Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

CONTINUITY-LIKE PROPERTIES OF SET-VALUED MAPPINGS+

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Theorems of the following type are proved: If $F: X \to Y$ is an upper semicontinuous mapping then (under some conditions on X and Y) F is lower semicontionuous at the points of some dense G_δ subset of X. The results are then applied to obtain new information and new proofs of known results about points of continuity and single-valuedness of (multivalued) monotone operators and (multivalued) metric projections. As corollaries we get also some results concerning differentiability of convex functions at the points of some dense G_δ subset of their domains of continuity.

0. Introduction. Let X and Y be topological spaces and $F: X \to Y$ be a multivalued mapping from X into Y (i. e. Fx, for every $x \in X$, is a non-empty subset of Y). The mapping F is said to be *upper semicontinuous* (usc) at the point $x_0 \in X$, if for every closed set $Z \subset Y$ with $Fx_0 \cap Z = \emptyset$, the set $\{x \in X: Fx \cap Z = \emptyset\}$ contains an open neighbourhood of x_0. F is said to be *lower semicontinuous* (lsc) at $x_0 \in X$ if, for every open $U \subset Y$ with $Fx_0 \cap U \neq \emptyset$, the set $\{x \in X: Fx \cap U \neq \emptyset\}$ contains an open neighbourhood of x_0. These two notions are entirely independent from each other. Simple examples show that $F: X \to Y$ may be usc at x_0 without being lsc at the same point. There are, of course, mappings F which are lsc at some points of X but are not usc at the same points of X. This is why it is rather surprising that, if $F: X \to Y$ is usc at every point of X, then in many cases F must be lsc at the "majority" of points of X. Similarly, if F is lsc on X, then F must be usc at the points of some "fat" subset of X. The first results of this kind seem to have been given by Hill [15] and Kuratowski [21]. Subsequently the results of Hill and Kuratowski were improved and generalized in different directions (see Polak [27], Choquet [8], Fort [11], Weinstein [32]). A very nice example of what one can expect in this area is the following general result (Fort [12]):

Let the multivalued mapping $F: X \to Y$, acting from the topological space X into the metrizable space Y, have compact images $Fx \subset Y$ for every $x \in X$. If F is usc (lsc) *at every point $x \in X$, then F is* lsc (usc) *at the points of some residual* subset of X.*

In 1955 Fort [13] proved also another result of the same type and gave many interesting applications to some problems from Analysis and Topology Our main concern here is with the phenomenon (we call it "continuity pheno

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+) This work was done while the author was at the university of Frankfurt am Main as a Research Fellow of the Alexander von Humboldt Foundation.

* The set $A \subset X$ is called "residual" in X if its complement $X \setminus A$ is a countable union $\bigcup \{B_i : i = 1, 2, \ldots\}$ of sets B_i whose closures $\bar{B}_i$ in X have no interior points. In another words, A is residual if $X \setminus A$ is of the first Baire category in X. The residual set $A \subset X$ may be empty if X is of first Baire category.

menon") which stands behind the above mentioned results. In studying this phenomenon we obtain results which are then shown to be the basis of many known and some new facts concerning points of continuity and single-valuedness of (multivalued) metric projections (see for instance, 2.17,2.18) and of (multivalued) monotone operators (see 1.7, 2.14). As corollaries we also obtain some known results concerning (Fréchet or Gâteaux) differentiability of a given convex function at the points of some dense G_δ-subset of its domain of continuity. Our main tool is a new notion called "lower almost continuity" which is close in spirit to the usual lower semicontinuity but is not so restrictive.

Though some results of 1 can be considered as particular cases of those contained in 2, we decided to single them out in order to demonstrate, in a situation free of technical details, what role the new notion "lower almost continuity" plays in the continuity phenomenon and its applications.

For a given $F:X\to Y$ we define $D(F)=\{x\in X:\ Fx\neq\emptyset\}$. Very often we consider mappings (from now on "mappings", "setvalued mappings", "multivalued mappings" and "multifunction" will be used as synonyms) for which $D(F)=X$ but sometimes X and $D(F)$ may differ. For these cases the definition of "usc" and "lsc" at a given point x_0 of X must be changed as follows: F is said to be usc (lsc) at $x_0\in X$ if either $Fx_0=\emptyset$ or $F:D(F)\to Y$ is usc (lsc) at x_0 in the sense of the definitions given in the beginning of the paper. For a given $W\subset Y$, $U\subset X$ and $F:X\to Y$ we will denote by $F^{-1}(W)$ the set $\{x\in X:\ Fx\cap W\neq\emptyset\}$ and by $F(U)$ the set $\bigcup\{Fx:x\in U\}$.

1. Countable systems. Proposition 1.1. *Let $F:X\to Y$ be a mapping from the topological space X into the set Y and let A be a subset of Y. Then the set $H(A)=\{x\in X$: a) $Fx\cap A\neq\emptyset$* and b) for every open $U, x\in U$, there exists an open, non-empty $U'\subset U$ with $F(U')\cap A=\emptyset\}$ *is nowhere dense in X.*

Proof. For every set $X_1\subset X$ denote by cl X_1 the closure of X_1 and by int X_1 the set of all interior points of X_1. I. e. int $X_1=\{x\in X_1$: there exists an open set $U\subset X$ for which $x\in U\subset X_1\}$. The set $X_1\setminus$int (cl X_1) is evidently nowhere dense. It remains to mention that $H(A)=F^{-1}A\setminus$int (cl $F^{-1}A$).

