3.

After this publication the perfection of $Pf_{2t}(M)$ was proved in some cases, under the natural restriction that the depth of this ideal reaches the maximal possible value. Józefiak and Pragacz in [7] prove the perfection in the case when M is a $2s\times 2s$ matrix, $2t=2s-2$ and depth of $Pf_{2t}(M)=6$, constructing a resolution over $R/Pf_{2t}(M)$. For this result they require $1/2\in R$. The same authors in [8] announce for a ring R of characteristic 0 (i. e. $Q\subset R$) and for each t a complex over $R/Rf_{2t}(M)$ which is exact in case of maximal depth of $Pf_{2t}(M)$. In particular, the perfection of $Pf_{2t}(M)$ in the considered case follows.

The present paper is a continuation of the researches along these lines. In fact we prove the perfection of a class of "mixed" ideals, generated by pfaffians and by minors. The main result of this paper is:

Theorem 1. *Let R be a commutative ring (Noetherian with identity). Let $M=(c_{ij})$ be an alternating $s\times s$ matrix with entries in R. Let $H=(s_0, s_1, \ldots, s_m)$ be a strictly increasing sequence of positive integers such that $s_h\geq 2h+1$ $(0\leq h\leq m-1)$, $s_m=s$. Let n be an integer, $s_0\leq n\leq s$. We write $M(i,j)$ for the intersection of the first i rows and the first (leftmost) j columns. Let $A=A_{H,n}=A_{H,n}(M)$ be the ideal of R generated by $(2t+2)\times(2t+2)$ pfaffians of $M(s_t, s_t)$ $(0\leq t\leq m)$, by $(t+t'+1)\times(t+t'+1)$ minors of $M(s_t, s_{t'})$ $(0\leq t<t'\leq m)$ and by $c_{12}, \ldots, c_{1n}$. Let $h=\min\{t: s_t\geq n\}$ and Σ_h denote the number of these indices g $(1\leq g\leq h)$ for which $s_{g-1}=2g-1$. Set $f_{H,n}=s(s-1)2-2sm+m(m-1)+s_0+\ldots+s_{m-1}+h+\Sigma_h$ and*

$$g_{H,n}=\begin{cases} f_{H,n-1}+1 & \text{for } n=s_t+1=2t+2 \text{ for some } t,\\ f_{H,n} & \text{in the other cases.}\end{cases}$$

Then $\max_{R,M}$ depth $A_{H,n}=g_{H,n}$.

If $n=s_t$ or $n=s_t+1$ or $n=s_t+2=2t+3$ for some t and depth $A_{H,n}=g_{H,n}$ *then $A_{H,n}$ is perfect.*

In case $R=K[X]$ for some Noetherian ring K and an alternating matrix X with indeterminates as entries and $A_{H,n}=A_{H,n}(X)$ the following additional statements are true:

1. depth $A_{H,n}=g_{H,n}$ *for $n=s_t$ or $n=s_t+1$ or $n=s_t+2=2t+3$.*
2. *if K is a domain, $A_{H,n}$ is radical and* depth $A_{H,n}=g_{H,n}$.
3. *if K is a domain and $n=s_t$ or $n=s_t+1$, $A_{H,n}$ is prime.*

An immediate consequence of this theorem is the following result which motivated the present investigation: set $H=(1, 3, 5, \ldots, 2m-3, s)$ and $n=1$.

Theorem 2. *Let R be a commutative Noetherian ring with identity and M be an alternating $s\times s$ matrix with entries in R. Assume $t\geq 1$ and $2t\leq s$. Then* $\max_{R,M}$ depth $Pf_{2t}(M)=d_t=(s-2t+1)(s-2t+2)/2$. *If* depth $Pf_{2t}(M)=d_t$, *then $Pf_{2t}(M)$ is perfect.*

Moreover if K is a Noetherian domain and $R=K[X]=K[x_{ij}: 1\leq i<j\leq s]$, where $X=(x_{ij})$ is an alternating $s\times s$ matrix and x_{ij} $(i<j)$ are indeterminates over K, then $Pf_{2t}(X)$ is a prime ideal and depth $Pf_{2t}(X)=d_t$.

The proof of Theorem 1 relies on the induction scheme introduced by Hochster and Eagon in [6]. The same scheme has been used by Kutz in [11]. In the proof of Hochster and Eagon we see three main moments: the proof that some ideals are radical, the construction of a generic point for the radicals of the "fundamental" ideals and the construction of a generic point for the ideals themselves.

In the present paper the above three moments are conserved but the construction of the first generic point is replaced by an inductive proof that the ideal is prime (Proposition 10). It must be noted that one of the fundamental difficulties in the work was the determination an "optimal" ideal class for which Theorem 1 can be proved.

The first part of the present paper consists of five paragraphs. In 1 the definition and some principal properties of the pfaffians are recalled and ideals generated by minors and pfaffians are briefly considered. In 2 we introduce the needed version of the method of Hochster and Eagon for indexing of the ideals by "descriptions". In 3 and 4 are proved the statements of Theorem 1 that some ideals are radical (resp. prime). Some technical statements required for this are established in 5. The contents of the second part is as follows. In 6 and 7 we compute the depth of the ideal in case K is a field and the matrix has algebraically independent entries, and in 8 the perfection is proved in the same case. The proof of the theorem is completed in 9.

The present paper was written on the basis of the author's master's thesis (University of Sofia, 1978). The main results were announced in [12]. I want to thank Luchezar Avramov for his help as adviser of my master's thesis and for his constant interest in this paper.

After finishing my work on this subject I learned that Dan Laksov and Hans Kleppe have proved Theorem 2. Their proof leans also on the method of Hochster and Eagon but they work in a different class of ideals ([10]). Using the result of Theorem 2 and some additional results from their paper, Avramov in [3] proved that $R[X]/Pf_{2t}(X)$ is a unique factorization domain if and only if R is a unique factorization domain. It follows that the ideals $Pf_{2t}(X)$, when perfect, are even Gorenstein, a result which was initially obtained by Kleppe and Laksov using a different method.

