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ON NEUMAN'S PROBLEM FOR A CLASS OF DEGENERATE PARABOLIC EQUATIONS

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In this paper the existence and uniqueness of the classical solution of Neuman's problem for a class of degenerate parabolic equations is proved. The method of parabolic regularization is used.

1. Introduction and conclusions. The aim of this paper is to investigate Neuman's problem for a class of degenerate parabolic equations

$$(1)\qquad Lu=\sum_{ij=1}^{n} a^{ij}(x, x_0)\, u_{x_i x_j}+\sum_{i=1}^{n} b^i(x, x_0)\, u_{x_i}-c(x, x_0)\, u_{x_0}+d(x, x_0)u=f(x, x_0)$$

in the cylinder $G=\Omega\times(0, T)$. Here Ω is a bounded domain in R^n, which is $C^{2l+4+\lambda}$ smoothly diffeomorphic to a ball, $l\geqq 2$ is an integer and $0<\lambda<1$. We consider the homogeneous boundary conditions

$$(2)\qquad Bu=\sum_{k=1}^{n}\sigma^k(x, x_0)\, u_{x_k}+\sigma(x, x_0)u=0 \quad\text{on}\quad \partial\Omega\times(0, T),\ u(x, 0)=0 \text{ on } \Omega$$

If $\vec{v}=(v^1, v^2, \dots, v^n)$ is the inner unit normal to $\partial\Omega\times(0, T)$, $\vec{\sigma}=(\sigma^1, \sigma^2, \dots, \sigma^n)$, suppose that $(\vec{\sigma}, \vec{v})>0$ and $\sigma(x, x_0)\leqq 0$ on $\partial\Omega\times[0, T]$.

Further we shall make the following assumptions regarding the operators L, B and the domain Ω:

(i) $\Sigma_{i,j=1}^{n} a^{ij}(x, x_0)\xi^i\xi^j\geqq\mu(x, x_0)\,|\xi|^2\geqq 0$ in $G'\supset\bar{G}$, $\xi\in\mathrm{R}^n$, $c(x, x_0)\geqq 0$ for $(x, x_0)\in G'$, $a^{ij}\in C^2(G')$, $c\in C^2(G')$ and $c(x, x_0)+\mu(x, x_0)>0$, $d(x, x_0)\leqq 0$, $(x, x_0)\in\bar{G}$.

(ii) The coefficients of the operator L and their derivatives $D_x^\alpha D_x^\beta$ of order $\alpha|+2\beta\leqq 2l+2$ are Hölder continuous with exponent λ in $\bar{G}$. Moreover, the coefficients of the boundary value operator B and their derivatives $D_x^\alpha D_{x_0}^\beta$ of order $|\alpha|+2\beta\leqq 2l+3$ are Holder continuous with exponent λ on $\partial\Omega\times[0, T]$.

(ii) The boundary $\partial\Omega\times[0, T]$ is noncharacteristic, i. e. $\Sigma_{ij=1}^{n} a^{ij}(x, x_0)v^i v^j>0$ on $\partial\Omega\times[0, T]$.

(iv) The compatibility conditions of the data $\dfrac{\partial^k f}{\partial x_0^k}(x, 0)=0$, $k=0, 1, \dots, l+1$, $x\in\partial\Omega$ hold.

Under these assumptions, we have the following principal result:

Theorem 1. *Suppose* (i)—(iv) *hold. If* $c(x, 0)\neq 0$ *for* $x\in\Omega$ *and if there exists a point* $P_\tau\in G_\tau=G\cap\{x_0=\tau\}$, $0\leqq\tau\leqq T$, *in which the operator* L *is strictly parabolic, then the boundary value problem* (1), (2) *has a unique classical solution* $u(x, x_0)\in C^l(\bar{G})$.

Let us consider some variants of Theorem 1 for a domain ω which is $C^{2l+4+\lambda}$ smoothly diffeomorphic to the annulus $r_2<|x|<r_1$. In this case the boundary conditions are slightly different. We define the operators.

(3)
$$B_1u=\sum_{k=1}^{n}\sigma^k(x, x_0)u_{x_k}+\sigma(x, x_0)u=0 \quad \text{on} \quad S_1,$$
$$B_2u=\sum_{k=1}^{n}\tau^k(x, x_0)u_{x_k}+\tau(x, x_0)u=0 \quad \text{on} \quad S_2,$$
$$u(x, 0)=0, \quad x\in\Omega,$$

on the boundary $S_1\cup S_2\cup\Omega$, where the coefficients $\sigma^k, \tau^k, \sigma, \tau$ and their derivatives $D_x^\alpha D_{x_0}^\beta$ of order $|\alpha|+2\beta\leqq 2l+3$ are Hölder continuous with exponent λ. Moreover, we suppose that $(\vec{\sigma}, \vec{\nu})>0$, $\sigma(x, x_0)\leqq 0$ on S_1, $(\vec{\tau}, \vec{\nu})>0$, $\tau(x, x_0)\leqq 0$ on S_2.

We can also consider Dirichlet-Neuman problem for the equation (1), e. g.

(4)
$$B_1u=\sum_{k=1}^{n}\sigma^k(x, x_0)u_{x_k}+\sigma(x, x_0)u=0 \quad \text{on} \quad S_1,$$
$$u(x, x_0)=0 \quad \text{on} \quad S_2, \quad u(x, 0)=0 \quad \text{on} \quad \omega.$$

Let us formulate the results corresponding to the boundary conditions (3), (4).

Theorem 1. *Suppose* (i)—(iv) *hold. If* $c(x, 0)\neq 0$ *for* $x\in\omega$ *and* $d(x, x_0)<0$ *in* $\bar{G}$, *then the boundary value problem* (1), (3) *has a unique classical solution* $u(x, x_0)\in C^l(\bar{G})$.

Theorem 1''. *Under the assumptions of Theorem* 1', *the boundary value problem* (1), (4) *has a unique classical solution* $u(x, x_0)\in C^l(\bar{G})$.