We note here that this proposition is valid even in the case when some (or all) sets Fx, $x\in X$, are empty.

Definition 1.2. *We say that F is A-lower almost continuous* (A-lac) *at some point x_0 of X if x_0 does not belong to $H(A)$. Equivalently, F is* A-lac *at x_0 if either $F(x_0)\cap A=\emptyset$ or there exists an open neighbourhood U of x_0 such that $Fx\cap A\neq\emptyset$ for all x from some dense subset of U. The same can be expressed also in the following way: F is* A-lac *at x_0 if from $F(x_0)\cap A\neq\emptyset$ it follows that the closure in X of the set $F^{-1}(A)$ is dense in some open neigbourhood of x_0.*

Theorem 1.3. *Let $\alpha=\{A_i\}_{i\geq 1}$ be a countable family of subsets of the set Y. Then every mapping $F:X\to Y$ from the topological space X into Y is* A-lac (*for every A from α*) *at the points of some residual subset of X.*

Proof. Let $F:X\to Y$ and $A\in\alpha$. By 1.1 the set $H(A)$ is nowhere dense. Therefore the set $\bigcup\{H(A):A\in\alpha\}$ is of the first Baire category. The mapping F is A-lac (for every $A\in\alpha$) at every point of the complement to $\bigcup\{H(A): A\in\alpha\}$. The theorem is proved.

We give immediately some applications. Let E be a Banach space. We will denote its norm topology by "n" and its weak topology by "w". By E^* we denote, as usual, the dual of E. I. e. E^* is the set of all continuous linear functionals on E. The value of $x^* \in B^*$ at the point $x \in E$ is denoted by $\langle x, x^* \rangle$. For $x_0 \in E$ and $r>0$, we also define $B[x_0, r]=\{x \in E: \|x-x_0\| \leqq r\}$ and $B(x_0, r) =\{x \in E: \|x-x_0\|<r\}$.

Theorem 1.4 (Namioka [25], see also Corollary 1-16). *Let K be a weakly compact subset of the separable Banach space E. Then the indentity mapping id: $(K, w) \to (K, n)$ is continuous at the points of some dense G_δ subset of (K, w).*

Proof. Let $\alpha=\{A_i\}_{i \geqq 1}$ be a countable family of closed convex subsets of (E, n) such that their interiors provide a topological base for (E, n). Such a family exists because (E, n) is separable. According to Theorem 1.3 the identity map id $:(K. w) \to (K, n)$ is A-lac for every $A \in \alpha$ at the points of some residual subset of (K, w). Since (K, w) is a compact space, the latter residual set will contain some dense G_δ-subset of (K, w). It remains to prove that the identity id: $(K, w) \to (K, n)$ is continuous at every $x_0 \in (K, w)$ where it is A-lac for every A from α. But this is almost trivial. For $r>0$ find A with $x_0 \in A \subset B(x_0, r)$. Since id is A-lac at x_0, the w-closure in K of the set $\text{id}^{-1}(A)=K \cap A$ will contain an open neighbourhood of x_0 in (K, w). On the other hand, the set A, as any convex closed subset of (E, n), is also w-closed. This means that $A \cap K$ is already w-closed in (K, w) and the above w-open neighbourhood of x_0 in (K, w) must be contained in $A \cap K \subset B(x_0, r) \cap K$. This completes the proof.

For some very interesting corollaries of this result, the reader is referred to the original paper of Namioka [25]. Later, Namioka [26] proved that Theorem 1.4 is valid for arbitrary, not only separable, Banach spaces.

We turn to another application of Theorem 1.3.

Definition 1.5. *The (multivalued) mapping $T: E \to E^*$ is called monotone if $\langle x_1-x_2, x_1^*-x_2^* \rangle \geqq 0$ whenever $x_i^* \in Tx_i$, $i=1, 2$. T is called maximal monotone if its graph is not properly contained in the graph of any other monotone mapping.*

By means of Zorn's lemma it is not difficult to see that the graph of any monotone mapping is contained in the graph of some maximal monotone mapping.

Information about the properties and applications of monotone mappings can be found in Minty [23], Browder [6;7] and Brezis [5]. Examples of monotone mappings will appear in the proofs of some theorems below.

Theorem 1.6 (Kenderov and Robert [20]). *Let $T: E \to E^*$ be a monotone mapping. Then for all x from some residual subset of E either $Tx =\emptyset$ or* $\inf \{\|y\|: y \in Tx\}=\sup \{\|y\|: y \in Tx\}$ *I. e. for "almost all" $x \in E$ the image Tx lies on the surface of some ball centered at* 0.

Proof. Let B^* be the closed unit ball of E^* and $r>0$ be an arbitrary rational number. Put $A_r=rB^*$ and $\alpha=\{A_r\}_r$. By Theorem 1.3 T is A-lac for every $A \in \alpha$ at the points of some residual subset of E. It remains to prove that, for every point x from the above residual subset, the set Tx lies on the surface of some ball. To do this we need the following very useful technical lemma.

