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ON THE BOCHNER CURVATURE TENSOR IN AN ALMOST HERMITIAN MANIFOLD

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We prove a classification theorem for RK-manifolds with linear dependence between invariants of an antiholomorphic plane in the tangent space. As a consequence we find a characteristic condition for an RK-manifold to be of pointwise constant antiholomorphic sectional curvature.

1. Introduction. Let M be a $2n$-dimensional almost Hermitian manifold, $n\geqq 3$, with metric g and almost complex structure I and let ∇ be the covariant differentiation on M. The curvature tensor R is defined by

$$R(X, Y, Z, U)=g(\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z, U)$$

for $X, Y, Z, U\in\mathfrak{X}(M)$. The manifold is said to be an RK-manifold [5] if

$$R(X, Y, Z, U)=R(IX, IY, IZ, IU)$$

for all $X, Y, Z, U\in\mathfrak{X}(M)$. In this paper we trate for simplicity only the case of an RK-manifold although one can made analogous considerations for an arbitrary almost Hermitian manifold.

Let E_i, $i=1,\dots, 2n$ be a local orthonormal frame field. The Ricci tensor S and the scalar curvature $\tau(R)$ are defined by

$$S(X, Y)=\sum_{i=1}^{2n} R(X, E_i, E_i, Y), \qquad \tau(R)=\sum_{i=1}^{2n} S(E_i, E_i).$$

Analogously we set

$$S'(X, Y)=\sum_{i=1}^{2n} R(X, E_i, IE_i, IY), \qquad \tau'(R)=\sum_{i=1}^{2n} S'(E_i, E_i).$$

We note that S and S' are symmetric and $S(X, Y)=S(IX, IY)$, $S'(X, Y)=S'(IX, IY)$.

The Bochner curvature tensor B [4] for M is defined by

$$B=R-\frac{1}{8(n+2)}(\varphi+\psi)(S+3S')-\frac{1}{8(n-2)}(3\varphi-\psi)(S-S')$$

$$+\frac{\tau(R)+3\tau'(R)}{16(n+1)(n+2)}(\pi_1+\pi_2)+\frac{\tau(R)-\tau'(R)}{16(n-1)(n-2)}(3\pi_1-\pi_2),$$

where φ, ψ, π_1 and π_2 are defined by

SERDICA Bulgaricae mathematicae publicationes. Vol. 9, 1983, p. 168–171.

$$\varphi(Q)(X, Y, Z, U)=g(X, U)Q(Y, Z)-g(X, Z)Q(Y, U)$$
$$+g(Y, Z)Q(X, U)-g(Y, U)Q(X, Z),$$
$$\psi(Q)(X, Y, Z, U)=g(X, IU)Q(Y, IZ)-g(X, IZ)Q(Y, IU)$$
$$+g(Y, IZ)Q(X, IU)-g(Y, IU)Q(X, IZ)$$
$$-2g(X, IY)Q(Z, IU)-2g(Z, IU)Q(X, IY),$$
$$\pi_1(X, Y, Z, U)=g(X, U)g(Y, Z)-g(X, Z)g(Y, U),$$
$$\pi_2(X, Y, Z, U)=g(X, IU)g(Y, IZ)-g(X, IZ)g(Y, IU)-2g(X, IY)g(Z, IU).$$

By a plane we mean a 2-dimensional linear subspace of the tangent space $T_p(M)$ of M in p. A plane α is said to be holomorphic (resp. antiholomorphic) if $I\alpha=\alpha$ (resp. $I\alpha$ is perpendicular to α).

A tensor field T of type (0.4) is said to be an LC-tensor if it has the properties:

1) $T(X, Y, Z, U)=-T(Y, X, Z, U)$,

2) $T(X, Y, Z, U)=-T(X, Y, U, Z)$,

3) $T(X, Y, Z, U)+T(Y, Z, X, U)+T(Z, X, Y, U)=0$.

We need the following lemma.

Lemma [1]. *Let M be a $2n$-dimensional almost Hermitian manifold, $n\geqq 2$. Let T be an* LC-*tensor, satisfying the conditions*:

1) $T(X, Y, Z, U)=T(IX, IY, IZ, IU)$,

2) $T(x, y, y, x)=0$, *where $\{x, y\}$ is a basis of any holomorphic or antiholomorphic plane.*

Then $T=0$.

In Section 2 we shall prove the following theorem.

Theorem. *Let M be a $2n$-dimensional* RK-*manifold, $n\geqq 3$, which satisfies*

(1.1) $\lambda R(x, y, y, x)+\mu(S(x, x)+S(y, y))+\nu(S'(x, x)+S'(y, y))=c(p)$

for each point $p\in M$ and for all unit vectors $x, y\in T_p(M)$ with $g(x, y)=g(x, Iy)=0$, where λ, μ, ν are constants, $(\lambda, \mu, \nu)\neq(0, 0, 0)$ and $c(p)$ does not depend on x, y. Then

1) *if $\lambda=0$, then*

$$\mu S+\nu S'=\frac{\mu\tau(R)+\nu\tau'(R)}{2n}g;$$

2) *if $\lambda\neq 0$, then M has vanishing Bochner curvature tensor and the tensor $((n+1)\lambda+2(n^2-4)\mu)S+(2(n^2-4)\nu-3\lambda)S'$ is proportional to the metric tensor*:

$$((n+1)\lambda+2(n^2-4)\mu)S+(2(n^2-4)\nu-3\lambda)S'$$
$$=\frac{1}{2n}\{((n+1)\lambda+2(n^2-4)\mu)\tau(R)+(2(n^2-4)\nu-3\lambda)\tau'(R)\}g.$$

An almost Hermitian manifold M is said to be of pointwise constant antiholomorphic sectional curvature if for each point $p\in M$ the curvature of an arbitrary antiholomorphic plane α in $T_p(M)$ does not depend on α.

