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ALMOST HERMITIAN MANIFOLDS WITH VANISHING GENERALIZED BOCHNER CURVATURE TENSOR

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We deal with fhe generalized Bochner curvature tensor and the Bochner curvature tensor introduced respectively in [1] and [11]. In section 2 we prove, that if an almost Hermitian manifold is a product of two almost Hermitian manifolds M_1 and M_2, then M_1 (resp. M_2) is of pointwise constant holomorphic sectional curvature μ (resp. $-\mu$). In section 3 we obtain a classification theorem for non Kähler nearly Kähler manifolds with vanishing Bochner curvature tensor and a classification theorem for nearly Kähler manifolds with vanishing Bochner curvature tensor and constant scalar curvature.

1. Preliminaries. Let M be a $2m$-dimensional almost Hermitian manifold with Riemannian metric g and almost complex structure J and let ∇ be the covariant differentiation on M. The curvature tensor R is defined by

$$R(X, Y, Z, U)=g(\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]} Z, U)$$

for $X, Y, Z, U \in \mathfrak{X}(M)$. From the curvature tensor R on may construct a tensor R^* [1] by

$$R^*(X, Y, Z, U)=\frac{3}{16}\{R(X, Y, Z, U)+R(X, Y, JZ, JU)$$

$$+R(JX, JY, Z, U)+R(JX, JY, JZ, JU)\}$$

$$+\frac{1}{16}\{R(JX, JZ, Y, U)-R(JY, JZ, X, U)+R(X, Z, JY, JU)-R(Y, Z, JX, JU)$$

$$+R(Y, JZ, JX, U)-R(X, JZ, JY, U)+R(JY, Z, X, JU)-R(JX, Z, Y, JU)\}.$$

The tensor R^* has the following properties:

1) $R^*(X, Y, Z, U)=-R^*(Y, X, Z, U)$,

2) $R^*(X, Y, Z, U)=-R^*(X, Y, U, Z)$,

3) $\underset{X,Y,Z}{\sigma} R^*(X, Y, Z, U)=0$,

where σ denotes the cyclic sum and R^* is the only tensor with these properties for which

$$R^*(X, Y, Z, U)=R^*(JX, JY, Z, U),\ R^*(X, JX, JX, X)=R(X, JX, JX, X).$$

Let $\{E_i;\ i=1, \ldots, 2m\}$ be an orthonormal local frame field. The Ricci tensor S and the scalar curvature $\tau(R)$ of M are defined by

$$S(X, Y)=\sum_{i=1}^{2m} R(X, E_i, E_i, Y), \ \tau(R)=\sum_{i=1}^{2m} S(E_i, E_i).$$

Analogously one denotes

$$S^*(X, Y)=\sum_{i=1}^{2m} R^*(X, E_i, E_i, Y), \quad \tau^*(R)=\sum_{i=1}^{2m} S^*(E_i, E_i)$$

and it is easy to see that $S^*(X, Y)=S^*(JX, JY)=S^*(Y, X)$.

The generalized Bochner curvature tensor B^* for M is defined by [1]

$$B^*=R^*-\frac{1}{2(m+2)}(\varphi+\psi)(S^*)+\frac{\tau^*(R)}{4(m+1)(m+2)}(\pi_1+\pi_2),$$

where

$$\varphi(Q)(X, Y, Z, U)=g(X, U)\,Q(Y, Z)-g(X, Z)\,Q(Y, U)$$
$$+g(Y, Z)Q(X, U)-g(Y, U)\,Q(X, Z),$$
$$\psi(Q)(X, Y, Z, U)=g(X, JU)\,Q(Y, JZ)-g(X, JZ)\,Q(Y, JU)$$
$$-2g(X,JY)\,Q(Z,JU)+g(Y,JZ)Q(X, JU)-g(Y,JU)Q(X, JZ)-2g(Z,JU)Q(X,JY),$$
$$\pi_1(X, Y, Z, U)=g(X, U)\,g(Y, Z)-g(X, Z)g(Y, U),$$
$$\pi_2(X, Y, Z, U)=g(X, JU)g(Y, JZ)-g(X, JZ)g(Y, JU)-2g(X, JY)g(Z, JU),$$

or an arbitrary tensor Q of type (0, 2).