1. Ideals generated by minors and pfaffians. Let $G=(g_{ij})$ be an alternating $n\times n$ matrix with entries in the ring R, that is $G^T=-G$ and all diagonal entries are 0.

There exists for each alternating matrix G a polinomial $PfG=Pf(g_{ij})$ of its entries known as the pfaffian of G which is uniquely determined by the following two conditions: $\det G=(PfG)^2$ and $Pf\ (A^TGA)=\det A.PfG$. The pfaffians have other properties which make them as useful for working with alternating matrices as determinants are for working with square matrices. The most useful property of the pfaffians is their expansion with respect to one row: For each $k\leq n$ we can expand PfG with respect to k'th row of G by the following formula $Pf(G)=\sum_{j=1}^{n}\sigma_{kj}\,g_{kj}\,Pf_{kj}$ where $\sigma_{kj}=\operatorname{sgn}\ (k, j, 1, 2, \ldots, \hat{k}, \ldots, \hat{j}, \ldots, n)$. (Here sgn denotes the number of inversions and $\hat{}$ denotes an absent element; Pf_{kj} is the pfaffian of the matrix formed from G by deletion of k'th and j'th rows and columns). For more information about pfaffians cf. [1, ch. III, §5]. A paraphrase in the language of polylinear algebra can be found in [4, § 2].

Let R be a commutative ring and M be an $r\times s$ matrix with entries in R. Let $t\geq 1$ be an integer. We write $I_t(M)$ for the ideal of R generated by $t\times t$ minors of M; in particular $I_t(M)=0$ if $t>\min\{r, s\}$. Obviously $I_{t+1}(M)\subset I_t(M)$.

If M is an alternating $s\times s$ matrix and $2t\leq s$ and if we delete $n-2t$ columns and the corresponding rows of M, an alternating $2t\times 2t$ matrix M' will be left; M' will be called an alternating submatrix of M. We write $Pf_{2t}(M)$ for the ideal of R generated by the pfaffians of all alternating $2t\times 2t$ submatrices of M; for $2t>s$ we set $Pf_{2t}(M)=0$. From the formula for row expansion of pfaffian it follows that $Pf_{2t+2}(M)\subset Pf_{2t}(M)$. It is evident too that $I_t(M)=I_t(M^T)$ and $Pf_{2t}(M)=Pf_{2t}(M^T)$ where M^T is the transposed matrix of M. Consequently the ideal A from Theorem 1 will be written with these notations as follows:

$$A=\sum_{i=0}^{m}Pf_{2i+2}\,(M(s_i, s_i))+\sum_{0\leq i<j\leq m}I_{i+j+1}\,(M(s_i, s_j))+(c_{12}, \ldots, c_{1n}).$$

From the relation $\det(M)=Pf^2(M)$ for each alternating matrix M follows the inclusion $Pf_{2t}(M)\subset \mathrm{Rad}\ (I_{2t}(M))$. Buchsbaum and Eisenbud [4, Corollary 2.6] prove the following:

Lemma 3. *Let M be an alternating matrix and $t\geqq 1$ be an integer. Then $I_{2t}(M)\subset I_{2t-1}(M)\subset Pf_{2t}(M)\subset \mathrm{Rad}\ (I_{2t}(M))$.*

Now let M be an alternating $s\times s$ matrix. Let a be an element of R and b be an invertible element in R. Let $1\leqq k\leqq s$, $1\leqq m\leqq s$, $k\neq m$. Add the m'th row multiplied by a to the k'th row multiplied by b and subsequently do the analogous operation on the columns.

Definition. *The described operation will be called an elementary transformation of the alternating matrix M and will be denoted by $E_a(k, m; a, b)$ or only by E_a.*

Evidently, the matrix $E_a(M)$ is also alternating and $Pf_{2t}(E_a(M))=Pf_{2t}(M)$ for each E_a, each t and each M.

2. Descriptions. We shall introduce a considerable number of conventions and notational devices for dealing with matrices, which will be in force for the rest of the paper.

Usually the letters $i, j, k, m, n, r, s, t, m, r'$ etc. will stand for elements of the set $\mathbf{N}$ of nonnegative integers. Let $K\subset R$ be rings. If M is an $r\times s$ matrix with entries in R (we permit "empty matrices" $\varnothing$ one or both of whose dimensions are 0) let $M(i, j)$ be the matrix formed by the intersection of the first i rows and the first (leftmost) j columns. $M(i,j)=\varnothing$ if $i=0$ or $j=0$, while $M(i,j)=M(r, j)$ if $i\geqq r$, and $M(i, j)=M(i, s)$ if $j\geqq s$. For a subring K of R we write $K[M]$ for $K[\{x: x \text{ is an entry of } M\}]\subset R$.

Now we shall observe some elementary properties of the ideals $I_t(M(i, j))$ (and of $Pf_{2t}(M(i, i))$, when M is alternating).

Let M be an $r\times s$ matrix. Then:

1) If $r\leqq i$, $I_t(M(i, j))=I_t(M(r, j))$. If $s\leqq j$, $I_t(M(i,j))=I_t(M(i, s))$;

2) If $t>\min\{i, j, r, s\}$, $I_t(M(i, j))=0$;

3) $I_{t+k}(M(i+k, j))\subset I_t(M(i, j))$ and $I_{t+k}(M(i, j+k))\subset I_t(M(i, j))$. Better: if $u\geqq t$ and $k+m-i-j\leqq u-t$, then $I_u(M(k, m))\subset I_t(M(i, j))$. Here is included the case $k\leqq i$, $m\leqq j$, $u=t$.