Lemma 1.7. *Let E be a Banach space, $T: E \to E^*$ be a monotone mapping and $A \subset E^*$ be convex and weak* compact. Suppose, further, the set $T^{-1}(A)=\{x \in E: Tx \cap A \neq \emptyset\}$ is dense in some open $U \subset E$. Then, for every $x \in U$, the set $Tx \subset A$. In particular, if T is A-lac at x_0, then $Tx_0 \subset A$.*

Proof of the lemma. Suppose the contrary: there exists some $x_1 \in U$ for which $Tx_1 \setminus A \neq \emptyset$. Take $x_1^* \in Tx_1 \setminus A$. Since A is weak* compact, the set $V := \{ e \in E : \langle e, x_1^* \rangle > \max \{ \langle e, x^* \rangle : x^* \in A \} \}$ is nonempty and open in (E, n). For every number $t > 0$ $tV = V$ and, since $x_1 \in U$, the open set $(x_1 + V) \cap U$ is nonempty. Since $T^{-1}(A)$ is dense in $U \ni x_1$, we must have some $x_2 \in (x_1 + V) \cap U \cap T^{-1}(A)$. In particular $Tx_2 \cap A \neq \emptyset$ and $x_2 = x_1 + e$ for some $e \in V$. This is already a contradiction because, by the monotonicity of $T : E \to E^*$, for every $x_2^* \in Tx_2 \cap A$ we have $\langle e, x_2^* \rangle \geq \langle e, x_1^* \rangle > \max \{ \langle e, x^* \rangle : x^* \in A \} \geq \langle e, x_2^* \rangle$. Lemma 1.7 is proved.

We turn back to the proof of Theorem 1.6. Suppose T is A-lac for every $A \in \alpha$ at x_0 and suppose that $t_1 := \inf \{ \| y \| : y \in Tx_0 \} < \sup \{ \| y \| : y \in Tx_0 \} =: t_2$. Take some rational number $r > 0$ from the open interval (t_1, t_2), Evidently $Tx_0 \cap rB^* \neq \emptyset$. By the above lemma $Tx_0 \subset rB^*$. This implies $t_2 \leq r$ which is a contradiction.

Corollary 1.8 (Kenderov [16; 17]). *Suppose that the Banach space E admits an equivalent norm whose dual is strictly convex. If $T : E \to E^*$ is a monotone mapping, then there is a dense G_δ subset S of (E, n) such that, for all $x \in S$, Tx contains at most one point.*

Proof. Without loss of generality we can assume that T is a maximal monotone mapping. Then Tx is convex for every $x \in E$ (otherwise T would not be maximal). On the other hand, by the definition of strict convexity, the only nonempty convex subsets of the surface of a strictly convex ball are the singletons. Therefore, it follows from Theorem 1.6 that for all x from some residual subset of (E, n) the set Tx contains at most one point.

Corollary 1.9 (Asplund [3]). *Let $f : E \to R$ be a real-valued continuous convex function defined in the Banach space E. Suppose that E admits an equivalent norm whose dual is strictly convex. Then f is Gâteaux differentiable at the points of some dense G_δ subset of E.*

Proof. It is easy to see that the subgradient $\partial_f : E \to E^*$ assigning to each $x_0 \in E$ the nonempty set $\partial(x_0) = \{ x^* \in X^* : f(x) - f(x_0) \geq \langle x - x_0, x^* \rangle$ for every $x \in E \}$ is a monotone mapping. By 1.8 $\partial : E \to E^*$ is single-valued at the points of some dense G_δ subset of E. Since it is known (see for instance Moreau [24]) that $f : E \to R$ is Gâteaux differentiable at some $x_0 \in E$ if and only if the subgradient $\partial : E \to E^*$ is single-valued at x_0, the proof is completed.

Remark 1.10. Using this approach we are also able to prove the result of Asplund [3, Theorem 1] about Fréchet differentiability of a given convex function $f : E \to R$ at the points of some dense G_δ subset of E.

Theorem 1.11 (Robert [28]). *Let E^* be separable relative to its norm topology. Then every monotone mapping $T : E \to E^*$ is single-valued and norm-to-norm upper semi continuous at the points of some dense G_δ subset of $(\text{int } D(T), n)$, where int Z, for some $Z \subset E$, means the norm interior of the set Z.*

Proof. Let $\{x_i\}_{i \geq 1}$ be a dense subset of (E^*, n^*), where n^* is the norm topology of E^*. Let B^* be the unit ball of the dual norm in E^* and $r > 0$ be an arbitrary rational number. Define $A(i, r) = x_1^* + rB^*$ and consider the countable system $\alpha = \{ A(i, r) : i = 1, 2, 3, \ldots ;\ r > 0$ and rational$\}$. By Theorem 1.3 T is A-lac for every $A \in \alpha$ at the points of some dense G_δ subset of (int D (T), n). Having in mind Lemma 1.7 and the fact that α forms a topological base for (E^*, n^*) we come to the conclusion that T is single-valued and norm-to-norm upper semi-continuous at the points of the above dense G_δ subset of (int $D(T)$, n).

We return to the general study of the continuity phenomenon. In order to explain what role "upper semicontinuity (usc)" and "lower semicontinuity (lsc)" play in this phenomenon, we need a definition.