An almost Hermitian manifold M which satisfies $(\nabla_X J)X=0$ for all $X\in\mathfrak{X}(M)$ is called a nearly Kähler manifold. It is well known that for such a manifold $R(X, Y, Z, U)=R(JX, JY, JZ, JU)$ holds good for all $X, Y, Z, U\in\mathfrak{X}(M)$ [3]. In general a manifold which satisfies this identity is said to be an RK—manifold. For an RK—manifold $S(X, Y)=S(JX, JY)$ holds. The Bochner curvature tensor B for an RK-manifold of dimension $2m\geqq 6$ is defined [12] by

$$B=R-\frac{1}{8(m+2)}(\varphi+\psi)(S+3S')-\frac{1}{8(m-2)}(3\varphi-\psi)(S-S')$$
$$+\frac{\tau(R)+3\tau'(R)}{16(m+1)(m+2)}(\pi_1+\pi_2)+\frac{\tau(R)-\tau'(R)}{16(m-1)(m-2)}(3\pi_1-\pi_2),$$

where

$$S'(X, Y)=\sum_{i=1}^{2m} R(X, E_i, JE_i, JY), \quad \tau'(R)=\sum_{i=1}^{2m} S'(E_i, E_i).$$

It is easy to check that for an RK-manifold $4S^*=S+3S'$.

Now let M be a nearly Kähler manifold and $X, Y, Z, U, V\in X(M)$. We shall use the following formulas (see [3; 6; 12]):

(1.1) $$R(X, Y, Z, U)-R(X, Y, JZ, JU)=-g((\nabla_X J)Y, (\nabla_Z J)U),$$

(1.2) $$2g((\nabla_X(\nabla_Y J))Z, U)=\underset{Y,U,Z}{\sigma} R(X, JY, U, Z),$$

(1.3) $$2(\nabla_X(S-S'))(Y, Z)=(S-S')((\nabla_X J)Y, JZ)+(S-S')(JY, (\nabla_X J)Z),$$

(1.4) $$X(\tau(R)-\tau'(R))=0,$$

$$\sum_{i,j=1}^{2m}(S-S')(E_i, E_j)(S-5S')(E_i, E_j)=0. \tag{1.5}$$

From the second Bianchi indentity

$$\underset{X,Y,Z}{\sigma}(\nabla_X R)(Y, Z, U, V)=0$$

it follows

$$\sum_{i=1}^{2m}(\nabla_{E_i} R)(X, Y, Z, E_i)=(\nabla_X S)(Y, Z)-(\nabla_Y S)(X, Z), \tag{1.6}$$

$$\sum_{i=1}^{2m}(\nabla_{E_i} S)(X, E_i)=\frac{1}{2}X(\tau(R)). \tag{1.7}$$

2. Product of almost Hermitian manifolds and the generalized Bochner curvature tensor.

Theorem 2.1. *Let an almost Hermitian manifold M be a product $M_1\times M_2$, where M_1 and M_2 are almost Hermitian manifolds. Then M has vanishing generalized Bochner curvature tensor if and only if M_1 (resp. M_2) is of pointwise constant holomorphic sectional curvature μ (resp. $-\mu$).*

Proof. If $B^*=0$ we find

$$\begin{aligned} R^*(X, Y, Z, U) &= \frac{1}{2(m+2)}\{g(X, U)S^*(Y,Z)-g(X, Z)S^*(Y,U) \\ &+g(Y, Z)S^*(X, U)-g(Y, U)S^*(X, Z)+g(X, JU)S^*(Y, JZ) \\ &-g(X, JZ)S^*(Y, JU)+g(Y, JZ)S^*(X, JU)-g(Y, JU)S^*(X, JZ) \\ &-2g(X,JY)S^*(Z, JU)-2g(Z, JU)S^*(X, JY)\} \\ &-\frac{\tau^*(R)}{4(m+1)(m+2)}\{g(X, U)g(Y, Z)-g(X, Z)g(Y,U)+g(X, JU)g(Y,JZ) \\ &-g(X, JZ)g(Y, JU)-2g(X, JY)g(Z, JU)\}. \end{aligned} \tag{2.1}$$