The same method could be used also in some cases, when the assumption $c(x, 0)\neq 0$ on ω is not satisfied. Let us, for example, consider the domain $G=\omega x(-T, T)$. We suppose that the following conditions hold:

(i') $\sum_{i,j=1}^{n} a^{ij}(x, x_0)\xi^i\xi^j\geqq\mu(x, x_0)|\xi|^2\geqq 0$ in the domain $G'\supset\bar{G}$, $\xi\in\mathbf{R}^n$, $a^{ij}(x, x_0)\in C^2(G')$

$$c(x, x_0)\geqq 0,\ \mu(x, x_0)+c(x, x_0)>0;\ c\in C^2(G'_+) \quad \text{in} \quad G'_+=G'\cap\{x_0>0\},$$
$$c(x, x_0)\leqq 0,\ \mu(x, x_0)-c(x, x_0)>0;\ c\in C^2(G'_-) \quad \text{in} \quad G'_-=G'\cap\{x_0<0\},$$
$$c(x, 0)=0,\ x\in\omega;\ d(x, x_0)<0 \quad \text{in} \quad \bar{G}.$$

Theorem 2. *Suppose* (i'), (ii), (iii) *hold. Then in the domain* $G=\omega\times(-T, T)$ *the boundary value problem*

(5)
$$B_1u=0 \quad on \quad \partial\omega_1\times(-T, T),$$
$$B_2u=0 \quad on \quad \partial\omega_2\times(-T, T),$$

for the equation (1) *has a unique classical solution* $u(x, x_0)\in C^l(\bar{G})$.

The following Theorem 2' is a version of Theorem 2.

Theorem 2'. *Suppose* (i'), (ii), (iii) *hold. Then in the domain* $G=\omega\times(-T, T)$ *the boundary value problem*

$$(6)\qquad \begin{aligned} B_1 u=0 \quad &on \quad \partial\omega_1\times(-T, T),\\ u=0 \quad &on \quad \partial\omega_2\times(-T, T), \end{aligned}$$

for the equation (1) *has a unique classical solution* $u(x, x_0)\in C^l(\bar{G})$.

Let us now consider some special boundary value problems which describe certain real processes. For convenience, we introduce the set $\Omega_0=\{x\in\Omega\,;\, c(x, 0)=0\}$. Let us formulate the following results:

Theorem 3. *Suppose* (i)—(iv) *hold. The boundary value operator*

$$(7)\qquad Bu=\sum_{k=1}^{n}\sigma^k(x, x_0)u_{x_k}+\sigma(x, x_0)u=0 \quad on \quad \partial\Omega\times(0, T),$$

$$u(x, 0)=0 \quad on \quad \Omega\setminus\Omega_0,$$

satisfies the assumptions of Theorem 1. We assume also that $D^{\alpha}_{xx_0}f(x, 0)=0$ *in some neighbourhood of* Ω_0 *for* $|\alpha|\leq l+1$. *If there exists a point* $P_\tau\in P_\tau=G\cap\{x_0=\tau\}$, $0\leq\tau\leq T$ *in which the operator* L *is strictly parabolic, then the boundary value problem* (1), (7) *has a unique classical solution* $u(x, x_0)\in C^l(\bar{G})$.

We can formulate theorems similar to Theorems 1', 1''.

In conclusion it should be mentioned, that many real processes can be described by means of the boundary value problems of the kind (1), (7) (see [3]), which justifies our interest in them. For instance, the equation describing the temperature distribution in case of steady laminar tube flow is

$$(8)\qquad v(y, z)\frac{\partial T}{\partial x}=\lambda\left(\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right), \quad \lambda=\text{const}>0,$$

where $v(y, z)\geq 0$ and $v(y, z)=0$ only on the boundary of the tube. The second boundary value problem might be considered for the equation (8). The equation describing the concentration distribution in case of pipe gas flow with a steady velocity profile is analogous to the equation (8). The third boundary value problem might be considered in this case.

M. Gevrey was the first to draw the attention to equations of the kind (8) (see [8; 9]). Later on G. Fateeva [7] proved the existence of a unique classical solution of the second boundary value problem for the equation (1), when $c(x, x_0)\equiv 1$. She also treats the quasilinear case, where results are obtained, provided $0<x_0<\delta$ and δ is small enough. In [10] P. Ippolito investigates the Dirichlet problem for the equation (1) and proved the existence of a unique classical solution.

Finally, let us state that our results are not contained in [7], where $c(x. x_0)\equiv 1$, e. g. the parabolic degeneracy is not considered at all. Unlike [10], where the first boundary value problem is treated for equation (1), this paper deals with Neuman's problem for the same equation. Besides, in [10] the case when $c(x, x_0)$ vanishes on the base of the cylinder is non considered.

In 2 the uniform boundedness of the solutions $u^\varepsilon(x, x_0)$ and their derivatives up to the order $l+1$ of the regularized boundary value problems (10), (2), (3), (4) is proved by means of Lemmas 1—5. Theorems 1, 1', 1'' are proved by a limiting process $\varepsilon\to 0$.

In 3 the case, when $c(x, 0)=0$ on the base of the cylinder, is investigated.

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2. We will use the following inequalities and identities (see [2; 6]):

$$L(v_1v_2)=v_1Lv_2+v_2Lv_1+2\sum_{i,j=1}^{n} a^{ij}(v_1)_{x_i}(v_2)_{x_j}-dv_1v_2$$

for any two functions $v_1, v_2 \in C^2(G)$;

$$[\sum_{i,j=1}^{n} a^{ij}\xi^i\eta^j]^2 \leq [\sum_{i,j=1}^{n} a^{ij}\xi^i\xi^j][\sum_{i,j=1}^{n} a^{ij}\eta^i\eta^j]$$

for any $\xi, \eta \in \mathsf{R}^n$,

$$(9)\qquad [\sum_{i,j=1}^{n} a^{ij}_{x_p} u_{x_ix_j}]^2 \leq M \sum_{i,j,k=1}^{n} a^{ij} u_{x_kx_i} u_{x_kx_j},$$

when $p=0, 1, \ldots, n$, under the assumption that (i) in 1 hold. Here the constant M depends on the maximum of the second derivatives of a^{ij}. The proof of (9) for $p=0$ follows with slight changes of Oleinik's proof for $p=1, 2, \ldots, n$ (see [2, *p.* 71]). Further we will use the short notations $u_k=u_{x_k}$, $b^i_{kl}=b^i_{x_kx_l}$ etc., and the summation convention is understood.

Of basic significance for the proof of Theorem 1 is the following regularized equation

$$(10)\qquad L^\varepsilon u=Lu+\varepsilon(\Delta u-u_{x_0})=f, \quad \varepsilon>0.$$

Let $u^\varepsilon(x, x_0)$ be a solution of (10), (2), which together with its derivatives $D^\alpha_x D^\beta_{x_0}$ of order $|\alpha|+2\beta\leq 2l+4$ is Hölder continuous with exponent λ (see [1]). By M_i, K_i we denote the constants which depend on the coefficients of the equation, the boundary value operator and the domain G, but not on ε.