Definition. 1.12. *Let A be a subset of the set Y and $F:X\to Y$ be a mapping from the topological space X into Y. We say that F is A-lower semicontinuous (A-upper semicontinuous) at the point $x_0\in X$ if either $Fx_0=\emptyset$ or (when $Fx_0\neq\emptyset$) from $Fx_0\cap A\neq\emptyset$ ($Fx_0\cap A=\emptyset$) it follows that the set $\{x\in X:Fx\cap A\neq\emptyset\}$ ($\{x\in X:Fx\cap A=\emptyset\}$) contains an open neighbourhood of x_0 in X. For brevity we will write A-*usc *and A-*lsc *instead of "A-upper semicontinuity" and "A-lower semicontinuity".*

It is clear that, when Y is a topological space, the mapping $F:X\to Y$ is lsc (in the usual sense) at x_0 if and only if F is A-lsc for every open $A\subset Y$. Analogously, F is usc at some x_0 if and only if F is A-usc for every closed $A\subset Y$. It is also clear that A-lower semicontinuity is a stronger property than the introduced in Definition 1.2 A-lower almost continuity. Nevertheless, we have

Proposition 1.13. *Let $A\subset Y$ and $F:X\to Y$ be A-*usc *at every point x of the topological space X. If F is A-*lac *at some $x_0\in X$, then it is A-*lsc *at x_0.*

Proof. If $Fx_0\cap A=\emptyset$, there is nothing to prove. Suppose $Fx_0\cap A\neq\emptyset$. Since F is A-lac at x_0, the set $F^{-1}(A)$ is dense in some open U, $U\ni x_0$. We prove that $U\subset F^{-1}(A)$. Indeed, if this is not the case, there would exist some $x_1\in U$ with $Fx_1\cap A=\emptyset$. Since F is A-usc at x_1 we find some open U_1, $x_1\in U_1\subset U$, such that $Fx\cap A=\emptyset$ for every $x\in U_1$. But this is a contradiction because $F^{-1}(A)=\{x\in X:Fx\cap A\neq\emptyset\}$ is dense in $U_1\subset U$.

Definition 1.14 (Arhangel'skii [2]). *The family $\alpha=\{A\}$ of subsets of the topological space Y is called a net if for every $y_0\in Y$ and every open $U\subset Y$, $U\ni y^0$, there exists some $A\in\alpha$ with $y_0\in A\subset U$.*

Theorem 1.15. *Let the space Y have two topologies t_1 and t_2 and let (Y,t_2) have a countable net $\alpha=\{A_i\}_{i\geq 1}$ consisting of t_1-closed sets. If $F:X\to Y$ is t_1-*usc *at every $x\in X$, then $F:X\to Y$ is t_2-*lsc *at the points of some residual subset of the topological space X.*

Proof. From Theorem 1.3 we deduce that F is A-lac for every $A\in\alpha$ at the points of some residual subset ot X. By Proposition 1.13 F will be also A-lsc for every $A\in\alpha$ at the same points. Since $\alpha=\{A_i\}_{i\geq 1}$ is a net for (Y,t_2) we obtain that F is t_2-lsc at the points of this residual set.

Corollary 1.16 (Alexiewicz and Orlicz [1]). *Suppose E is a separable normed space and $f:X\to(E,w)$ is a continuous single-valued mapping. Then $f:X\to(E,n)$ is continuous at the points of some residual subset of X.*

Proof. Let $\{x_i\}_{i\geq 1}$ be dense in (E,n) and $r>0$ be a rational number. Consider the set $A(i,r)=x_i+rB$ where B is the (closed) unit ball of E. The countable family $\alpha=\{A(i,r):i=1,2,3,\ldots;\ r$ positive and rational$\}$ is evidently a net for the norm topology in E. This net consists of weak-closed sets and we can apply Theorem 1.15 for $t_1=w$ and $t_2=n$.

Remark 1.17. Theorem 1.15 contains in essence the main result of Fort from [13]. In the paper of Fort the mapping F is supposed to be single-valued.

We would also like to note the connection between our Theorem 1.3 and the famous "interior mapping principle" of Banach: *If $T:E\to G$ is a continuous* *linear map from the Banach space E onto the Banache space G, then the image TU of any open $U\subset E$ is an open subset of G.* Usually (see Dunford and Schwartz [10, 55—57]) the proot of this fact consists of two important steps and the first one is to show that the closure $\overline{TB}$ of the set TB (B being the closed unit ball of E) contains $0\in G$ in its interior. This follows from Theorem 1.3 if we put $X:=G$, $Y:=E$, $F:=T^{-1}$ and $\alpha=\{nB:n=1,2,3,\ldots\}$, where B is the closed unit ball of E.

2. General case. We need here a generalization of the notions introduced in 1.

Definition 2.1. *Let $F:X\to Y$ be a (multivalued) mapping from the toppological space X into the set Y and let $\lambda=\{A\}$ be a collection of subsets of Y. We say that F is λ-lac (λ-lower almost continuous) at some $x_0\in X$ if there exists an open U, $U\ni x_0$, such that whenever $Fx_0\cap A\neq\emptyset$ for some $A\in\lambda$ the set $\overline{F^{-1}(A)}$ (the closure in X of $F^{-1}(A)$) contains U.*

In the case when λ consists of only one A, λ-lower almost continuity at x_0 coincides with A-lower almost continuity at x_0. As the next example shows the assertion "*F is λ-lac at x_0*" is in general stronger than "*F is A-lac at x_0 for every $A\in\lambda$*". Suppose $X=Y=E$ is a Banach space and $F:X\to Y$ is the identity map. Put λ equal to the collection of all open $U\ni x_0$, $U\subset Y$. Then F is obviously U-lac at x_0 for every $U\in\lambda$, but it is not λ-lac at x_0.