We denote by R_1 the curvature tensor of M_1. Analogously we have R_1^*, S_1^* $\tau^*(R_1)$. Note that $R=R_1$, $R^*=R_1^*$ and $S^*=S_1^*$ on M_1. Let $X\in\mathfrak{X}(M_1)$ and $\{E_i;\ i=1, \ldots, 2k\}$ be an orthonormal local frame field on M_1. In (2.1) we put $U=X$, $Y=Z=E_i$ and adding for $i=1, \ldots, 2k$ we obtain

$$S_1^*(X, X)=\left\{\frac{\tau^*(R)}{2(m-k)}-\frac{(k+1)\tau^*(R)}{2(m+1)(m-k)}\right\}g(X, X). \tag{2.2}$$

From (2.1) it follows for a unit field $X\in\mathfrak{X}(M_1)$

$$R_1(X, JX, JX, X)=\frac{4}{m+2}S_1^*(X, X)-\frac{\tau^*(R)}{(m+1)(m+2)}. \tag{2.3}$$

Because of (2.2) and (2.3) M_1 is of pointwise constant holomorphic sectional curvature μ. Hence

$$S_1^*(X, X)=\frac{k+1}{2}\mu\, g(X, X) \tag{2.4}$$

for $X\in\mathfrak{X}(M_1)$ and

$$\tau^*(R_1)=k(k+1)\mu. \tag{2.5}$$

Analogously M_2 is of pointwise constant holomorphic sectional curvature μ' and

$$\tau^*(R_2)=(m-k)(m-k+1)\mu'. \tag{2.6}$$

Using (2.3)—(2.6) and $\tau^*(R)=\tau^*(R_1)+\tau^*(R_2)$ we find $\mu'=-\mu$.

The converse is a simple calculation by using the fact, that M is of pointwise constant holomorphic sectional curvature μ if and only if $R^*=\frac{\mu}{4}(\pi_1+\pi_2)$, see [1].

Corollary 2.2. *Let an almost Hermitian manifold M be a product of more than two almost Hermitian manifolds. Then M has vanishing generalized Bochner curvature tensor if and only if it is of zero holomorphic sectional curvature.*

For the case of a Kähler manifold M in Theorem 2.1 and Corollary 2.2 see [10].

3. Nearly Kähler manifolds with vanishing Bochner curvature tensor.

Lemma. *Let M be a $2m$-dimensional nearly Kähler manifold, $m>2$. If the Bochner curvature tensor of M vanishes, then the tensor $S-S'$ is parallel.*

Proof. Let $\{E_i;\ i=1,\ldots,2m\}$ be a local orthonormal frame field. From (1.1) we find

$$(S-S')(X,Y)=\sum_{i=1}^{2m} g((\nabla_X J)E_i,\ (\nabla_Y J)E_i),$$

which implies

$$(\nabla_X(S-S'))(Y,Y)=2\sum_{i=1}^{2m} g((\nabla_X(\nabla_Y J))E_i,\ (\nabla_Y J)E_i).$$

Hence, using (1.2) we obtain

$$(\nabla_X(S-S'))(Y,Y)=\sum_{i=1}^{2m}\{R(X,JY,(\nabla_Y J)E_i,E_i)+R(X,J(\nabla_Y J)E_i,E_i,Y)+R(X,JE_i,Y,(\nabla_Y J)E_i)\}. \tag{3.1}$$

From $B=0$, (3.1) and (1.3) we derive $\nabla(S-S')=0$.

Theorem 3.1. *Let M be a $2m$-dimensional non Kähler nearly Kähler manifold, $m>2$. If M has vanishing Bochner curvature tensor, it is locally isometric to one of:*

a) *the sixth sphere* S^6;

b) $CD^1(-c)\times S^6(c)$ *where* $CD^1(-c)$ (*resp.* $S^6(c)$) *is the one-dimensional complex hyperbolic space of constant sectional curvature $-c$ (resp. the sixth sphere of constant sectional curvature c).*

Proof. According to the lemma the tensor $S-S'$ is parallel. Let M be locally a product $M_1(\lambda_1)\times\ldots\times M_k(\lambda_k)$, where $S-S'=\lambda_i g$ on $M_i(\lambda_i)$ and $\lambda_i\neq\lambda_j$ for $i\neq j$. As it's easy to see $M_i(\lambda_i)$ is a nearly Kähler manifold for $i=1,\ldots,k$. From $B=0$ it follows $B^*=0$. So Theorem 2.1, Corollary 2.2 and [9] imply that if $k>1$ M is either of zero holomorphic sectional curvature, or locally a product of two nearly Kähler manifolds of constant holomorphic sectional curvature $-\mu$ and μ, respectively, $\mu>0$. Since M is non Kähler, the former is impossible and the latter occurs only when M is locally isometric to $CD^{m-3}(-c)$

$\times S^6(c)$ [5]. On the other hand $CD^{m-3}(-c)\times S^6(c)$ has nonvanishing Bochner curvature tensor, if $m>4$. Indeed it doesn't satisfy the condition $R(x, y, z, u)=0$ for all x, y, z, u spanning a 4-dimensional antiholomorphic plane.