Lemma 1. *Under the assumptions of Theorem 1 the following estimates* $\max_{x_0=0, x\in\bar\Omega} |D^\alpha_{xx_0} u^\varepsilon(x, x_0)|\leq K'_\alpha$, $|\alpha|\leq l+1$ *hold.*

Proof. The proof of Lemma 1 follows directly from the boundary value operator (2), the equation (10) and its derivatives up to the order l.

Without loss of generality, in order to prove Theorem 1, we assume that G is a cylinder with a base Ω, which is a ball, its centre and radius being respectively 0 and R. Besides, the operator L is strictly parabolic on the axis of the cylinder G. Let G_0 is a cylinder with base ω_0, and ω_0 is a concentric to Ω ball. The radius r of ω_0 is small enough, so that the operator L is strictly parabolic in $\bar G_0$. Let $u=vw$, where $w=\exp(bx_0)[2-\exp(-\alpha|x|^2)]>0$ in $\bar G$, and let us consider the operator (see [10])

$$\tilde L v=(L(vw))/w=\sum_{i,j=1}^{n} a^{ij}v_{x_ix_j}+\sum_{i=1}^{n}[b^i+(2\sum_{j=1}^{n} a^{ij}w_{x_j})/w]v_{x_i}-cv_{x_0}+[(Lw)/w]w=f/w.$$

The inequality

$$Lw\leq\{-[4\alpha^2a^{ij}x_ix_j-2\alpha a^{ii}-2\alpha b^ix_i+d]\exp(-\alpha|x|^2)-\beta c[2$$
$$-\exp(-\alpha|x|^2)]\}\exp(\beta x_0)\leq-[4\alpha^2\mu(x, x_0)|x|^2-O(\alpha)+\beta c]\exp(-\alpha|x|^2)<0$$

holds in $\bar{G}$, when α, β are sufficiently large. Analogously we consider the boundary value operator

$$\tilde{B}v=(B(vw))/w=\sum_{k=1}^{n}\sigma^{k}v_{x_k}+[\sigma+(\sum_{k=1}^{n}\sigma^{k}w_{x_k})/w]v=0,$$

where

$$\sum_{k=1}^{n}\sigma^{k}w_{x_k}=(2\alpha\sum_{k=1}^{n}\sigma^{k}x_k)\exp(-\alpha R^2+\beta x_0)=-2\alpha R\sum_{k=1}^{n}\sigma^{k}\nu^{k}.\exp(-\alpha R^2+\beta x_0)<0$$

on $\bar{S}$. The operators $\tilde{L}$ and $\tilde{B}$ satisfy the conditions (i)—(iv) in 1. Consequently, if we preserve the previous notations, without loss of generality we may assume that $d(x, x_0)<0$ in $\bar{G}$ and $\sigma(x, x_0)<0$ on $\bar{S}$.

In the following Lemmas 2—5 our aim will be to prove the uniformly boundedness of the derivatives up to the order $l+1$ of the solutions $u^{\varepsilon}(x, x_0)$ of (12), (2) with constants independent of ε.

Lemma 2. *Under the assumptions of Theorem 1 the following estimates* $\max_{(x,x_0)\in\bar{G}}|u^{\varepsilon}(x, x_0)|\leq K_0$ *hold.*

Proof. We consider the auxiliary function $v^0(x, x_0)=(u^{\varepsilon})^2-N$. The estimates

$$Lv^0=2a^{ij}u_i^{\varepsilon}u_j^{\varepsilon}+2u^{\varepsilon}f-d(u^{\varepsilon})^2-dN\geq a^{ij}u_i^{\varepsilon}u_j^{\varepsilon}+1 \text{ in } \bar{G}, \tag{11}$$

$$Bv^0=2u^{\varepsilon}Bu^{\varepsilon}-\sigma(u^{\varepsilon})^2-\sigma N\geq 1 \text{ on } S,\ v^0<0 \text{ on } \Omega$$

hold, when N is sufficiently large. From the maximum principle it follows that $v^0(x, x_0)$ can not attain a positive maximum in $\bar{G}$. Consequently, $|u^{\varepsilon}(x, x_0)|\leq N^{1/2}$ and Lemma 2 is proved.

Lemma 3. *Under the assumptions of Theorem 1 the estimates*

$$\max_{(x,x_0)\in\bar{G}}|D^{\alpha}u^{\varepsilon}|\leq K_1 \tag{12}$$

for $|\alpha|=1$ *hold.*

Proof. From Lemma 2 and the inner a priori estimates of Bernstein (see [6]) we obtain the estimates (12) in $\bar{G}_0$. In order to prove (12) in G_1, where $G_1=G\setminus\bar{G}_0$ we make a polar change of the x-variables, and for convenience we preserve the previous notations considering that $x_1, x_2, \ldots, x_{n-1}$ are angular variables and x_n is a radial variable. In the new variables the assumption about the vector field $(\sigma^1, \sigma^2, \ldots, \sigma^n)$ denotes that $\sigma^n<0$ on $\bar{S}$. In our further calculations, for convenience, we will omit the index ε.

Let us introduce the auxiliary functions:

$$v^1(x, x_0)=(n_1z^1(x, x_0)+u_{x_0}^2)\exp(-\eta_1x_0)+N_0v^0(x, x_0),$$

$$z^1(x, x_0)=[m_1\sum_{k=1}^{n-1}u_k^2+2u_n^2+u_nTu+m_1]\exp((R-x_n)\xi_1).$$

Here $Tu=4\Sigma_{k=1}^{n-1}\theta^k(x, x_0)u_k+4\theta(x, x_0)u$ and θ^k, θ are smooth extensions into $\bar{G}_1$ of the functions σ^k/σ^n, σ/σ^n respectively, which are defined on S, so that their derivatives in $\bar{G}_1$, $D_x^{\alpha}D_{x_0}^{\beta}$ of order $|\alpha|+2\beta\leq 2l+3$ are Hölder continuous with exponent λ. The positive constant m_1 is chosen so that

$$m_1 \sum_{k=1}^{n-1} u_k^2 + 2u_n^2 + u_n Tu + m_1 \geqq \sum_{k=1}^{n} u_k^2,$$

$$2m_1 \sum_{k=1}^{n-1} a^{ij} u_{ki} u_{kj} + 4a^{ij} u_{ni} u_{nj} + a^{ij} u_{nj} (Tu)_i$$

$$\geqq (3m_1/2) \sum_{k=1}^{n-1} a^{ij} u_{ki} u_{kj} + 3a^{ij} u_{ni} u_{nj} - M_1 \sum_{k=1}^{n} u_k^2 - M_2,$$

$$m_1 \geqq \max(2, (4nH_1)^2), \quad H_1 = \max_{(x,x_0)\in \overline{G}_1} \Sigma(\theta^k)^2)^{1/2}.$$