Corollary. *Let M be a $2n$-dimensional RK-manifold, $n\geq 3$. Then M has pointwise constant antiholomorphic sectional curvature if and only if M has vanishing Bochner curvature tensor and*

$$(n+1)S-3S'=\frac{1}{2n}((n+1)\tau(R)-3\tau'(R))g.$$

This is an analogue of a well known theorem of Schouten and Struik [2], see also [3].

2. Proof of the theorem. Let e_i, Ie_i, $i=1,\dots,n$, be an arbitrary orthonormal basis $T_p(M)$, $p\in M$. In (1.1) we put $X=e_1$, $Y=e_i$ or $Y=Ie_i$, $i=2,\dots,n$. Adding on i we find

$$\lambda R(e_1, Ie_1, Ie_1, e_1)-(\lambda+2(n-2)\mu)S(e_1, e_1)-2(n-2)\nu S'(e_1, e_1)$$
$$=\mu\tau(R)(p)+\nu\tau'(R)(p)-2(n-1)c(p)$$

and since we can take for e_1 an arbitrary unit vector in $T_p(M)$ we have

$$\lambda H(x)-(\lambda+2(n-2)\mu)S(x, x)-2(n-2)\nu S'(x, x)$$
$$=\mu\tau(R)(p)+\nu\tau'(R)(p)-2(n-1)c(p) \tag{2.1}$$

for each unit vector $x\in T_p(M)$, where $H(x)$ is the curvature of the holomorphic plane spanned by x, Ix, i. e. $H(x)=R(x, Ix, Ix, x)$.

If $\lambda=0$ (2.1) takes the form

$$\mu S(x, x)+\nu S'(x, x)=\frac{2(n-1)c-\mu\tau(R)(p)-\nu\tau'(R)(p)}{2(n-2)}.$$

We put $x=e_i$, $x=Ie_i$ and adding on i we obtain

$$c=\frac{\mu\tau(R)+\nu\tau'(R)}{n}$$

and case 1) is proved.

If $\lambda\neq 0$, (1.1) and (2.1) take the form

$$R(x, y, y, x)+\mu_1(S(x, x)+S(y, y))+\nu_1(S'(x, x)+S'(y, y))=c_1, \tag{2.2}$$

$$H(x)-(1+2(n-2)\mu_1)S(x, x)-2(n-2)\nu_1 S'(x, x)$$
$$=\mu_1\tau(R)(p)+\nu_1\tau'(R)(p)-2(n-1)c_1(p), \tag{2.3}$$

where $\mu_1=\mu/\lambda$, $\nu_1=\nu/\lambda$, $c_1=c/\lambda$.

From (2.2) $R(x, y, y, x)=R(x, Iy, Iy, x)$ and consequently

$$R\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right)=R\left(\frac{x+y}{\sqrt{2}}, \frac{Ix-Iy}{\sqrt{2}}, \frac{Ix-Iy}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right)$$

which gives $H(x)+H(y)=4R(x, y, y, x)-2R(x, Iy, Iy, x)+2R(x, Ix, Iy, y)+2R(x, Iy, Ix, y)$.

Hence it is easy to find

$$(n+2)H(x)+\sum_{i=1}^{n}H(e_i)=S(x, x)+3S'(x, x) \tag{2.4}$$

and

$$\sum_{i=1}^{n} H(e_i)=\frac{\tau(R)(p)+3\tau'(R)(p)}{4(n+1)} \tag{2.5}$$

From (2.4) and (2.5) we obtain

$$H(x)-\frac{1}{n+2}(S(x,x)+3S'(x,x))=-\frac{\tau(R)(p)+3\tau'(R)(p)}{4(n+1)(n+2)} \tag{2.6}$$

Using (2.3) and (2.6) we get

$$(2(n-2)\mu_1-\frac{n+1}{n+2})S(x,x)+(2(n-1)\nu_1-\frac{3}{n+2})S'(x,x) \tag{2.7}$$

$$=2(n-1)c_1(p)-\mu_1\tau(R)(p)-\nu_1\tau'(R)(p)-\frac{\tau(R)(p)+3\tau'(R)(p)}{4(n+1)(n+2)}.$$

Hence by a simple calculation we obtain

$$c_1=(\frac{\mu_1}{n}+\frac{2n+1}{8n(n^2-1)})\tau(R)+(\frac{\nu_1}{n}-\frac{3}{8n(n^2-1)})\tau'(R). \tag{2.8}$$

The substitution of (2.8) in (2.7) gives

$$(\mu_1+\frac{n+1}{2(n^2-4)})S(x,x)+(\nu_1-\frac{3}{2(n^2-4)})S'(x,x) \tag{2.9}$$

$$=\frac{1}{2n}\{(\mu_1+\frac{n+1}{2(n^2-4)})\tau(R)(p)+(\nu_1-\frac{3}{2(n^2-4)})\tau'(R)(p)\}.$$

From (2.2), (2.8) and (2.9) it follows

$$R(x,y,y,x)-\frac{n+1}{2(n^2-4)}(S(x,x)+S(y,y))+\frac{3}{2(n^2-4)}(S'(x,x)+S'(y,y)) \tag{2.10}$$

$$=-\frac{2n^2+3n+4}{8(n^2-1)(n^2-4)}\tau(R)(p)+\frac{9n}{8(n^2-1)(n^2-4)}\tau'(R)(p).$$

According to the lemma, (2.6) and (2.10) imply that the Bochner curvature tensor B for M vanishes. The rest of the theorem follows from (2.2) and (2.10).

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