Let us also mention that, according to our definition, F is λ-lac at some $x_0\in X$ if $Fx_0\cap A=\emptyset$ for every $A\in\lambda$. In particular, this is so when $Fx_0=\emptyset$.

The set of points in X where $F:X\to Y$ is not λ-lac will be denoted by $H(\lambda)$. Therefore $H(\lambda)=\{x\in X$: for every open $U\subset X$, $U\ni x$, there exist an $A\in\lambda$ and a nonempty open $U'\subset U$ such that $Fx\cap A\neq\emptyset$ but $F(U')\cap A=\emptyset\}$.

Clearly $H(\lambda)\supset H(A)$ for every $A\in\lambda$. However, as the example above shows, $H(\lambda)$ can be bigger than $\bigcup\{H(A):A\in\lambda\}$.

We give here sufficient condititions for the set $H(\lambda)$ to be nowhere dense. It is convenient for us to describe first one general construction which can be repeated infinitely many times if the set $H(\lambda)$ is dense in some open subset of X. The sufficient conditions we have in mind are of such a nature that they, in fact, forbid the unlimited step by step iteration of this basic construction. Thus the simultaneous fulfillment of both the *sufficient conditions* and the assumption *$H(\lambda)$ is somewhere dense* is impossible and leads to a contradiction.

Let us now be more precise and consider a mapping $F:X\to Y$ from the topological space X into the set Y, and a collection $\lambda=\{A\}$ of subsets of Y. Define as above the set $H(\lambda)$ and put $H:=H(\lambda)$.

Basic construction 2.2. *If H is dense in some nonempty open $U_0\subset X$, then there exist sequences of open sets $\{U_i\}_{i\geq 0}$ in X, of elements $\{x_i\}_{i\geq 0}$ of X and of members $\{A_i\}_{i\geq 0}$ of the collection $\lambda=\{A\}$ such that for every nonnegative integer i:*

1. $U_{i+1}\subset U_i$,
2. $x_i\in U_i$,
3. $Fx_i\cap A_i\neq\emptyset$,
4. $F(U_{i+1})\cap A_i=\emptyset$.

Remark 2.3. It is clear from 2, 3 and 4 that $x_i\neq x_j$ whehever $i\neq j$. The conditions 1, 2, 3 and 4 imply also that U_{i+1} is a proper subset of U_i ($x_i\in U$ but $x_i\notin U_{i+1}$). It is easy also to see that $A_i\neq A_j$ for $i\neq j$.

Proof of 2.2. We proceed by induction with respect to the index i. Let H be dense in the nonempty and open set $U_0 \subset X$. Take $x_0 \in U_0 \cap H$. By the definition of H there will exist an $A_0 \in \lambda$ and an open nonempty $U' \subset U_0$ with $F_{x_0} \cap \neq A_0 \emptyset$ and $F(U') \cap A_0 = \emptyset$. Thus we have defined x_0, A_0 and U_0. Put $U_1 := U'$. Clearly 1, 2, 3 and 4 are fulfilled for $i=0$. Suppose now that $\{x_j : j=0, 1, \dots, n\}$, $\{A_j : j=0, 1, \dots, n\}$, $\{U_j : j=0, 1, \dots, n+1\}$ have already been constructed so that 1, 2, 3 and 4 are satisfied for $0 \leq i \leq n$. Since H is dense in U_{n+1} we choose $x_{n+1} \in H \cap U_{i+1}$. By the definition of H, there will exist $A_{n+1} \in \lambda$ and $U' \subset U_{n+1}$, U' open and nonempty, such that $Fx_{n+1} \cap A_{n+1} \neq \emptyset$ but $F(U') \cap A_{n+1} = \emptyset$. We put now $U_{n+2} := U'$ and see that the requirements 1—4 are satisfied with $0 \leq i \leq n+1$. This completes the proof.

Lemma 2.4 *If $H(\lambda)$ is dense in U_0, then $F(U_0) \cap A \neq \emptyset$ for infinitely many different $A \in \lambda$.*

Proof. Trivial from 2.2 and 2.3.

Lemma 2.5. *If $F(U)$, where U is an open subset of X, intersects only finitely many members of $\lambda = \{A\}$, then $H(\lambda)$ cannot be dense in U.*

Proof. Trivial from 2.2.

As a corollary we get the following assertion containing 1.1 as a particular case.

Corollary 2.6. *If collection $\lambda = \{A\}$ is finite, then $H(\lambda)$ is nowhere dense.*

More generally, we have

Proposition 2.7. *Let $F: X \to Y$ be a mapping from the topological space X into the set Y and let $\lambda = \{A\}$ be a system of subsets of Y. Suppose, further, the set $\{x \in X:$* there exists an open $U \ni x$ with $F(U) \cap A \neq \emptyset$ for only a finite number of members $A \in \lambda\}$ *is dense in X. Then $H(\lambda)$ is nowhere dense in X.*

Proof. By hypothesis and 2.5, every open $U_0 \subset X$ contains an open subset U in which H is not dense. This observation completes the proof.

We want to show now how 2.7 "works" in concrete situations.