We will show that $v^1(x, x_0)$ can not attain a positive maximum on S A simple computation gives

$$Bv^1 \geqq \{n_1 Bz^1 + 2u_{x_0}((Bu)_{x_0} + [B, \partial/\partial x_0](u)) - \sigma u_{x_0}^2\} \exp(-\eta_1 x_0) + N_0 Bv_0.$$

From the boundary value condition (2) it follows that $(Bu)_{x_0} = 0$ on S. When ξ_1 is sufficiently large the estimate $Bz^1 \geqq M_3 \Sigma_{k=1}^n u_k^2 - M_3'$ holds (see [12]). Since $[B, \partial/\partial x_0]$ is an operator of the first order and does not depend on $\partial/\partial x_0$, when n_1, N_0 are sufficiently large we have $Bv^1 \geqq 1$ on S. From the maximum principle it follows that $v^1(x, x_0)$ can not attain a positive maximum on S.

We will show, that $v^1(x, x_0)$ can not attain a positive maximum in the domain G_1 as well, when the constants η_1, N_0 are sufficiently large. For Lv^1 we have

$$Lv^1 = n_1(I_1 + I_2) \exp((R - x_n)\xi_1 - \eta_1 x_0) + (I_3 + I_4) \exp(-\eta_1 x_0) + N_0 L v_0,$$

where

$$I_1 = (a^{nn}\xi_1^2 - b^n \xi_1) z^1 \exp((x_n - R)\xi_1) - 2 \sum_{j=1}^{n} a^{nj} \xi_1 (2m_1 \sum_{k=1}^{n-1} u_k u_{kj}$$

$$+ 4u_n u_{nj} + u_n (Tu)_j + u_{nj} Tu) + 2m_1 \sum_{k=1}^{n-1} a^{ij} u_{ki} u_{kj} + 4a^{ij} u_{ni} u_{nj} + 2a^{ij} u_{nj} (Tu)_i$$

$$+ 2m_1 \sum_{k=1}^{n-1} u_k [-a_k^{ij} u_{ij} - b_k^i u_i - d_k u + f_k] + (Tu + 4u_n)[-a_n^{ij} u_{ij} - b_n^i u_i - d_n u + f_n]$$

$$+ u_n(L(Tu)) - 4u_n (\sum_{k=1}^{n-1} \theta^k c_k u_{x_0} + c\theta u_{x_0}) - d[m_1 \sum_{k=1}^{n-1} u_k^2 + 2u_n^2 + Tu \,.\, u_n]$$

$$+ dm_1 + \eta_1 c z^1 \exp((x_n - R)\xi_1).$$

The estimate

$$I_1 \geqq \sum_{k=1}^{n} a^{ij} u_{ki} u_{kj} + \eta_1 c \sum_{k=1}^{n} u_k^2 - M_4 \sum_{k=1}^{n} u_k^2 - M_5$$

can be proved in the same way as in 2 of [12].

For I_2, I_3, I_4 it is clear that

$$I_2 = 2m_1 \sum_{k=1}^{n-1} c_k u_k u_{x_0} + 4c_n u_n u_{x_0} + 4u_n (\sum_{k=1}^{n-1} c_k \theta^k u_{x_0} + c\theta u_{x_0}) + c_n u_{x_0} Tu$$

$$\leq M_6 cu_{x_0}^2 + M_7 \sum_{k=1}^{n} u_k^2 + M_8,$$

$$I_3 = 2a^{ij} u_{ix_0} u_{jx_0} + \eta_1 cu_{x_0}^2,$$

$$I_4 = 2u_{x_0}(-a_{x_0}^{ij} u_{ij} - b_{x_0}^i u_i + c_{x_0} u_{x_0} - d_{x_0} u + f_{x_0}) - du_{x_0}^2$$

$$\geq -M_9 \sum_{k=1}^{n} a^{ij} u_{ki} u_{kj} - M_{10} \sum_{k=1}^{n} u_k^2 - \frac{d}{2} u_{x_0}^2 - M_{11} cu_{x_0}^2 - M_{12}.$$

We choose the positive constant n_1, so that $n_1 \geq M_9 + 1$. Using (11) when η_1, N_0 are sufficiently large, we have

$$(13) \qquad Lv^1 \geq (\sum_{k=0}^{n} a^{ij} u_{ki} u_{kj}) \exp(-\eta_1 T) + 1 \text{ in } \bar{G}_1.$$

Consequently $v^1(x, x_0)$ does not attain a positive maximum in G_1 and on the upper base of the cylinder G_1. When N, N_0 are sufficiently large, it follows from the proof of (12) in $\bar{G}_0$, that v^1 does not attain a positive maximum on $S_0 = \partial\omega_0 \times (0, T)$ and from Lemma 1, it follows that v^1 does not attain a positive maximum on $\Omega \setminus \omega_0$ also. Consequently $v^1 \leqq 0$ in $\bar{G}_1$ and the proof of Lemma 3 is based on the choice of v^1 and the relation between Cartesian and polar coordinates.

