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A REMARK ON THE KRASNOSIELSKII'S TRANSLATION OPERATOR ALONG TRAJECTORIES OF ORDINARY DIFFERENTIAL EQUATIONS

G. DYLAWERSKI, L. GÓRNIEWICZ

The study of periodic solutions for differential equations are extensively developed by several authors. For topological arguments see [3, 4, 5, 7, 8, 9, 10]. One of the most remarkable topological methods used in the study of this problem belongs to Krasnosielskii (comp. [3, 4, 5]). He considered the translation operator along trajectories of differential equations and reduced the problem of existence of periodic solutions to calculation of the topological degree of this operator (for this method see also [7, 8]). Observe, that the above translation operator was defined as a single-valued map.

We are going to show that even in very simple case of ordinary differential equations is more natural to consider it as a multi-valued map. So we will define the multi-valued translation operator along trajectories of ordinary differential equations. Moreover, by using the topological degree theory for admissible multi-valued maps (comp. [2] or [6]) we are able to find periodic solutions. Note, that this paper contained only the case of differential equations of first order. Some other equations can be studied by using of this method but we will present it in next papers.

1. Admissible multi-valued maps. For details concerning admissible multi-valued maps see [2]. In this section we will formulate only some definitions and facts which we will use in next sections.

By homology we will understand the Čech homology functor with compact carriers and rational coefficients. Let X and Y be two metric spaces. A continuous map $p\colon Y\to X$ is called *a Vietoris map*, if the following conditions are satisfied:

(i) p is a proper map, i. e., for every compact $A\subset X$ the counter image $p^{-1}(A)$ is a compact set,

(ii) p is onto,

(iii) for every point $x\in X$ the set $p^{-1}(x)$ is acyclic.

In what follows the symbols φ, ψ, χ will be reserved for multi-valued maps; the single-valued maps will be denoted by $f, g, h, p, q, \ldots$.

A multi-valued map $\varphi\colon X\to Y$ is called *upper semi continuous* (u. s. c), if

(i) $\varphi(x)$ is a compact set, for each $x\in X$,

(ii) for each open $V\subset Y$ the set $\varphi^{-1}(V)=\{x\in X;\ \varphi(x)\subset V\}$ is open.

An u. s. c. map $\varphi\colon X\to Y$ is called acyclic provided for each $x\in X$ the set $\varphi(x)$ is acyclic.

An u. s. c. map $\varphi\colon X\to Y$ is called admissible (comp. [2]) if there exists a metric space Z and two (single-valued) maps $p\colon Z\to X$, $q\colon Z\to Y$ such that the following conditions are satisfied:

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(i) p is a Vietoris map,
(ii) $q(p^{-1}(x)) \subset \varphi(x)$, for each $x \in X$;
in this case the pair (p, q) we will call a selected pair for φ.

Observe, that any acyclic map is admissible. Moreover arbitrary composition of admissible maps is admissible, so the class of admissible maps is quite general.

Two admissible maps $\varphi, \psi : X \to Y$ are called *homotopic*, if there exists an admissible map $\chi : X \times [0, 1] \to Y$ such that $\chi(x, 0) \subset \varphi(x)$ and $\chi(x, 1) \subset \psi(x)$.

Some important properties of admissible maps are summarized in the following proposition:

Proposition 1.1 (1.1.1) *If $\varphi : X \to Y$ and $\psi : Y \to Z$ are admissible maps, then the composition $\psi \circ \varphi : X \to Z$ of φ and ψ is admissible, too.*

(1.1.2) *If $\varphi, \psi : X \to R^n$ are admissible maps, then the map $\chi : X \times [0, 1] \to R^n$ given by the formula: $\chi(x, t) = t . \varphi(x) + (1-t) . \varphi(x)$, is admissible, where R^n denote the n-adimensional euclidean space and*

$$t . \varphi(x) + (1-t) . \psi(x) = \{u \in R^n ;\ u = t . y + (1-t) . z,\ y \in \varphi(x) \text{ and } z \in \psi(x)\}.$$

(1.1.3) *If $\varphi : X \to Y$ and $\psi : X_1 \to Y_1$ are two admissible maps, then the product map $\varphi \times \psi : X \times X_1 \to Y \times Y_1$ is an admissible map.*

For the proof of Proposition 1.1 see [1] or [2].