Definition 2.8. *The mapping $F: X \to Y$, where X and Y are topological spaces, is said to be lower almost continuous* (lac) *at $x_0 \in X$ if F is W*-lac *at x_0 for every open $W \subset Y$.*

Theorem 2.9. *Let X be a topological, Y be a metrizable space and $F: X \to Y$ be a mapping. If the set $L = \{x \in X: Fx$* is a nonempty compact subset of Y and F is upper semicontinuous at $x\}$ *is dense in X, then F is lac at the points of some residual subset of X.*

Proof. According to a theorem of A. Stone [30] the metrizable space Y is paracompact and has a topological base $\alpha = \{\lambda_i\}_{i \geq 1}$ consisting of locally finite coverings λ_i of open subsets of Y. It is enough to prove that the set $\bigcup \{H(\lambda_i): i = 1, 2, \dots\}$ is of the first Baire category. Therefore the next lemma completes the proof.

Lemma 2.10. *The set $H(\lambda_i)$ is nowhere dense in X.*

Proof. By 2.5 and the hypothesis of the theorem it suffices to show that for every $x_0 \in L$ there exists some open $U \subset X$, $U \ni x_0$, with $F(U) \cap A \neq \emptyset$ for not more that finitely many $A \in \lambda_i$. Consider the collection $\bar{\lambda}_i = \{\bar{A}\}$ of closures in Y of the sets from λ_i. $\bar{\lambda}_i$ is again a locally finite collection and therefore the set $G = \bigcup \{\bar{A} : Fx_0 \cap \bar{A} = \emptyset\}$ is closed in Y. Since F is usc at $x_0 \in L$, there exists some open set $U \subset X$, $U \ni x_0$, such that $Fx \cap G = \emptyset$ for every

$x \in U$. This means that, whenever $x \in U$, Fx intersects only those $\bar{A} \in \bar{\lambda}_i$ for which $Fx_0 \cap \bar{A} \neq \emptyset$. Since $\bar{\lambda}_i$ is locally finite and Fx_0 is compact, there exist only a finite number of sets $\bar{A} \in \bar{\lambda}_i$ with $Fx_0 \cap \bar{A} \neq \emptyset$. It remains to apply Proposition 2.7. The lemma is proved.

As a particular case we get the following result

Theorem 2.11 (Fort [12]). *If $F:X \to Y$ is an upper semicontinuous mapping with compact and nonempty images from the topological space X into the metrizable space Y, then F is lower semicontinuous at the points of some residual subset of X.*

Proof. This follows immediately from 2.9 (with $L = X$) and the next general fact which shows the roles played by "semicontinuity" and "almost semicontinuity" in the continuity phenomenon.

Lemma 2.12. *If $F:X \to Y$ is* usc *at every point of X and if F is* lac *at $x_0 \in X$, then F is* lsc *at x_0.*

Indeed, take some open $V \subset Y$ with $Fx_0 \cap V \neq \emptyset$. By lower almost continuity of F at x_0 the closure in X of the set $\{x \in X: Fx \cap V \neq \emptyset\}$ will contain some open $U \ni x_0$. It is enough to show that $U \subset F^{-1}(\bar{V})$. Suppose this is not so and take $x_1 \in U$ with $Fx_1 \cap \bar{V} = \emptyset$. Since F is usc at x_1, we have $Fx \cap \bar{V} = \emptyset$ for all x from some open $U_1 \ni x_1$, $U_1 \subset U$. This is a contradiction because the set $F^{-1}(V)$ is dense in $U \supset U_1$ and for some $\bar{x} \in U_1$ we must have $F\bar{x} \cap V \neq \emptyset$.

We will now use 2.9 to provide new proofs of known results as well as to get new results which, it seems to us, can not be obtained directly from the theorem of Fort 2.10. We start with

Proposition 2.13 (Asplund [3]). *If the continuous convex function $f:E \to R$, defined on the Banach space E, is Fréchet differentiable at the points of some dense subset of E, then f is Fréchet differentiable at the points of some dense G_δ subset of E.*

Proof: Consider the subgradient $\partial:E \to E^*$ of the convex function f (the definition of ∂ is given here in the proof of 1.9). It is a mapping which is norm-to-norm usc and single-valued at every point where f is Fréchet differentiable (the latter can be taken as an equivalent definition of Frechet differentiability at some point $x_0 \in E$; see Asplund and Rockafellar [4]). We apply 2.9 to $\partial:E \to E^*$ and get that ∂ is norm-to-norm lac at the points of some dense G_δ subset of (E, n). The rest of the proof is contained in the following lemma because every subgradient ∂ is a monotone mapping.

Lemma 2.14. *If the monotone mapping $T:E \to E^*$ is norm-to-norm* lac *at some $x_0 \in E$, then T is single-valued and norm-to-norm* usc *at x_0.*

Proof. Let V be an open ball in E^* with $Tx_0 \cap V \neq \emptyset$. By the hypothesis, the set $T^{-1}(V)$ is dense in some open $U \ni x_0$. Since the closure $\bar{V}$ in E^* is a weak*-compact subset of E^* we can apply 1.7 to get that $T(U) \subset \bar{V}$. In particular $Tx_0 \subset \bar{V}$. Since $\bar{V}$ is an arbitrary closed ball in E^*, Tx_0 is a singleton and $T:E \to E^*$ is norm-to-norm usc at x_0. The proof of 2.14 and of 2.13 is finished.