Lemma 4. *Under the assumptions of Theorem 1 the estimates*

$$(14) \qquad \max_{(x, x_0) \in \bar{G}} |D^\alpha u^\varepsilon (x, x_0)| \leqq K_2,$$

$|\alpha| = 2$, *hold.*

Proof. In $\bar{G}_0$ the proof of (14) is a consequence of Lemma 2 and the inner a priori estimates of Bernstein. In order to prove (14) in $\bar{G}_1$, we make a polar change of the variables, using the notations introduced in Lemma 3. We consider the function

$$v^2(x, x_0) = (n_2 z^2(x, x_0) + u_{x_0 x_0}^2) \exp(-\eta_2 x_0) + N_1 v^1(x, x_0),$$

where

$$z^2(x, x_0) = (m_2 \sum_{k=0, l=1}^{n-1} u_{kl}^2 + 2 \sum_{k=0}^{n} u_{kn}^2 + \sum_{k=0}^{n} u_{kn} T_k u + m_2) \exp(\xi_2(R - x_n)).$$

We define the operators T_k, $k = 0, 1, \ldots, n$,

$$T_k u = 4(\sum_{i=1}^{n-1} \theta^i u_i + \theta u)_k, \quad k = 0, 1, \ldots, n-1,$$

$$T_n u = 4[-\sum_{i=1}^{n-1} A^{ni} (\sum_{k=1}^{n-1} \theta^k u_k + \theta u)_i + \sum_{i,j=1}^{n-1} A^{ij} u_{ij} + \sum_{i=1}^{n-1} B^i u_i$$

$$-B^n(\sum_{k=1}^{n-1} \theta^k u_k + \theta u) - Cu_{x_0} + Du - F].$$

The functions θ^k, θ are introduced in Lemma 3. The functions A^{ij}, B^i C, D, F are smooth extensions respectively of a^{ij}/a^{nn}, b^i/a^{nn}, c/a^{nn}, d/a^{nn}, f/a^{nn} from S

into $\bar{G}_1$, so that their derivatives $D_x^\alpha D_{x_0}^\beta$ of order $|\alpha|+2\beta\leqq 2l+3$ are Hölder continuous with exponent λ. The positive constant m_2 is chosen so that

$$m_2\sum_{k=0,l=1}^{n-1}u_{kl}^2+2\sum_{k=0}^{n}u_{kn}^2+\sum_{k=0}^{n}u_{kn}T_ku+m_2\geqq\sum_{k=0,l=1}^{n}u_{kl}^2,$$

$$2m_2\sum_{k=0,l=1}^{n-1}a^{ij}u_{kli}u_{kli}+4\sum_{k=0}^{n}a^{ij}u_{kni}u_{knj}+\sum_{k=0}^{n}a^{ij}u_{kni}(T_ku)_j\geqq\frac{3m_2}{2}\sum_{k=0,l=1}^{n-1}a^{ij}u_{kli}u_{klj}$$

$$+3\sum_{k=0}^{n}a^{ij}u_{kni}u_{knj}-M_{13}\sum_{k=0,l=1}^{n}u_{kl}^2-M_{14},\ m_2\geqq\max(2,(4n^2H_2)^2),$$

where H_2 is the maximum of the coefficients of the derivatives of the highest order in T_k, $k=0,1,\dots,n$.

We will prove that $v^2(x,x_0)$ can not attain a positive maximum in $\bar{G}_1$, when ξ_2, η_2, N_1 are sufficiently large. Indeed, on S we have the estimate

$$Bv^2\geqq n_2\{-\sigma^n\xi_2\sum_{k=0,l=1}^{n}u_{kl}^2+2m_2\sum_{k=0,l=1}^{n-1}u_{kl}((Bu)_{kl}+[B,\partial^2/\partial x_k\partial x_l](u)$$

$$+4\sum_{k=0}^{n}u_{kn}Bu_{kn}+\sum_{k=0}^{n}Bu_{kn}T_ku+\sum_{k=0}^{n}u_{kn}(T_k(Bu)+[B,T_k](u))+2m_2\sigma\}\exp(-\eta_2x_0)$$

$$+\{2u_{x_0x_0}((Bu)_{x_0x_0}+[B,\partial^2/\partial x_0^2](u)-\sigma u_{x_0x_0}^2\}\exp(-\eta_2x_0)+N_1Bv^1.$$

If we use the boundary condition (2) and the definition of the operators T_k, $k=0,1,\dots,n$, we obtain that $(Bu)_{kl}=0$ for $k,l=0,1,\dots,n-1$ and $T_ku+4u_{kn}=0$ for $k=0,1,\dots,n$ on S. It is clear that the commutators are operators of second order, which do not depend on $\partial^2/\partial x_0^2$. Therefore, when ξ_2, N_1 are large enough we have

(15) $$Bv^2\geqq 1\quad\text{on}\quad S.$$

From the maximum principle, it follows that $v^2(x,x_0)$ can not attain a positive maximum on S.

We will show that $v^2(x,x_0)$ can not attain a positive maximum in the domain G_1 as well, when n_2, η_2, N_1 are large enough. For Lv^2 we have

$$Lv^2=\{n_2(I_1+I_2)\exp(\xi_2(R-x_n))+(I_3+I_4)\}\exp(-\eta_2x_0)+N_1Lv^1,$$

where

$$I_1=(a^{nn}\xi_2^2-b^n\xi_2)z^2\exp(\xi_2(x_n-R))-2\sum_{j=1}^{n}a^{nj}\xi_2[2m_2\sum_{k=0,l=1}^{n-1}u_{kl}u_{klj}+4\sum_{k=0}^{n}u_{kn}u_{knj}$$

$$+\sum_{k=0}^{n}u_{kn}(T_ku)_j+\sum_{k=0}^{n}u_{knj}T_ku]+2m_2\sum_{k=0,l=1}^{n-1}a^{ij}u_{kli}u_{klj}+4\sum_{k=0}^{n}a^{ij}u_{kni}u_{knj}$$

$$+2\sum_{k=0}^{n}a^{ij}u_{knj}(T_ku)_i+2m_2\sum_{k=0,l=1}^{n-1}u_{kl}(-a_k^{ij}u_{lij}-a_l^{ij}u_{kij}-a_{kl}^{ij}u_{ij}-b_k^iu_{li}-b_l^iu_{ki}+c_ku_{lx_0}$$

$$+c_{kl}u_{x_0}-d_ku_l-d_lu_k-d_{kl}u+f_{kl})+2m_2\sum_{k,l=1}^{n-1}c_lu_{kx_0}u_{kl}+\sum_{k=0}^{n}(4u_{kn}+T_ku)(-a_k^{ij}u_{nij}$$

$$-a_n^{ij}u_{kij}-a_{kn}^{ij}u_{ij}-b_k^i u_{ni}-b_n^i u_{ki}+c_k u_{nx_0}+c_{kn}u_{x_0}-d_k u_n$$

$$-d_n u_k-d_{kn}u+f_{kn})+\sum_{k=1}^{n} c_n u_{kx_0}(4u_{kn}+T_k u)+\sum_{k=0}^{n} u_{kn}L(T_k u)$$

$$-4\sum_{l=1}^{n-1}\theta^l c_l u_{x_0x_0}u_{nx_0}-dz^2\exp(\xi_2(x_n-R))+2m_2 d+\eta_2 cz^2\exp(\xi_2(x_k-R).$$

The estimate

$$I_1\geq\sum_{k=0,l=1}^{n-1}a^{ij}u_{kli}u_{klj}+\eta_2 c\sum_{k=0,l=1}^{n}u_{kl}^2-M_{15}\sum_{k=0,l=1}^{n}u_{kl}^2-M_{16}$$

can be proved in the same way as in [12].