Let K_r^n denote the closed ball in R^n with the center 0 and radius r, let S_r^{n-1} denote the boundary of K^n in R^n, $n \geqq 2$.

We will consider admissible maps of the form $\varphi : K_r^n \to R^n$ such that $\varphi(S_r^{n-1}) \subset R^n / \{0\}$. We will use the following notation:

$$A(K_r^n, R^n) = \{\varphi : K_r^n \to R^n ;\ \varphi \text{ is admissible and } \varphi(S^{n-1}) \subset R^n / \{0\}\}.$$

It is well known that the Brouwer degree can be extended to $A(K^n, R^n)$. We will formulate it in the following theorem:

Theorem 1.2. *There exists a multi-valued map* Deg: $A(K^n, R^n) \to Q$ *such that:*

(1.2.1) *If $f \in A(K_r^n, R^n)$ is a single valued map, then* $\mathrm{Deg}(f) = \{\deg(f)\}$, *where* $\deg(f)$ *denote the Brouwer degree of f,*

(1.2.2) *If* $\mathrm{Deg}(\varphi) \neq \{0\}$, *then there exists $x \in K_r^n$ such that $0 \in \varphi(x)$,*

(1.2.3) *If φ and ψ are homotopic, then* $\mathrm{Deg}(\varphi) \cap \mathrm{Deg}(\psi) \neq \emptyset$ *(here homotopy χ has additionally the following property $\chi(S_r^{n-1} \times [0, 1])$ is contained in $R^n \setminus \{0\}$), where Q denotes the field of rationals.*

For details concerning Theorem 1.2 see last chapter in [2]. We will use the following complementation to the property (1.2.3).

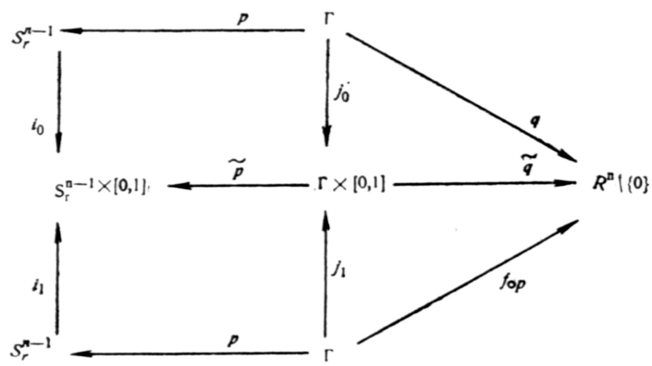
Proposition 1.3. *Let $f, \varphi \in A(K_r^n, R^n)$, f is single-valued. Assume that the map χ given by the formula $\chi(x, t) = t \cdot f(x) + (1-t) \cdot \varphi(x)$ for every $x \in K_r^n$ and $t \in [0, 1]$ satisfies the following condition $\chi(S_r^{n-1} \times [0, 1]) \subset R^n \setminus \{0\}$. Then* $\mathrm{Deg}(\varphi) = \{\deg(f)\}$; *i. e.* $\mathrm{Deg}(\varphi)$ *is a singleton.*

Proof. Let (p, q) be a selected pair for φ. We have the following diagram:

$$S_r^{n-1} \xleftarrow{p} \Gamma \xrightarrow{q} R^n \setminus \{0\}$$

in which p is a Vietoris map. For the proof it is sufficient to show that $q_{*n-1} \circ p_{*n-1}^{-1} = f_{*n-1}$.

But it immediately follows from the commutativity of the following diagram:



in which $\tilde{p}(y,t)=(p(y),t)$, $\tilde{q}(y,t)=t\cdot f(p(y))+(1-t)q(y)$ for every $y\in\Gamma$ and $t\in[0,1]$ and $i_0(x)=(x,0)$, $i_1(x)=(x,1)$, $j_0(y)=(y,0)$, $j_1(y)=(y,1)$, for each $x\in S_r^{n-1}$ and $y\in\Gamma$.