We discuss now another application of 2.9. Let E be a normed space and $M \subset E$. The so-called "metric projection generated by M" is a multivalued mapping $P_M:E \to M$ assigning to each $x \in E$ the (possibly empty) set $P_M(x) = \{y \in M: \|x-y\| = \inf \{\|x-z\|: z \in M\}\}$. In [29] Stechkin has proved that, in several cases, the multivalued metric projection $P_M:E\to M$ has empty or one-point images for the points of some dense G_δ subset of (E, n). We will prove now something more. Namely, the metric projections involved in the case of Stechkin are not only (not more than) single-valued at the points of some dense G_δ subsets of (E, n), but also usc at these points.

Definition 2.15. *The Banach space E is called locally uniformly convex if for every sequence $\{x_i\}_{i\geq 0}$ in E with $\|x_i\|=1$, $i=0, 1, 2, \ldots$, and $\lim_{i\to\infty}\|x_0+x_i\|=2$, it follows that $\lim_{i\to\infty} x_i=x_0$.*

We need also a result of Zhivkov [33] (see also Deutsch and Lambert [9. Theorem 2.6]):

Lemma 2.16. *Let M be a subset of the locally uniformly convex normed space E and let y_0 belongs to $P_M(x_0)=\{x\in M: \|x-x_0\|=\inf\{\|x_0-z\|: z\in M\}\}$. Then at every point of the set $\{x=(1-t)x_0+ty_0: 0<t\leq 1\}$ the metric projection $P_M:E\to M$ is both single-valued and norm-to-norm* usc.

Theorm 2.17. *Let M be a subset of the locally uniformly convex Banach space E and $P:E\to M$ be the metric projection generated by M. Then P is (not more that) single-valued and norm-to-norm* usc *at all points of some dense G_δ subset of (E, n).*

Proof. Note first that the set L of points of the type $x=(1-t)x_0+ty_0$, where $0<t\leq 1$ and $y_0\in Px_0$, is norm dense in $D(P)=\{x\in E: Px\neq\emptyset\}$. Lemma 2.16 shows that we can apply Theorem 2.9 to the mapping $P:(D(P), n)\to(M,n)$. Thus the map $P:D(P)\to M$ is norm-to-norm lac at the points of some residual subset of $(D(P), n)$. Since, by definition, the metric projection $P:E\to M$ is usc at every point outside $D(P)$, the remainder of the proof is contained n the following lemma.

Lemma 2.18. *Let E, M and P be as above. If $P:D(P)\to M$ is norm-to-norm* lac *at some $x_0\in D(P)$, theu it is single-valued and norm-to-norm* usc *at x_0.*

Proof. Let V be some open ball in E with $Px_0\cap V\neq\emptyset$. Since P is lac at x_0, the set $P^{-1}(V)$ is dense in some open $U\subset(D(F), n)$, $x_0\in U$. It suffices to prove that, for every $x\in U$, the set Px lies in the closure $\bar{V}$ of V in (E,n) Suppose this is not so and take $\bar{x}\in U$, $\bar{y}\in P\bar{x}$ with $\bar{y}\notin\bar{V}$. Take the number $t>0$ so small that $x_1=(1-t)\bar{x}+t\bar{y}$ still belongs to the open set U. It is not difficult to see that $Px_i=\bar{y}$. By 2.16 $P:D(P)\to M$ is norm-to-norm usc at x_1 and therefore, for some open $U_1\subset U$, $U_1\ni x_1$, there follows $P(U_1)\cap\bar{V}=\emptyset$. However, this is a contradiction because $P^{-1}(V)$ is dense in $U\supset U_1$ and, for at least one $z\in U_1$, we will have $\emptyset\neq Pz\cap V\subset P(U_1)\cap\bar{V}=\emptyset$. This completes the proof of 2.18 and 2.17.

Up to now most of the results in 2 depended in one or another way on the fact that the corresponding collections λ_i, $i=1, 2, \ldots$, were locally finite and formed a base $\alpha=\{\lambda_i\}_{i\geq 1}$ for the topological space Y. Roughly speaking, this is the case only when Y is metrizable. We find this situation rather restrictive and want to show one possible way to gain more freedom in the choice ot the space Y. The price one must pay for more freedom in the choice of Y is less freedom in the choice of X. However, we feel that the price we are going to pay is not too high. The restrictions on X we have in mind are not severe and are quite natural. Under this mild restriction on X we will be able to get rid of the unpleasant hypothesis, appearing in a number of our results, that the images of $F:X\to Y$ must be compact on a dense subset of X.

Let $\gamma=\{W\}$ be an open covering of the topological space X. The set $S\subset X$ is said to be γ-small if its closure $\bar{S}$ is contained in some W, $W\in\gamma$.

Definition 2.19 (Frolik [14]). *The space X is called strongly countably complete if it has a countable system $\{\gamma_i\}_{i\geq 1}$ of open coverings γ_i satisfying the requirement: every decreasing seguence of nonempty closed sets $\{S_i\}_{i\geq 1}$, where S_i is γ_i-small, $i=1, 2, \ldots$, has nonempty intersection.*

The class of strongly countably complete spaces is very large. It contains all complete metric spaces as well as all (locally) countably compact spaces. Every residual subset of a strongly countably complete space contains a (non-, empty) dense G_δ subset.