Now let M be an alternating $s\times s$ matrix, Then:

1) If $s\leqq i$, $Pf_{2t}(M(i, i))=Pf_{2t}(M)$;

2) If $2t>\min\{i, s\}$, $Pf_{2t}(M(i, i))=0$;

3) $Pf_{2(t+j)}(M(i+j, i+j))\subset Pf_{2t}(M(i, i))$. Better: if $u\geqq t$ and $j-i\leqq u-t$, then $Pf_{2u}(M(j, j))\subset Pf_{2t}(M(i, i))$. (Property 3 for pfaffians can be proved by repeated use of the formula for the expansion with respect to a row.)

We shall denote m-tuple of numbers by $(s_1, s_2, \ldots, s_m)$. Let M be an alternating $s\times s$ matrix.

Definition. *We shall call a description (for M) the $m+1$-tuple $H=(s_0, s_1, \ldots, s_m)$, when s_i are integers and $0\leqq s_0<s_1<\ldots<s_m\leqq s$. The description H will be called a standard description when $s_t\geqq 2t+1$ $(0\leqq t\leqq m-1)$ and $s_m=s$.*

Proposition 4: *Let R be a commutative ring with identity. $M=(c_{ij})$ be an alternating $s\times s$ matrix with entries in R, $H=(s_0, s_1, \ldots, s_m)$ be a description, $0\leqq n\leqq s$ and k be an odd positive number. Let*

$$A^k_{H,n}(M)=\sum_{j=0}^{m} Pf_{2i+k+1}(M(s_i,s_i))+\sum_{0\leqq i<j\leqq m} I_{i+j+k}(M(s_i, s_j))+(c_{12}, \ldots, c_{1n}).$$

$A^1_{H,n}$ will be denoted also by $A_{H,n}$.

Then there exists a standard description $H'=(s'_0, s'_1, \ldots, s'_{m'})$ *such that* $A^k_{H,n}(M)=A_{H',n}(M')$.

Proof. We shall introduce three types of operations substituting a description H' for the description H, but conserving the ideal,

Operation 1. Let $h=\max\{t: s_t<2t+k\}$, $s'_h=\min\{s, 2h+k\}$. Then we construct H' by $(s_0, \ldots, s_{h-1}, s'_h, s_{h+1}, \ldots, s_m)$ deleting the repeated numbers.

Operation 2. It is applied to H such that $s_t\geq 2t+k\,(0\leq t\leq m-1)$, $s_m=\min\{s, 2m+k\}$. Then $H'=(s'_0, \ldots, s'_{m+k_1})$, where $k_1=(k-1)/2$ and

$$s'_i=\begin{cases} 2i+1, & 0\leq i<k_1 \\ s_{i-k_1}, & k_1\leq i\leq m+1. \end{cases}$$

Operation 3. It is applied for $k=1$ and to H such that $s_t\geq 2t+1$ $(0\leq t\leq m-1)$, $s_m=\min\{s, 2m+1\}$. Then $H'=(s'_0, \ldots, s'_{m+q})$, where $q=s-s_m$ and

$$s'_i=\begin{cases} s_i, & 0\leq i\leq m, \\ s_{m+i-m}, & m\leq i\leq m+q. \end{cases}$$

Apply to the initial description operation 1 so many times, as possible (that is, while h exists). After that apply once operation 2, followed by 3. Finish by applying again operation 1 several times (while it is possible). We leave to the reader the verification that this leads to a standard description. To complete the proof of the proposition we must show that every operation conserves the ideal. For operations 1 and 2 this follows from the relations between $I_t(M(i,j))$, $Pf_{2t}(M(i,i))$ with various values of the parameters: see the beginning of the paragraph. For operation 3 we have:

$$A_{H',n}=A_{H,n}+\sum_{i=m}^{m+q} Pf_{2i+2}(M(s'_i, s'_i))+\sum_{\substack{0\leq i<j\leq m+q\\ m<j}} I_{i+j+1}(M(s'_i, s'_j)).$$

For $m\leq i\leq m+q$ we have

$$Pf_{2i+2}(M(s'_i, s'_i))\subset Pf_{2i}(M(s'_{i-1}, s'_{i-1}))\subset \ldots \subset Pf_{2m+2}(M(s'_m, s'_m))=Pf_{2m+2}$$
$$(M(s_m, s_m))\subset A_{H,n}.$$

Suppose $0\leq i<j\leq m+q$ and $m<j$. By property 3 for minors $I_{i+j+1}(M(s'_i, s'_j))\subset I_{i+m+1}(M(s'_i, s_m))$. Hence, when $m<i$, $I_{i+j+1}(M(s'_i, s'_j))\subset I_{2m+1}(M(s_m, s_m))\subset Pf_{2m+2}(M(s_m, s_m))\subset A_{H,n}$ (by Lemma 3). When $i\leq m$, $I_{i+j+1}(M(s'_i, s'_j))\subset I_{i+m+1}(M(s_i, s_m))\subset A_{H,n}$. Therefore $A_{H',n}=A_{H,n}$.

Definition. *By a generic alternating matrix over* R *we mean* $X=(x_{ij})$, *where* x_{ij} $i<j$ *are algebraically independent over* R, $x_{ij}=0$ *for* $i=j$, $x_{ij}=-x_{ij}$ *for* $i>j$. *If* X *is a generic alternating* $s\times s$ *matrix over* R *we denote* $R[X]=R[x_{ij}: 1\leq i<j\leq s]$.

Now we shall prove that in the cases which are interesting to us we can assume without loss of generality that $n<s$. By $B_{H,s}$ we denote the radical of the ideal $A_{H,s}$.

Lemma. 5. *Let* R *be a Noetherian ring,* X *be a generic alternating matrix over* R *and* X' *be the matrix constructed by deleting the first row and column of* X. *Let* $H=(s_0, \ldots, s_m))$ *be a description such that* $s_0>0$. *Set* H'

$=(s_0-1,\ldots,s_m-1)$. *Then* $A_{H,s}(X)$ *is prime (resp. radical) if and only if* $A_{H',0}(X')$ *is prime (resp. radical) ideal. The radical* $B_{H,s}(X)$ *is prime if and only if* $B_{H',0}(X')$ *is prime. If* R *is a field,* $A_{H,s}(X)$ *is perfect if and only if* $A_{H',0}(X')$ *is perfect, and also* depth $A_{H,s}(X)=$ depth $A_{H',0}(X')+s-1$.