Let $k=1$. Then

$$S-S'=\frac{1}{2m}\{\tau(R)-\tau'(R)\}g. \tag{3.2}$$

Now (3.2) and (1.5) imply

$$(\tau(R)-\tau'(R))(\tau(R)-5\tau'(R))=0.$$

If $\tau(R)(p)-\tau'(R)(p)=0$ for some $p\in M$ then $\tau(R)-\tau'(R)=0$ because of (1.4). Then from (3.2) and $B=0$ we obtain

$$R=\frac{1}{2(m+2)}(\varphi+\psi)(S)-\frac{\tau(R)}{4(m+1)(m+2)}(\pi_1+\pi_2).$$

Hence $R(X, Y, Z, U)=R(X, Y, JZ, JU)$ holds for all $X, Y, Z, U\in\mathfrak{X}(M)$ and so M is a Kähler manifold [4], which is a contradiction. Consequently

$$\tau(R)-5\tau'(R)=0. \tag{3.3}$$

Now (1.4) implies that $\tau(R)$ and $\tau'(R)$ are global constants. From $B=0$, (3.2) and (3.3) we find

$$R=\frac{1}{2(m+2)}(\varphi+\psi)(S)-\frac{(4m+3)\,\tau(R)}{10m(m+1)(m+2)}(\pi_1+\pi_2)+\frac{\tau(R)}{20m(m-1)}(3\pi_1-\pi_2). \tag{3.4}$$

Using (1.6), (1.7), (3.4) and $X(\tau(R))=0$ we obtain

$$(2m+3)\{(\nabla_X S)(Y, Y)-(\nabla_Y S)(X, Y)\}=3(\nabla_{JY}S)(X, TY)-3S((\nabla_X J)Y, JY) \tag{3.5}$$

In particular $(\nabla_X S)(X, X)=0$ and hence

$$(\nabla_X S)(Y, Y)+2(\nabla_Y S)(X, Y)=0. \tag{3.6}$$

From (3.5) and (3.6) we derive $(\nabla_X S)(Y, Y)=0$ which implies $\nabla S=0$. Since M is not locally isometric to $CD^1(-c)\times S^6(c)$, it follows that $S=\tau(R)/g(2m)$. Now (3.4) takes the form

$$R=\frac{5m+1}{20m(m^2-1)}\tau(R)\pi_1+\frac{m-3}{20m(m^2-1)}\tau(R)\pi_1.$$

Consequently M is of constant holomorphic sectional curvature. Since M is not Kähler, it is locally isometric to S^6 [5].

From [8] and Theorem 3.1 we obtain

Theorem 3.2. *Let M be a $2m$-dimensional nearly Kähler manifold, $m>2$, with vanishing Bochner curvature tensor and constant scalar curvature. Then M is locally isometric to one of the following:*

a) *the complex Euclidian space* CE^m;

b) *the complex hyperbolic space* CD^m;

c) *the complex projective space* CP^m;

d) *the sixth sphere* S^6;

e) *the product* $CD^1(-c)\times S^6(c)$;

f) *the product* $CD^{m_1}(-c)\times CP^{m_2}(c)$, $m_1+m_2=m$.

We note that Theorems 3.1 and 3.2 can be proved also by using Theorem 2.1, Corollary 2.2 and [6, Theorem 4.11].

Since for an almost Hermitian manifold the condition of constant antiholomorphic sectional curvature implies $B=0$, see [7], we obtain

Corollary 3.3 [2]. *Let M be a 2m-dimensional nearly Kähler manifold, $m>2$. If M is of pointwise constant antiholomorphic sectional curvature* ν, *it is locally isometric to one of the following:*

a) *the complex Euclidian space* CE^m;

b) *the complex hyperbolic space* $CD^m(4\nu)$;

c) *the complex projective space* $CP^m(4\nu)$;

d) *the sixth sphere* $S^6(\nu)$.

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