Definition 2.20. *The collection $\lambda=\{A\}$ of subsets of the topological space Y is called countably conservative if $\overline{\bigcup\{A : A\in\lambda'\}}=\bigcup\{\bar{A} : A\in\lambda'\}$ for every countable subcollection $\lambda'\subset\lambda$.*

Prorosition 2.21. *Let $F:X\to Y$ be an* usc *set-valued mapping from the strongly countably complete space X into the space Y and let $\lambda=\{A\}$ be a countably conservative collection of subsets of Y. Then the set* (see 2.2) *$H(\bar{\lambda})$, where $\bar{\lambda}=\{\bar{A}\}$ consists of the closures of A in Y, is nowhere dense in X.*

Proof. Let $\{\gamma_i\}_{i\geq 1}$ be a sequence of open coverings of X with respect to which X is strongly countably complete and suppose that $H(\bar{\lambda})$ is dense in some open $U_0\subset X$, $U_0\neq\emptyset$. In a manner similar to that used in 2.2, we construct three sequences $\{U_i\}_{i\geq 0}\subset X$. $\{A_i\}_{i\geq 0}\subset\lambda$ and $\{x_i\}_{i\geq 0}\subset X$ so that for every integer $i\geq 0$:

1. $\bar{U}_{i+1}\subset U_i$;
2. $x_i\in U_i$;
3. $Fx_1\cap\bar{A}_1\neq\emptyset$;
4. $F(U_{i+1})\ \bar{A}_i=\emptyset$;
5. $\bar{U}_{i+1}$is γ_{i+1}—small.

We observe now, by 1, 2 and 5, that the "tails" $S_n=\overline{\{x_i\}}_{i\geq n}$, $n=1, 2, 3, \ldots$, form a decreasing sequence of γ_n—small closed subsets of X. Since X is countably, strongly complete, the set $\bigcap\{S_n : n\geq 1\}\neq\emptyset$. Let $z_0\in\bigcap\{S_n : n\geq 1\}$. From 2 and 1 we see that $z_0\in U_i$, $i\geq 1$. By 4 $Fz_0\cap\bar{A}_i=\emptyset$ for every $i\geq 1$. Since λ is countably conservative, the set $A:=\bigcup\{\bar{A}_i : i\geq 1\}$ is closed and we have $Fz_0\cap A=\emptyset$. As F is usc at x_0 we find an open V, $z_0\in V$, such that $F(V)\cap A=\emptyset$. This is a contradiction because, as a neighbourhood of z_0, the open set V must contain some x_i and by 3 we would have $\emptyset\neq Fx_i\cap\bar{A}_i\subset F(V)\cap A=\emptyset$.

Theorem 2.22. *Let t_1 and t_2 be two topologies in the space Y and let $\lambda_i=\{A\}$, $i=1. 2, 3, \ldots$, be t_1-countably conservative collections of t_1-closed sets in Y. Suppose that $\alpha=\{\lambda_i\}_{i\geq 1}$ forms a net in Y* (see Definition 1.14) *for the topology t_2. Then every t_1-usc mapping $F: X\to Y$ from the strongly countably complete space X into Y must be t_2-*lsc *at the points of some dense G_δ subset of X.*

Proof. From the previous proposition, the set $H:=\bigcup\{H(\bar{\lambda}_i) : i\geq 1\}$ is of the first Baire category in X. We prove now that F is t_2-lsc at every $x_0\notin H$. Take an arbitrary open $V\subset(Y, t_2)$ with $Fx_0\cap V\neq\emptyset$ and let $A\in\alpha$ be such that $A\subset V$ and $Fx_0\cap A\neq\emptyset$. Since $x_0\notin H$, $F^{-1}(A)$ will be dense in some open

$U \subset X$, $U \ni x_0$. It is enough to prove that $Fx \cap A \neq \emptyset$ for every $x \in U$. Suppose the contrary: $Fx_1 \cap A = \emptyset$ for some $x_1 \in U$. Since the set A is t_1-closed and since F is t_1-usc at x_1, there will exist an open $U_1 \subset U$, $U_1 \ni x_1$, with $F(U_1) \cap A = \emptyset$. But this is a contradiction because $F^{-1}(A)$ is dense in the open set $U \supset U_1$.

Corollary. 2.23. *Let X be a strongly countably complete space, Y be a space admitting a net $\alpha = \{\lambda_i\}_{i \geq 1}$ of countably conservative collections λ_i, $i = 1, 2, \ldots$, and $F: X \to Y$ be an* usc (*multivalued*) *mapping. Then F is* lsc *at the points of some dense G_δ subset of X.*

Proof. Put $t_1 = t_2$ in 2.22.

Remark 2.24. a) For the particular case when Y is a metrizable space corollary 2.23, together with an application to the theory of metric projections, can be found in Kenderov [18, 19].

b) The class of spaces Y admitting "σ-countably conservative net" (as in Corollary 2.23) is very large. Since the image under continuous and closed (single-valued) mappings of any conservative collection is again a conservative collection, the closed and continuous image Y of any metrizable space Z admits "σ-conservative net" and belongs to this class. Among the latter spaces Y there are such (see Lašnev [22], Stricklen [31]) no point of which has a countable base of neighbourhoods.

c) In this paper we studied only one part of the continuity phenomenon. Generally speaking, we gave sufficient conditions, based on the notion "lower almost continuity", in order that the usc mapping $F: X \to Y$ be lsc at some points of X. It is also possible to introduce the notion "upper almost continuity" and to use it in analogous way in order to prove that, under some conditions, every lsc mapping has points of upper semicontinuity.

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Received 21.4. 1981