Proof. Obviously $R[X]/A_{H,s}(X)\cong R[X']/A_{H',0}(X')$ as rings. From here the first three statements follow at once. The following famous lemma of Rees gives immediately that $A_{H,s}(X)$ and $A_{H',0}(X')$ are perfect or not alike.

Lemma 6 [6, Proposition 19]. *Let* R *be a field,* $S=R[x_1,\ldots,x_n]$ *and* $\mathbf{a}$ *be a homogeneous ideal of* S. *(Here* $x_1,\ldots,x_n$ *are indeterminates over* R*). Then* $S/\mathbf{a}$ *is a Cohen-Macaulay ring if and only if* $\mathbf{a}$ *is perfect.* To finish the proof of Lemma 5, we compute

depth $A_{H,s}(X)=$ ht $A_{H,s}(X)=\dim R[X]-\dim R[X]/A_{H,s}(X)$

$=s-1+\dim R[X']-\dim R[X']/A_{H',0}(X')=s-1+htA_{H',0}(\mathbf{X})=s-1$ $+$ depth $A_{H',0}(X')$.

3. The radical of $A_{H,n}$ is prime if $n=s_h+1=2h+2$ or $n=s_h$. By using the technical Lemmas 7,8, and 9 we shall prove the statement in the headline for generic alternating matricies over a Noetherian domain.

Lemma 7. *In the ring* R *let* $\mathbf{b}$ *be an ideal, let* $\mathbf{q}$ *be a prime ideal and* $\mathbf{p}_1,\ldots,\mathbf{p}_n$ *be incomparable prime ideals, such that the following condition is satisfied:* $\mathbf{a}=\bigcap_{i=1}^{n}\mathbf{p}_i\subset\mathbf{q}\subsetneqq\mathbf{b}$. *Then:*

1) $\mathbf{b}\not\subset\mathbf{p}_i$ *for* $1\leq i\leq n$;

2) *if there exists a* x *in* R, *such that* $\mathbf{a}_x$ *in prime, then there is a unique index* i, *such that* $x\notin\mathbf{p}_i$.

The elementary proof is omitted.

Most ideals with which we shall work in the rest of the paper (e. g. $I_t(M)$, $Pf_{2t}(M)$ or their sums) have natural systems of generators. When we refer about a generating system for such an ideal we mean such a set of generators.

Lemma 8. *Let* R *be a ring and* X *a generic alternating* $s\times s$ *matrix over* R. *Let* $H=(s_0,\ldots,s_m)$ *be a description,* $0\leq n\leq s$, *and* $k>0$ *be an odd number. Assume* $0\leq i<j\leq m$ *and* $(i,j)\neq(1,2),(1,3),\ldots,(1,n)$; *if* $k=1$, *we assume in addition* $s_0<j$. *Then there does not exist any minor or pfaffian* $\bar{M}$ *of the generating system of* $A^k_{H,n}(X)$ *such that* $\bar{M}=\pm x^p_{ij}+\bar{M}'$, *where* $p=0,1,\ldots$, *and* $\bar{M}'$ *is a polynomial not containing a pure power of* x_{ij} *in its expansion.*

Proof. Suppose the contrary and set $x=x_{ij}$.

The equality $p=0$ is impossible because $\bar{M}$ is a form of positive degree. If $p=1$, $\bar{M}$ is a form of degree 1. Therefore $k=1$ and $\bar{M}$ is a 2×2 pfaffian in $X(s_0,s_0)$. More precisely $\bar{M}=x_{ij}$. Therefore $j\leq s_0$ — a contradiction to the hypothesis.

If $p\geq2$, $\bar{M}$ is not a pfaffiafan because in any of the pfaffians the degree of x is not greater than 1. If $p=2$, deg $\bar{M}=2$, i. e. $\bar{M}$ is a 2×2 minor in $X(s_0,s_1)$ (and $k=1$) and this minor contains the entries x_{ij} and $-x_{ij}$. Just one such 2×2 minor exists; it is formed by the intersection of i'th and j'th rows and the corresponding columns. Hence $\bar{M}=\det\begin{pmatrix}0 & x\\ -x & 0\end{pmatrix}$. But the matrix of

$\bar{M}$ is not contained in $X(s_0, s_1)$. Therefore this case is impossible as well. If $p>2$, $\bar{M}$ is a $p\times p$ minor and $\bar{M}=\pm x^p+\dots$ is impossible because x is contained as variable just 2 times in the expansions of the entries of X. Thus the lemma is proved.

Corollary. *Under hypotheses of Lemma 8 $x_{ij}\notin$ rad $A^k_{H,n}(X)$.*

Proof. Set $A=A^k_{H,n}(X)$, $B=$ rad A and $x=x_{ij}$. By the assumption that x is an element of B, it follows that $x^a=\Sigma_q f_q M_q+g$, where $f_q\in R[X]$, M_q are minors or pfaffians from the generating system of A,

$$g\in a=\begin{cases}(x_{12},\dots,x_{1n}) \text{ for } k>1,\\ (x_{12},\dots,x_{1n})+(x_{rt}:1\leq r<t\leq s_0) \text{ for } k=1.\end{cases}$$

Therefore there exists such an u, that $f_u M_u=cx^a+\dots$ $(c\in R,\ c\neq 0)$. Since cx^a is a sum of products of two factors, one of which is a summand in the canonical expansion of M_u as a determinant or a pfaffian, such a factor may be only of the type $\pm x^p$ where $p=0, 1,\dots, a$. We get $M_u=\pm x^p+\dots$ —a contradiction by Lemma 8.

In 1 we noted that the transformations E_a conserve the ideals Pf_{2t}. But only some of these E_a conserve the ideals $A_{H,n}$.