2. The translation operator. In this section we will generalize the Krasnosielskii's translation among trajectories of differential equations operator (comp. [3, 4, 5]). We are going to define it as a multi-valued operator for which the topological degree is defined i. e., as an admissible multi-valued map (comp. section 1).

Let $f\colon [a,b]\times R^n\to R^n$ be a continuous map. In what follows we will assume also that there are real numbers $\alpha\geq 0$ and $\beta\geq 0$ such that:

(2.1) $$\|f(t,x)\|\leq\alpha+\beta\cdot\|x\|, \text{ for each } t\in[a,b] \text{ and } x\in R^n.$$

Remark 2.2. It is well known that if f is continuous and satisfies (2.1), then for arbitrary $t_0\in[a,b]$ and $x_0\in R^n$ the Cauchy problem:

$$y'(t)=f(t,y(t)),$$
$$y(t_0)=x_0,$$

has at least one solution on $[a,b]$.

Now, we are interested to consider the following ordinary differential equation:

(2.3) $$y'(\cdot)=f(\cdot,y(\cdot)).$$

Let us fix $t_1,t_0\in[a,b]$ such that $t_1>t_0$. For given t_1 and t_0 we can associate with problem (2.3) the following two multi-valued maps: $\psi_{t_0}\colon R^n\to C([a,b],R^n)$ given by putting:

$$\psi_{t_0}(x)=\{y\in C([a,b],R^n);\ y(t_0)=x \text{ and } y'(t)=f(t,y(t)), \text{ for each } t\},$$

where $C([a,b],R^n)$ denote the space of all continuous maps from $[a,b]$ into R^n with the supremum norm;

$$\varphi_{t_1,t_0}\colon R^n\to R^n$$

given as follows:

$$\varphi_{t_1, t_0}(x) = \{z \in R^n;\ \text{there exists } y \in \psi_{t_0}(x) \text{ such that } z = y(t_1)\}.$$

Definition 2.4 (comp. [3]). *The operator* $\Phi_{t_1, t_0}\colon R^n \to R^n$ *defined by the following formula*: $\Phi_{t_1, t_0}(x) = x - \varphi_{t_1, t_0}(x)$ *is called the translation along trajectories of equation* (2.3) *operator, where* $x - \varphi_{t_1, t_0}(x) = \{u \in R^n;\ u = x - z,\ z \in \varphi_{t_1, t_0}(x)\}$.

Recall, the following well known fact ([6]).

Proposition 2.5. *If* $f\colon [a, b] \times R^n \to R^n$ *is a continuous map which satisfies condition* (2.1), *then* $\psi_{t_0}\colon R^n \to C([a, b], R^n)$ *is an acyclic map.*

Because the map φ_{t_1, t_0} is the composition of ψ_{t_0} with the single-valued continuous map e_{t_1} it is an admissible map (comp. (1.1)), where $e_{t_1}\colon C([a, b], R^n \to R^n$ is given by putting $e_{t_1}(y) = y(t_1)$. Therefore from (2.5) we obtain:

Proposition 2.6. *If* $f\colon [a, b] \times R^n \to R^n$ *is a continuous map and satisfies* (2.1), *then for arbitrary* t_1, t_0 *the translation operator* Φ_{t_1, t_0} *is an admissible map.*

Remark 2.7. Assume moreover that $f\colon [a, b] \times R^n \to R^n$ is a ω-periodic map with respect to first variable i. e., for each $x \in R^n$ and $t \in [a, b]$, if $(t + \omega) \in [a, b]$, then $f(t, x) = f(t + \omega, x)$. It is easy to see that $0 \in \Phi_{a+\omega, a}(x)$ for some $x \in R^n$ implies that there exists a ω-periodic solution of (2.3). In fact, then we can find $y_0 \in \psi_a(x)$ such that $y_0(a) = x$ and $y_0(a + \omega) = x$, so by simple calculation we obtain that the map $y_1\colon [a, b] \to R^n$ defined as follows: $y_1(t + n\omega) = y_0(t)$, where $t \in [a, a + \omega]$ and $(t + n \cdot \omega) \in [a, b]$, is a ω-periodic solution of (2.3).