For I_2, I_3, I_4 it is clear that

$$I_2=2m_2\sum_{l=1}^{n-1}c_l u_{lx_0}u_{x_0x_0}+(4u_{nx_0}+T_0u)\,c_n u_{x_0x_0}+4\sum_{l=1}^{n-1}\theta^l c_l u_{x_0x_0}u_{nx_0}$$

$$\leq M_{17}cu_{x_0x_0}^2+M_{18}\sum_{k=0,l=1}^{n}u_{kl}^2+M_{19},$$

$$I_3=2a^{ij}u_{ix_0x_0}u_{jx_0x_0}+\eta_2 cu_{x_0x_0}^2,$$

$$I_4=2u_{x_0x_0}(-2a_{x^0}^{ij}u_{ijx_0}-a_{x_0x_0}^{ij}u_{ij}-2b_{x_0}^i u_{ix_0}-b_{x_0x_0}^i u_i+2c_{x_0}u_{x_0x_0}$$

$$+c_{x_0x_0}u_{x_0}-d_{x_0x_0}u-2d_{x_0}u_{x_0}+f_{x_0x_0})-du_{x_0x_0}^2\geq -M_{20}\sum_{k=1}^{n}a^{ij}u_{kix_0}u_{kjx_0}$$

$$-M_{20}'\sum_{k=0,l=1}^{n}u_{kl}^2-M_{21}\,cu_{x_0x_0}^2-\frac{d}{2}u_{x_0x_0}^2-M_{22}.$$

We choose the positive constant n_2 so that $n_2\geq M_{20}+1$. When η_2, N_1 are sufficiently large, from (13) we have

$$Lv^2\geq\Big(\sum_{k,l=0}^{n}a^{ij}u_{kli}u_{klj}\Big)\exp(-\eta_2 T)+1. \tag{16}$$

From the proof of (14) in G_0, it follows that $v^2(x, x_0)$ can not attain a positive maximum on S_0, when N_1 is large enough.

We observe also that $v^2(x, x_0)$ does not attain a positive maximum on $\Omega\setminus\omega_0$, from Lemma 1. Like in Lemma 3, from (15), (16) it follows that $v^2(x, x_0)$ does not attain a positive maximum on S, in the domain and on the upper base of the cylinder. Consequently, $v^2\leq 0$ in $\bar{G}_1$, and the proof of Lemma 4 is based on the choice of v^2 and the relation between Cartesian and polar coordinates.

L e m m a 5. *Under the assumptions of Theorem 1, the estimates*

$$\max_{(x,x_0)\in\bar{G}}|D^\alpha u^\varepsilon|\leq K_\mu,\ |\alpha|=\mu,\ 3\leq\mu\leq l+1,$$

hold

P r o o f. Lemma 5 is inductively proved by means of functions of the kind

$$v^\mu(x, x_0)=[n_\mu z^\mu(x, x_0)+(D_{x_0}^\mu u)^2]\exp(-\eta_\mu x_0)+N_{\mu-1}v^{\mu-1}(x, x_0),$$

$$z^{\mu}(x, x_0)=[m_{\mu}\sum_{|\alpha|=\mu,\alpha_0\neq\mu}(D^{\alpha}_{x'}u)^2+2\sum_{|\alpha+\beta|=\mu,\beta\neq 0}(D^{\alpha}_{x'}D^{\beta}_{x_n}u)^2$$

$$+\sum_{|\alpha+\beta|=\mu,\beta\neq 0}T_{\alpha\beta}(u)\,.\,D^{\alpha}_{x'}D^{\beta}_{x_n}u+m_{\mu}]\exp(\xi_{\mu}(R-x_n)),$$

where $x'=(x_0, x_1, \ldots, x_{n-1})$, $D^{\alpha}_{x'}=D^{\alpha_0}_{x_0}D^{\alpha_1}_{x_1}\ldots D^{\alpha_{n-1}}_{x_{n-1}}$. The operators $T_{\alpha\beta}=T_{\alpha\beta}(x_0, x, u, u_{x_0}, \ldots u_{x_{n-1}})$ are defined on S according to the condition $T_{\alpha\beta}(u)=-4D^{\alpha}_{x'}D^{\beta}_{x_n}u$. The derivatives $D^{\beta}_{x_n}u$ are substituted by their equivalent expressions, which include only derivatives of the variables $x_0, x_1, \ldots, x_{n-1}$ by means of the boundary value operator B, the equation (10) and the derivatives of equation (10) up to the necessary order. In $\bar{G}_1$ the coefficients of $T_{\alpha\beta}$ are smoothly extended, so that their derivatives $D^{\alpha}_{x}D^{\beta}_{x_0}$ of order $|\alpha|+2\beta\leq 2l+3$ are Hölder continuous with exponent λ. The positive constant m_{μ} is chosen so that

$$z^{\mu}(x, x_0)\exp(\xi_{\mu}(x_n-R))\geqq\sum_{|\alpha|=\mu,\alpha_0\neq\mu}(D^{\alpha}_{xx_0}u)^2,$$

$$2m_{\mu}\sum_{|\alpha|=\mu,\alpha_0\neq\mu}a^{ij}(D^{\alpha}_{x'}u)_i\,(D^{\alpha}_{x'}u)_j+4\sum_{|\alpha+\beta|=\mu,\beta\neq 0}a^{ij}(D^{\alpha}_{x'}D^{\beta}_{x_n}u)_i\,(D^{\alpha}_{x'}D^{\beta}_{x_n}u)_j$$

$$+\sum_{|\alpha+\beta|=\mu,\beta\neq 0}a^{ij}(T_{\alpha\beta}u)_i\,(D^{\alpha}_{x'}D^{\beta}_{x_n}u)_j\geqq\frac{3m_{\mu}}{2}\sum_{|\alpha|=\mu,\alpha_0\neq\mu}a^{ij}(D^{\alpha}_{x'}u)_i\,(D^{\alpha'}_{x}u)_j$$

$$+3\sum_{|\alpha+\beta|=\mu,\beta\neq 0}a^{ij}(D^{\alpha}_{x'}D^{\beta}_{x_n}u)_i\,(D^{\alpha}_{x'}D^{\beta}_{x_n}u)_j-M_{\mu,1}\sum_{|\alpha|=\mu,\alpha\neq\mu}(D^{\alpha}_{xx_0}u)^2-M_{\mu,2}$$

$$m_{\mu}\geqq\max(2,(4n^{\mu}H_{\mu})_2),$$

where $D^{\alpha}_{xx_0}=D^{\alpha_0}_{x_0}D^{\alpha_1}_{x_1}\ldots D^{\alpha_n}_{x_n}$ and H_{μ} is the maximum of the coefficients of the derivatives of the highest order in $T_{\alpha\beta}$.