Lemma 9. *Let R be a ring, M be an alternating $s\times s$ matrix with entries in R, $H=(s_0, s_1,\dots, s_m)$ be a description and $0\leq n\leq s$. Suppose j and j' satisfy at least one of the following conditions:*

1) $1\leq j<j'\leq s$;
2) $s_{t-1}<j'<j\leq s_t$ *for some* $t\geq 1$ *and* $n<j'$;
3) $s_{t-1}<j'<j\leq s_t$ *for some* $t\geq 1$ *and* $j\leq n$;
4) $1\leq j'<j\leq s_0$ *and* $n<j'$;
5) $1\leq j'<j\leq s_0$ *and* $j\leq n$.

Then $A^k_{H,n}(E_a(M))=A^k_{H,n}(M)$ where $E_a=E_a(j', j;\dots,\dots)$.

The proof consists of some elementery verifications of invariance with respect to E_a of the ideals $I_t(\dots)$, $Pf_{2t}(\dots)$ and $(x_{12},\dots,x_{1n})$ the sum of which equals $A^k_{H,n}$.

Difinition. *The operation described in Lemma 9 will be called an admissible elementary operation (or $A^k_{H,n}$—admissible) if it satisfies the conditions of the lemma for some description of $A^k_{H,n}$.*

Proposition 10. *Let K be a Noetherian domain and let X be a generic alternating $s\times s$ matrix over K. Let $H=(s_0, s_1,\dots, s_m)$ be a standard description such that $n=s_h$ or $n=s_h+1=2h+2$. Then $B_{H,n}=$ rad $A_{H,n}(X)$ is prime.*

Proof. We proceed by induction on s and by secondary induction on $s-n$. Denote $A=A_{H,n}(X)$ and $B=$ rad A.

Let us consider the case $s_0=s$, i. e. $m=0$. Then $H=(s_0)=(s)$. Hence $A_{H,n}(X)=Pf_2(X)=(x_{12}, x_{13},\dots, x_{1n}, x_{23},\dots, x_{2n},\dots, x_{n-1n})$. Therefore A is a prime ideal and, hence, $B=A$ is prime. Consequently in the rest of the proof we can assume that $s_0<s$.

Beginning of the induction:

If $s=1$, $X=0$ then $A=0$ and hence $B=0$. If $s=2$. Then $X=\begin{pmatrix}0 & x\\ -x & 0\end{pmatrix}$ and H can be only equal to $(1, 2)$. Therefore $A_{H,0}=A_{H,1}=0$. It follows that $B_{H,0}=B_{H,1}=0$ and $A_{H,2}=(x)$ is a prime ideal.

Induction hypothesis: $s \geq 3$ and for each $s' < s$ the proposition is true. We shall prove the proposition for s.

Beginning of secondary induction: Assume $n = s$. Then by Lemma 5 $B_{H,n}(X)$ is prime if and only if $B_{H',0}(X')$ is prime. But X' is an $(s-1)\times(s-1)$ matrix and by induction hypothesis $B_{H',0}(X')$ is prime.

Hypothesis of secondary induction: $n < s$ and for each $n' > n$ where $n' = s_{h'}$ or $n' = s_{h'} + 1 = 2h' + 2$ the proposition is true.

Now we shall prove the proposition for the given $s > 2$ and $n < s$. Assume $x_{i_0 j_0} \notin A$ and one of the following conditions is satisfied:

1.1) $i_0 = 1,\ j_0 \in \{n+1,\ n+2, \ldots, s_{h+1}\}$
(1) 1.2) $s_0 > 1,\ i_0 = 2,\ j_0 = s_0 + 1$
1.3) $s_0 = 1,\ n > 1,\ i_0 = 2,\ j_0 = 3$

Note these i_0, j_0 $(i_0 < j_0)$ have the following property: for each i such that $1 \leq i \leq s$, $i \neq i_0$ either $E_a(i, i_0; \ldots, \ldots)$ is $A_{H,n}$-admissible or $x_{ij_0} \in A$; for each j such that $1 \leq j \leq s$, $j \neq j_0$ either $E_a(j, j_0; \ldots, \ldots)$ is $A_{H,n}$-admissible or $x_{i_0 j} \in A$ (for x_{23} the property holds because $A_{H,n} = A^3_{H'n}$ where $H' = (s_1, s_2, \ldots, s_m)$). The entry x will be called strategic for $A_{H,n}$ if it satisfies one of conditions (1).

Let $\mathbf{c}$ be generated by all the variables which are elements of the generating system of A and lie on the same row or on the same column as x. More precisely

$$\boldsymbol{c} = \begin{cases} (x_{12}, \ldots, x_{1n}) & \text{for } i_0 = 1, \\ (x_{12}, x_{23}, x_{24}, \ldots, x_{2s_0}, x_{1s_0+1}) & \text{for } x \text{ satisfying 1.2) and } n > s_0, \\ (x_{12}, x_{23}, x_{24}, \ldots, x_{2s_0}) & \text{for } x \text{ satisfying 1.2) and } n = s_0, \\ (x_{12}) & \text{for } x \text{ satisfying 1.3) and } n = 2, \\ (x_{12}, x_{13}) & \text{for } x \text{ satisfying 1.3) and } n > 2. \end{cases}$$