Therefore as a simple consequence of (1.2) we obtain

Theorem 2.8. *If* $f\colon [a, b] \times R^n \to R^n$ *is a continuous,* ω*-periodic map which satisfies* (2.1), *and for some closed ball* K^n *the translation operator* $\Phi_{a+\omega, a}$ *satisfies the following conditions:*

(i) $$\Phi_{a+\omega, a}(S^{n-1}) \subset R^n \setminus \{0\},$$

(ii) $$\operatorname{Deg}(\Phi_{a+\omega, a} K^n) \neq \{0\},$$

then the problem (2.3) *has a* ω*-periodic solution.*

Remark 2.9. Assume only that $f\colon [a, b] \times R^n \to R^n$ is continuous and satisfies (2.1). It for some $t_1 > t_0$ there is $x \in R^n$ such that $0 \in \Phi_{t_1, t_0}(x)$, then we obtain a solution of (2.3) which has the same values for t_1 and t_0, so we obtain a solution of some boundary value problem connected to (2.3).

Observe that in this case theorem (2.8) remain true (without assumption ω-periodic).

3. Applications. Several applications of (2.8) and (2.9) are possible (comp. [7]). We would like, follow [5], to present one of them. In this section we will assume that $f\colon [a, b] \times R^n \to R^n$ is a continuous ω-periodic map which satisfies (2.1).

Definition 3.1. *A* C^1*-map* $V\colon R^n \to R$ *is called a direct potential for* f *iff there exists* $r_0 > 0$ *such that the following conditions are satisfied*:

(i) $$\text{if } \|x\| \geq r_0, \text{ then } \operatorname{grad} V(x) \neq 0,$$

and

(ii) $$(\operatorname{grad} V(x), f(t, x)) > 0, \text{ for each } t \in [a, b] \text{ and } \|x\| \geq r_0.$$

Remark. It is easy to see that the topological degree of the map grad V is well defined with respect to every ball K_r^n, $r \geq r_0$. Moreover deg (grad V) is the same for every $r \geq r_0$. It allows to define the index Ind V of direct potential V by putting:

Ind $V = \deg$ (grad V), where deg (grad V) is considered with respect to ball K_r^n, $r \geq r_0$.

To formulate the main result of this section we will need some lemmas.

Lemma 3.2. *Assume that the map* $f\colon [a, b] \times R^n \to R^n$ *is continuous, satisfies* (2.1) *and has a direct potential* V. *Then for every* $t_1 \in [a, b]$ *there is* $r_1 > r_0$ *such that for every* $r > r_1$ *and for every* $t \in [a, t_1]$ *the following condition is satisfied:* $\Phi_{t,a}(S_r^{n-1}) \subset R^n \setminus \{0\}$.

The proof of Lemma 3.2 is strictly analogous to respective lemma in (5, p. 70].

Lemma 3.3. *Assume that the map* $f\colon [a, b] \times R^n \to R^n$ *is continuous, satisfies* (2.1) *and the following condition:*

$$f(a, x) \neq 0, \text{ for each } x \in S_r^{n-1}, \quad r > 0.$$

Then there exists $\delta > 0$ *such that* $\chi_t(S_r^{n-1} \times [0, 1]) \subset R^n \setminus \{0\}$, *where* $\chi(x, s) = -sf(a, x) + (1-s)\Phi_{t,a}(x)$, *for each* $x \in S_r^{n-1}$, $t \in (a, a+\delta)$.

Proof. Assume contrary that for each $\delta > 0$ there is $x \in S_r^{n-1}$, $t \in (a, a+\delta)$ and $s \in [0, 1]$ such that $0 \in \chi_t(x)$. By putting $\delta = 1, 1/2, \ldots,$ we obtain the following sequences $t_n \in (a, a+\delta)$, $s_n \in [0, 1]$, $x_n \in S_r^{n-1}$ and $y_n\colon [a, b] \to R^n$ such that: $0 = -s_n \cdot f(a, x_n) + (1-s_n)(-y_n(t_n) + x_n)$, for each n and y_n is a solution of the Cauchy problem with $y_n(a) = x_n$. So for every n we have:

$$x_n - y_n(t_n) = p_n f(a, x_n), \text{ where } p_n = +\frac{s_n}{1-s_n} \geq 0.$$