Proof of Theorem 1. Let u_1, u_2 be two classical solutions of (1), (2) and $u=u_1-u_2$. Then $u(x, x_0)$ is a solution of the homogenious boundary value problem

(17) $$L^{\varepsilon}u=0 \quad\text{in}\quad G,\ Bu=0 \quad\text{on}\quad S,\ u(x, 0)=0,\ x\in\Omega,$$

and according to the maximum principle it follows that if $u(x, x_0)$ attains a positive maximum in a certain inner point P_1 of the domain G or upon the upper base of the cylinder, then $(L^{\varepsilon}u)(P_1)<0$ which contradicts (17). If $u(x, x_0)$ attains a positive maximum in a point $P_2\in S$, then $(Bu)(P_2)<0$, which is impossible because of (17). When $x_0=0$, then $u(x, 0)=0$ for $x\in\Omega$ according to (17). In the same way we prove that $-u(x, x_0)$ can not attain a positive maximum in $\bar{G}$ therefore, the classical solution of (1), (2) is unique.

By means of the a priori estimates proved in Lemmas 2, 3, 4 and 5 we have the result that the solutions $u^{\varepsilon}(x, x_0)$ of (10), (2) and their derivatives $D^{\alpha}_{xx_0}$ of order $|\alpha|\leq l+1$ are uniformly bounded by the constants which do not depend on ε. Using the Ascoli-Arzela theorem and a diagonalization argument, we can find a subsequence which converges uniformly in $\bar{G}$ to the desired solution.

When proving Theorems 1′, 1″, the techniques in Theorem 1 in this paper is once again applicable, with certain modifications due to the boundary conditions (3) and (4). Therefore we will omit the proofs of Theorems 1′, 1″.

Corollary 1. *Suppose* (i)—(iv) *in* 1 *hold and we also have*

$$\sum_{i,j=1}^{n} a^{ij}(x, x_0)\xi^i\xi^j \geqq \mu\,|\,\xi\,|^2,\ \mu>0 \quad for \quad (x, x_0)\in \bar{G},\ \xi\in \mathbb{R}^n.$$

In this case it is not necessary for the conditions $c(x, x_0)\geqq 0$ *in* G', $c(x, x_0)\in C'(G')$ *to be fulfilled. It is enough for* $c(x, x_0)\geqq 0$ *to be valid in* $\bar{G}$.

3. In this paragraph we consider some boundary value problems, which describe real processes. The details are similar to those in Theorem 1 and are not carried out here.

In order to prove Theorem 2 we make a polar change of the x-variables, using the notations introduced in Lemma 3 in 2. In the new variables the assumption about the vector fields $(\sigma^1, \sigma^2, \ldots, \sigma^n)$, $(\tau', \tau^2, \ldots, \tau^n)$ denotes that $\sigma^n<0$ on S_1, $\tau^n>0$ on S_2. We suppose that the domain ω is the annulus $r_2<X_n<r_1$. We consider, as in 2, the operators $\tilde{L}v$, $\tilde{B}_1v$, $\tilde{B}_2v$, where $\tilde{L}v=(L(vw))/w$, $\tilde{B}_1v=(B_1(vw))/w$, $\tilde{B}_2v=(B_2(vw))/w$ and $w=\exp(\beta x_0)[\exp(2\alpha(r_1-r_2)^2)-\exp(\alpha(r_1-x_n)(x_n-r_2))]>0$ in $\bar{G}$. From the inequalities $Lw<0$ in $\bar{G}$, $B_1w<0$ on $\bar{S}_1$, $B_2w<0$ on $\bar{S}_2$, without loss of generality, we assume that the operators L, B_1, B_2 satisfy the conditions $d(x, x_0)<0$ in $\bar{G}$, $\sigma(x, x_0)<0$ on $\bar{S}_1$, $\tau(x, x_0)<0$ on $\bar{S}_2$.

Proof of Theorem 2. Let the solution of boundary value problem (1), (6) be of the kind $w^0(x)+u^1(x, x_0)$, where $w^0(x)$, $u^1(x, x_0)$ are solutions respectively of the boundary value problems

$$\sum_{i,j=1}^{n} a^{ij}(x, 0)w^0_{ij}+\sum_{i=1}^{n} b^i(x, 0)w^0_i+d(x, 0)w^0=f(x, 0),$$

$$\sum_{k=1}^{n} \sigma^k(x, 0)w^0_k+\sigma(x, 0)w^0=0, \quad x\in\partial\Omega_1,$$

$$\sum_{k=1}^{n} \tau^k(x, 0)w^0_k+\tau(x, 0)w^0=0, \quad x\in\partial\Omega_2,$$

$$Lu^1=x_0 f^1(x, x_0),\ B_1u^1=0,\ B_2u^1=0. \tag{18}$$

Let the solution $u^1(x, x_0)$ of (18) be of the kind $x_0w^1(x)+u^2(x, x_0)$, where $w^1(x)$, $u^2(x, x_0)$ are solutions of the boundary value problems

$$\sum_{i,j=1}^{n} a^{ij}(x, 0)w^1_{ij}+\sum_{i=0}^{n} b^i(x, 0)w^1_i+(d(x, 0)-c_{x_0}(x, 0))w^1=f^1(x, 0),$$

$$\sum_{k=1}^{n} \sigma^k(x, 0)w^1_k+\sigma(x, 0)w^1=0, \quad x\in\partial\Omega_1,$$

$$\sum_{k=1}^{n} \tau^k(x, 0)w^1_k+\tau(x, 0)w^1=0, \quad x\in\partial\Omega_2,$$

$$Lu^2=x_0^2 f^2(x, x_0),\ B_1u^2=0,\ B_2u^2=0.$$

By induction we define the functions $w^0(x), w^1(x), \ldots, w^{l+1}(x)$ as being solutions of the boundary value problems

$$\sum_{ij=1}^{n} a^{ij}(x, 0)w^m_{ij} + \sum_{i=1}^{n} b^i(x, 0)w^m_i + (d(x, 0) - c_{x_0}(x, 0)w^m = f^m(x, 0),$$

$$\sum_{k=1}^{n} \sigma^k(x, 0)w^m_k + \sigma(x, 0)w^m = 0, \quad x \in \partial\Omega_1,$$

$$\sum_{k=1}^{n} \tau^k(x, 0)w^m_k + \tau(x, 0)w^m = 0, \quad x \in \partial\Omega_2.$$