Obviously $\boldsymbol{c}$ is a prime ideal. Let $\bar{X}$ be the image of X in $K[X]/\boldsymbol{c}$ under the canonical homomorphism. Then $K[X]/\boldsymbol{c} \cong K[\bar{X}]$, $A_{H,n}(X)/\boldsymbol{c} = A_{H,n}(\bar{X})$ and the entries $\bar{x}_{ij} = x_{ij} + \boldsymbol{c}$ which don't equal 0 are independent variables over K. Denote $\tilde{K} = K\,[\bar{x}_{ij} : \{i, j\} \cap \{i_0\ j_0\} \neq \emptyset\}_{\bar{x}}$, i. e. the described polynomial ring localized at the system $\{\bar{x}^n\}_0^\infty$. Marking by tilde images under localization, it is true that $K[\bar{X}]_{\bar{x}} = \hat{K}[\tilde{x}_{ij} : \{i, j\} \cap \{i_0, j_0\} = \emptyset]$ and $(A_{H,n}(\bar{X}))^\sim = A_{H,n}(\tilde{X})$. Since $\tilde{x}$ is an invertible strategic entry all entries on its row and column, which are not equal to 0, can be anihilated by $A_{H,n}$-admissible elementary operations. In the matrix so obtaind the other entries are of the type $\tilde{x}_{ij} - (\tilde{x}_{ij_0}\tilde{x}_{i_0 j} - \tilde{x}_{j_0 j}x_{ii_0})/\tilde{x}$ and the different from 0 among these entries form a set of independent variables over $\tilde{K}$ and generate $K[\bar{X}]_{\bar{x}}$ over $\tilde{K}$. Denote this matrix by $\check{Y}$. Hence the equality $A_{H,n}(\tilde{X}) = A_{H,n}(\check{Y})$ holds in $K[\bar{X}]_{\bar{x}}$.

Let Y be obtained by deleting the i_0'th and j_0'th rows and the corresponding columns of $\check{Y}$. Hence the entries of Y which are not equal to 0 (i. e. the nondiagonal entries) are independant variables over $\tilde{K}$ and they generate $K[\bar{X}]_{\bar{x}}$. It is clear that $A_{H,n}(\tilde{X}) = A_{H',n'}(Y)$ where $H' = (s'_0, s'_1, \ldots, s'_{m-1})$ and

for $i_0 = 1$, $s'_t = \begin{cases} s_t - 1, & t < h \\ s_{t+1} - 2, & t \geq h \end{cases}$ and $n' = 0$

for $i_0=2$, $s'_t=s_{t+1}-2$ and $n'=\begin{cases} 0 \text{ for } h=0,1 \\ n-2 \text{ for } h>1. \end{cases}$

Note that we can use the induction hypothesis for $A_{H',n'}(Y)$ because Y is a generic alternating $(s-2)\times(s-2)$ matrix and moreover either $n'=0$ or H' is a standard description and n' satisfies the hypothesis of the proposition. Since Y is a generic alternating $(s-2)\times(s-2)$ matrix over the domain $\tilde{K}$, by the induction hypothesis on s, $B_{H',n'}(Y)=B_{H,n}(\bar{X})_{\bar{x}}$ is prime. In view of the fact that by Lemma 8 $x\notin B_{H,n}(X)$, $\bar{x}\notin B_{H,n}(\bar{X})$ and hence the hypothesis of Lemma 7, 2) hold for $\boldsymbol{a}=B_{H,n}(\bar{X})$. By this lemma there exists a unique prime ideal $P_n/C\subset K[\bar{X}]$ from the Noether-Lasker decomposition of $B_{H,n}(\bar{X})=\cap P_i/C$, for $x\notin P_u$ set $u=u(x)$.

Now we shall show that for each index i there exists a strategic entry x for which $i=u(x)$. Denote

$$\boldsymbol{q}=\begin{cases} B_{H,2} \text{ for } n=s_0=1 \\ B_{H,s_{h+1}} \text{ for the other cases.} \end{cases}$$

By the secondary induction hypothesis $\boldsymbol{q}$ is prime. Denote

$$x_0=\begin{cases} x_{13} \text{ for } n=s_0=1 \\ x_{23} \text{ for } s_0=1,\ n>1, \\ x_{2,s_0+1} \text{ for } s_0>1. \end{cases}$$

Therefore x_0 is a strategic entry and by the corollary to Lemma 8 $x_0\notin\boldsymbol{q}$. Consequently $B=\bigcap_i P_i\subset\mathbf{q}\underset{\neq}{\subset}\mathbf{b}=\operatorname{rad}(A+(\{x:x \text{ is strategic entry of } A\}))$. By Lemma 7,1) $\boldsymbol{b}\not\subset P_i$. Therefore a P_i containing all the strategic entries doesn't exist.

The proof of the theorem will be finished by the following:

L e m m a 11. *$u(x)$ does not depend on x.*

The proof of the lemma is given in the last paragraph of this part.

4. Proof that the ideals $A_{H,n}(X)$ are radical. In this paragraph K will be a Noetherian domain and X will be a generic alternating $s\times s$ matrix over K.

P r o p o s i t i o n 12. *Let $H=(s_0,\ldots,s_m)$ be a standard description, $s_h<n<s_{h+1}$, $H'=(s_0,\ldots,s_{h-1},n,s_{h+1},\ldots,s_m)$, $n'=s_{h+1}$ Then $B_{H,n}=B_{H',n}\cap B_{H,n'}$.*

P r o o f. By the inclusions $A_{H,n}\subset A_{H',n}$ and $A_{H,n}\subset A_{H,n'}$ it is clear that $B_{H,n}\subset B_{H',n}$ and $B_{H,n}\subset B_{H,n'}$ hence $B_{H,n}\subset B_{H',n}\cap B_{H,n'}$. The opposite inclusion will follow from the relation $A_{H',n}\cap A_{H,n'}\subset B_{H,n}$, for which it is sufficient to show that $A_{H',n}\cdot A_{H,n'}\subset A_{H,n}$. But

$$A_{H',n}=A_{H,n}+Pf_{2h+2}(X(n,n))+\sum_{0\leq i<h} I_{i+h+1}(X(s_i,n))+\sum_{h<i\leq m} I_{i+h+1}(X(n,s_i))$$

and $A_{H,n'}=A_{H,n}+(x_{n+1},\ldots,x_{n'})$ where $x_k=x_{1k}$. Hence it suffices to show that

1) for $n+1\leq k\leq n'=s_{h+1}$, if P is any $(2h+2)\times(2h+2)$ pfaffian of $X(n,n)$, $x_kP\in A_{H,n}$;

2) for $n+1\leq k\leq n'$, if M is any $(i+h+1)\times(i+h+1)$ minor of $X(s_i,n)$, $(0\leq i<h)$, $x_kM\in A_{H,n}$:

3) for $n+1\leq k\leq n^i$, if M is any $(i+h+1)\times(i+h+1)$ minor of $X(n,s_i)$, $(h<i\leq m)$, $x_kM\in A_{H,n}$.