It implies that:

$$\text{(i)} \qquad \int_a^{t_n} (f(\tau, y_n(\tau)), \quad f(a, x_n))d\tau \leq 0.$$

We can assume, without loss of generality, that $\lim x_n = x_0$. We choose τ_n such that:

$$I_n = (f(\tau_n, y_n(\tau_n)), f(a, x_n)) = \min_{a \leq a\tau \leq t_n} (f(\tau, y_n(\tau)), f(a, x_n)).$$

Obviously $\lim \tau_n = a$. Now, by simple calculation we obtain: $\lim y_n(\tau_n) = x_0$. So as a consequence we have $\lim I_n = \| f(a, x_0) \|^2 > 0$, but it is a contradiction with (i). The proof is completed.

By using Lemma 3.2 we have $\Phi_{a+\omega, a}(S_r^{n-1}) \subset R^n \setminus \{0\}$ for some $r > r_0$ and ω-periodic f which is continuous, satisfies (2.1) and has a direct potential V. Because V is a direct potential then from condition (ii) in Definition 3.1 follows that $f(a, x) \neq 0$ for every $x \in S_r^{n-1}$. Consequently, by (3.3) we have

Ind$(V) = \deg(-f(a, \cdot)) = \mathrm{Deg}(\Phi_{t,a})$ for arbitrary t which is near of a (comp. (3.3)). Let $\chi\colon S_r^{n-1} \times [0,1] \to R^n\{0\}$ be a multi valued map given by putting:

$$\chi(x, s) = \Phi_{st+(1-a)a+\omega,\, a}(x).$$

It is evident that for t near of a the map χ is an admissible homotopy between $\Phi_{t,a}$ and $\Phi_{a+\omega,\, a}$. So by (1.2) we have $\mathrm{Deg}(\Phi_{t,a}) \subset \mathrm{Deg}(\Phi_{a+\omega,\, a})$ and because $\mathrm{Ind}(V) = \mathrm{Deg}(\Phi_{t,a})$ we obtain:

$$\mathrm{Ind}(V) \subset \mathrm{Deg}(\Phi_{a+\omega,\, a}). \tag{3.4}$$

Now, as a consequence of (3.4), (2.8) and (2.9) we can formulate the following result.

Theorem 3.5. *Let $f: [a, b] \times R^n \to R^n$ be a continuous, ω-periodic with respect to first variable map which satisfies* (2.1). *Assume that f has a direct potential V such that* $\mathrm{Ind}(V) \neq 0$. *Then equation* (2.3) *has a ω-periodic solution.*

REFERENCES

1. J. Bryszewski, L. Górniewicz. A Poincáre type coincidence theorem for multi-valued maps. *Bull. Acad. Polon. Sci.*, **8**, 1976, 593-598.
2. L. Górniewicz. Homological methods in fixed point theory of multi-valued maps. *Dissert. Math.*, **129**, 1976, 1-71.
3. M. A. Krasnosielskii. Translation along trajectories of differential equations. Providence, 1968.
4. M. A. Krasnosielskii. The theory of periodic solutions of a non-autonomus differential eguations. *Russian Math. Surveys*, **21**, 1966, 53-74.
5. М. А. Красносельский, П. Забрейко. Геометрические методы в нелинейном функциональном анализе. М., 1975.
6. J. M. Lasry, R. Robert. Analyse non linéaire multivoque. Centre de Recherche de Math., de la Décision, No 7611, Paris-Dauphine.
7. J. Mawhin. Recent results on periodic solutions of differential equations.— In: *Int. Conf. Diff. Equat.* New York, 1975.
8. J. Mawhin. L_2-estimates and periodic solutions of some non-linear differential equations. *Bollettino UMI*, **10**, 1974, 343-354.
9. J. Mawhin, C. Minoz. Application du degré topologique à l'estimation du nombre des solutions périodiques d'équations différentielles. *Ann. Math. Pura Appl.*, **96**, 1973, 1-19.
10. H. O. Peitgen. On continua of periodic solutions for functional differential equations. *Rocky Mount. J. Math.*, **3**, 1977, 609-617.

Institute of Mathematics
University of Gdansk
Gdansk Poland

Department of Mathematics
Higher Pedagogical School of Stupsk
76200 Stupsk Poland

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