Let the solution of (1), (6) be of the kind $w(x, x_0) + \Sigma_{m=0}^{l+1} x_0^m w^m(x)$, where $w(x, x_0)$ is a solution of the boundary value problem

(19) $$Lw = x_0^{l+2} f^{l+2}(x, x_0),\ B_1 w = 0,\ B_2 w = 0.$$

Let $w^{\pm}(x, x_0)$ are solutions of the boundary value problems

(20) $$Lw^{\pm} = x_0^{l+2} f^{l+2}(x, x_0) \quad \text{in} \quad G^{\pm},$$

$$B_1 w^{\pm} = 0 \quad \text{on} \quad S_1^{\pm},\ B_2 w^{\pm} = 0 \quad \text{on} \quad S_2^{\pm},$$

where $G^+ = G \cap \{x_0 > 0\}$, $G^- = G \cap \{x_0 < 0\}$

$$S_i^+ = S_i \cap \{x_0 > 0\},\ S_i^- = S_i \cap \{x_0 < 0\}, \quad i = 1, 2.$$

For that purpose we consider the regularized boundary value problems

(21) $$L^{\varepsilon} w_{\varepsilon}^{\pm} = L w_{\varepsilon}^{\pm} + \varepsilon(\Delta w_{\varepsilon}^{\pm} - (\partial/\partial x^0) w_{\varepsilon}^{\pm}) = x_0^{l+2} f^{l+2}(x. x_0),$$

$$B_1 w_{\varepsilon}^{\pm} = 0,\ B_2 w_{\varepsilon}^{\pm} = 0,\ w_{\varepsilon}^{\pm}(x, 0) = 0,\ x \in \Omega.$$

The boundary value problems (21) satisfy the compatibility conditions of the data up to the order $l+1$ (see 1) and from the boundary value operators, the equations (21) and the derivatives of equations (21) up to the order l we obtain the result $D^{\alpha}_{xx_0} W_{\varepsilon}^{\pm}(x, 0) = 0$ for $x \in \Omega$, $|\alpha| \leqq l+1$. In the same way as in 2, $w_{\varepsilon}^{\pm}(x, x_0)$ might be proved to be uniformly bounded by constants independent of ε altogether with their derivatives up to the order $l+1$ in $\overline{G}^{\pm}$. Therefore, using Ascoli-Arzela theorem and a diagonalization argument, we can find that the boundary value problems (20) have solutions $w^{\pm}(x, x_0)$ of class $C^l(\overline{G^{\pm}})$, so that $D^{\alpha}_{xx_0} w^{\pm}(x, 0) = 0$ for $x \in \Omega$, $|\alpha| \leqq l$. Hence the function $w(x, x_0)$, $w(x, x_0) = w^+(x, x_0)$ in G^+, $w(x, x_0) = w^-(x, x_0)$ in G^- and $w(x, 0) = 0$ for $x_0 = 0$ is a solution of the boundary value problem (19) of class $C^l(\overline{G})$.

Let u_1, u_2 be two classical solutions of (1), (6) and $u = u_1 - u_2$ be a solution of the homogenious boundary value problem $Lu = 0$ in G, $B_1 u = 0$ on S_1, $B_2 u = 0$ on S_2. It can be proved that $u(x, x_0)$ can not attain a positive maximum for $x_0 \neq 0$, as in Theorem 1. When $x_0 = 0$, since the operator L is strictly elliptic, from the maximum principle of the elliptic equations it follows that $u(x, x_0)$ again can not attain a positive maximum. Analogously considering $-u(x, x_0)$ we can prove the uniqueness of the classical solution of the boundary value problem (1), (6). Theorem 2 is proved in the same way as Theorem 2.

P r o o f of Theorem 3. We consider the regularized boundary value problem

$$L^\varepsilon u^\varepsilon = Lu^\varepsilon + \varepsilon(\Delta u^\varepsilon - \partial/\partial x_0 u^\varepsilon) = f, \quad \varepsilon > 0,$$

$$Bu^\varepsilon = 0 \quad \text{on} \quad S, u^\varepsilon(x, 0) = 0 \quad \text{for} \quad x \in \Omega.$$

Since $|D^\alpha_{x_0} u^\varepsilon(x, x_0)||_{x_0=0} \leqq M_{23}$, like in Theorem 1 it can be proved that for u^ε Lemmas 2—5 hold. Hence, there exists a classical solution $u(x, x_0) \in C^l(\bar{G})$ of the boundary value problem (1), (7).

If u_1, u_2 are two classical solutions of (1), (7) then $u = u_1 - u_2$ is a classical solution of the homogenious boundary value problem. As in Theorem 1 it can be proved that $u(x, x_0)$ can not attain a positive maximum on $S, \Omega \setminus \Omega_0$, in the domain and on the upper base of the cylinder G. If $u(x, x_0)$ attains a positive maximum in a point $P \in \Omega_0$, then $P \notin \Omega \cap \partial\Omega_0$, hence $P \in \partial\Omega \cap \partial\Omega_0$ or P is an inner point for Ω_0 and this contradicts the maximum principle for elliptic equations.

Corollary 2. *Let the operator L satisfy the assumptions of Theorem 3 and in addition*

$$\sum_{i,j=1}^{n} a^{ij}(x, x_0)\xi^i\xi^j \geqq \mu|\xi|^2, \ \mu > 0 \quad \textit{for} \quad (x, x_0) \in \bar{G}.$$

In this case, as in Corollary 1, it is not necessary for the condition $c(x, x_0) \geqq 0$ *in* G', $c(x, x_0) \in C^2(G')$, *to be fulfilled. It is enough for* $c(x, x_0) \geqq 0$ *to be valid in* $\bar{G}$.

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