These three cases can be proved in similar ways. For example let us prove the third case. If the first column of M coincides with the first column of X this is obvious, for then $M\in(x_1,\ldots,x_n)\subset A_{H,n}$. Otherwise, let M' be the $(i+h+2)\times(i+h-2)$ minor constructed by adding to M the k'th row and the first column. $M'\in A_{H,n}$ (here for $i=h+1$ we use Lemma 3). The expansion by minors of M' with respect to the first column is a sum containing $\pm x_k M$, and all the other terms are in $(x_1,\ldots,x_n)$. Hence, $x_k M\in A_{H,n}$, as required.

Proposition 13. $A_{H,n}(X)$ is radical $(s_0\leqq n\leqq s)$.

Proof. Consider the rings $\{K[Y]:Y$ is an alternating submatrix of $X\}$ and order the Y's by inclusion. For a fixed Y, order the ideals $A_{H,n}(Y)$ (where H is a standard description) by inclusion of the specified generating sets. By [6, Proposition 24], it suffices to show that for each Y, H, n one of the following conditions holds:

1) $A_{H,n}(Y)$ is radical.

2) There are Y', H', and n' such that $Y'<Y$, H' is a standard description of Y' and if $A_{H',n'}(Y')$ is radical, then $A_{H,n}(Y)$ is radical.

3) $B_{H,n}(Y)$ is a prime ideal and there exists a form of positive degree $x\in K[Y]$ such that: a) $x\notin B_{H,n}(Y)$ and b) $A_{H,n}(Y)+(x)=A_{\tilde{H},\tilde{n}}(Y)$ for some $\tilde{H}$ and $\tilde{n}$.

4) There exist $x\in K[Y]$, standard descriptions $\tilde{H}$ and $\hat{H}$, integers $\tilde{n}$ and $\hat{n}$ such that: a) $A_{H,n}(Y)+(x)=A_{\tilde{H},\tilde{n}}(Y)$ and b) $A_{\hat{H},n}(Y)\subset A_{H,n}(Y):(x)$ and $B_{H,n}(Y):(x)\subset B_{\hat{H},\hat{n}}(Y)$.

By Lemma 5 we can use 1), 2) to reduce to the cases where $n<s$. Let $x=x_{1,n+1}$. If $n=s_h$, 3) holds by Proposition 10 and the corollary to Lemma 8. If $s_h<n<s_{h+1}$ we have $B_{H,n}=B_{H',n}\cap B_{H,n'}$ by Proposition 12 and by the proof of Proposition 12 (x). $A_{H',n}\subset A_{H,n}$. Since $x\in B_{H,n'}\setminus B_{H',n}$, $B_{H,n}:(x)\subset B_{H',n}$ (because $B_{H',n}$ is prime by Proposition 10).

Proposition 14. *Let* $H=(s_0,\ldots,s_m)$ *be a standard description. Then*:

1) *If* $n=s_h$ *or* $n=s_h+1=2h+2$, $A_{H,n}$ *is prime.*

2) *If* $s_h<n<s_{h+1}$, $A_{H,n}=A_{H',n}\cap A_{H,n'}$ *(*H' *and* n' *are defined in Proposition* 12*) is the decomposition of the radical ideal as an intersection of primes.*

The proof follows from Propositions 10, 12 and 13.

5. Proofs of some technical results. Proof of Lemma 11. (All the notations from the proof of Proposition 10 will be used below.)

1. First we shall prove for strategic x and y that, if they both lie on one row or column, or if $h>0$, for no minor or pfaffian $\bar{M}$ in the generating set of $A_{H,n}(X)$ can the equality $\bar{M}=\pm x^p y^q+\ldots$ hold $(p,q=0,1,\ldots)$ (The arguments here are very similar to those in the proof of Lemma 8). Suppose the contrary. We encounter x as a variable only on 2 places in entries of X and these places are symmetric with respect to the diagonal. Therefore the power of x in the expansion of the pfaffians is not greater than 1, and in that of the minors – not greater than 2. The same holds for y. Hence, the possible cases are $p, q=0, 1, 2$. By Lemma 8, the case $p=0$ does not occur, and by analogy $q=0$ is not possible either. Having in mind the symmetry between x and y (and hence between p and q) in this proof, the cases to consider are $(p,q)=(1,1),(1,2),(2,2)$.

Further it will be useful to know the entries of X in which occur the variables x and y and to know exactly the place in the expansion of $\bar{M}$ where $\pm x^b y^q$ comes out. For this purpose we shall denote by x, y the varibles in

$K[X]$ but by x_{ij} — the entries of X (i. e. x_{ij} may be with $i \geq j$). Thus, every variable appears with iudias regularly ordered ($x_{ij}: i<j$) or irregularly ($-x_{ij}: i>j$). Hence, the given monomial can appear in the expansion in the pfaffians and minors in 0,1 or more ways according to the appearence of different variables, taking part in it, with regularly or irregularly ordered indices. Let $x = x_{i_1 j_1}$ and $y = x_{i_2 j_2}$.

1. Suppose x and y lie on the same row or column, i. e. that either $i_1 = i_2$ or $j_1 = j_2$.

a) Let $p=1$ or $p=2$; $q=2$; then xy^2 equals either $-x_{i_1 j_1} x_{i_2 j_2} x_{j_2 i_2}$ or $x_{j_1 i_1} x_{i_2 j_2} x_{j_2 i_2}$, and $x^2 y^2 = x_{i_1 j_1} x_{j_1 i_1} x_{i_2 j_2} x_{j_2 i_2}$. These are all possible ways $x^p y^q$ can appear in the expansion of $\bar{M}$ and always there are two entries lying on the same row or column. But this is impossible in the expansion of a determinant or a pfaffian.

b) Let $p=q=1$; then $xy = x_{i_1 j_1} x_{i_2 j_2}$ or $xy = x_{j_1 i_1} x_{j_2 i_2}$ are impossible for the same reason as in case a).

Consequently one has either: $xy = -x_{i_1 j_1} x_{j_2 i_2}$ or $xy = -x_{j_1 i_1} x_{i_2 j_2}$. In this case $\bar{M}$ has degree 2. Hence, $\bar{M}$ is a 2×2 minor or 4×4 pfaffian. Two of the indices i_1, i_2, j_1, j_2 are identical. Therefor $\bar{M}$ cannot be a pfaffian. Then $\bar{M}$ is a 2×2 minor of $X(s_0, s_1)$. Hence, either $j_2 \leqq s_0$ or $j_1 \leqq s_0$, which contradicts the definition of strategic entry.

2. Suppose $h>0$ and x, y do not lie on the same row or column.

a) If $p=1, q=2, \bar{M}$ is not a pfaffian. Hence, $\bar{M}$ is a 3×3 minor of $X(s_0, s_2)$, But the expression is $xy^2 = -x \cdot x_{i_2 j_2} x_{j_2 i_2}$. Therefore $j_2 \leqq s_0$, which contradicts the definition of strategic entry.

b) If $p=q=2$, $\bar{M}$ is not a pfaffian. Therefore it is a 4×4 minor of $X(s_0, s_3)$ or $X(s_1, s_2)$. If $\bar{M}$ is a minor of $X(s_0, s_3)$, the equality $-x^2 y^2 = x^2 x_{i_2 j_2} x_{j_2 i_2}$ implies that $j_2 \leqq s_0$: contradiction.

If $\bar{M}$ is a minor of $X(s_1, s_2)$, because of $x^2 y^2 = x_{i_1 j_1} x_{j_1 i_1} x_{i_2 j_2} x_{j_2 i_2}$, it follows that $i_1, j_1, i_2, j_2 \leqq s_1$. Since $h>0$, this is possible only under conditions 1.2) and 1.3). But they are incompatible. Therefore x and y both satisfy the same condition, i. e. $x=y$, which contradicts Lemma 8.

c) Let $p=q=1$; then $\bar{M}$ is a 4×4 pfaffian or 2×2 minor, If $\bar{M}$ is a pfaffian, it is in $X(s_1, s_1)$. Hence $x_{i_1 j_1}, x_{j_1 i_1}, x_{i_2 j_2}, x_{j_2 i_2}$ are entries of $X(s_1, s_1)$. Therefore $i_1, i_2, j_1, j_2 \leqq s_1$ — contradiction (as in the case b)). If $\bar{M}$ is a minor, it is in $X(s_0, s_1)$. Therefore $i_1, i_2, j_1, j_2 \leqq s_1$ — contradiction.

II. Let both x and y be strategic entries. Suppose $u(x) \neq u(y)$. Therefore $y \in P_{u(x)}$ and $x \in P_i$ for each $i \neq u(x)$. Consequently $xy \in P_i$ for each i, and $xy \in B$. Hence, $x^k y^k = \sum_i f_i M_i + g$ where $f_i \in K[X]$, with M_i minors or pfaffians in the generating system of A and $g \in (x_{12}, \ldots, x_{1n})$. It follows that one of the M_i should be of the form $\pm x^p y^p + \ldots$. By part I above this cannot happen if either x and y both lie on the same row or column, or if $h>0$. Therefore in case $n = s_0 = 1$ the lemma is proved.

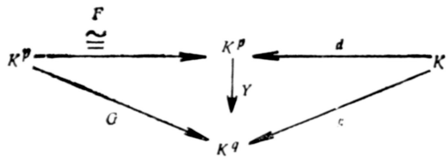
Let $u_0 = u(x_{1 j_0})$ where $j_0 \in \{n+1, \ldots, s_{h+1}\}$. In the case $h=0$ it follows, by all proved, that for $s_0 = 1$; $n = 1, 2$ we have $u(x_{23}) = u(x_{13}) = u_0$, and for $s_0 > 1$ and $n = s_0$ we have $u(x_{2, s_0+1}) = u(x_{1, s_0+1}) = u_0$.

Therefore $u(x) = u_0$ for each strategic x, as required.

Lemma 15. *Let F be an invertible $p\times p$ matrix, Y be a $p\times q$ matrix, d be an $1\times p$ matrix. Let $G = FY$ and $c = dY$. Then c is a linear combina-*

tion of the vector-rows of G. If d is a linear combination of the vector-rows of F, c is the same combination for G.

Proof. If $A=(a_{ij})$ is an $m\times n$ matrix and in K^m and in K^n are fixed the bases $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$, let us denote by $K^m \xrightarrow{A} K^h$ the map $e_j \longrightarrow \sum_{j=1}^{n} a_{ij} f_j$. Then we see all statements from the following commutative diagram:



Lemma 16. *Let* $a_i=(a_{i1}, a_{i2}, \ldots, a_{ip})$, $(i=1, 2, \ldots, k; k\leq p)$, $b=(b_1, b_2, \ldots, b_p)$, $c=(c_1, c_2, \ldots, c_p)$ *be vectors in* K^p *and let* $r_1, \ldots, r_k$, $s_1, \ldots, s_k$, t, $t_1, \ldots, t_k$ *be elements of* K, $b=\sum_i r_i a_i$, $c=\sum_i s_i a_i$, *and also* $c_j=\sum_{i=1}^{k} t_i a_{ij}+tb_j$, *for k different values of j between* $1, 2, \ldots, p$.

If the $k\times k$ *matrix* (a_{ij}) *is invertible, then* $c=\sum_i t_i a_i+tb$.

The proof is straightforward linear algebra.

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