Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

CONVOLUTIONS, MULTIPLIERS AND COMMUTANTS CONNECTED WITH MULTIPLE DIRICHLET EXPANSIONS

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An explicit convolutional representation of a class of operators having certain invariant subspaces and commuting with the partial differentiations $\partial/\partial z_1, \ldots, \partial/\partial z_n$ is found. An application of these results to the multiple Dirichlet expansions of locally holomorphic function is made.

0. Introduction. Let D_j, $j=1,\ldots,n$ be a finite convex domain in the complex z_j-plane, let $\bar{D}_j$ be the closure of D_j and let $0\notin\bar{D}_j$ for all j. Let $D=D_1\times\cdots\times D_n$. The elements of $\mathbf{C}^n$ are denoted by letters without indices as $z=(z_1,\ldots,z_n)$. By $H(\bar{D})$ it is denoted the space of all functions $f(z)$ holomorphic on $\bar{D}$, endowed with the usual inductive topology [1, 378—381]. Let Φ_j, $j=1,\ldots,n$ be an arbitrary non-zero continuous linear functional in $H(\bar{D}_j)$. It is known [1, 378—381] that Φ_j can be represented in the form

$$(1)\qquad \Phi_j f=\frac{1}{2\pi i}\int_{\Gamma_j} f(\zeta_j)\gamma_j(\zeta_j)d\zeta_j,\quad f\in H(\bar{D}_j)$$

with a holomorphic function $\gamma_j(z_j)$ on the complement of $\bar{D}_j$ such that $\gamma_j(\infty)=0$ and where Γ_j is a contour lying in the domain of analyticity of f enclosing $\bar{D}_j$. In the whole paper the integration contours are considered with counterclockwise orientation. Conversely for every $\gamma_j(z_j)$ of such kind formula (1) defines a continuous linear functional in $H(\bar{D}_j)$. Let

$$(2)\qquad H_{\Phi_j}\overset{\mathrm{def}}{=}\{f\in H(\bar{D}):\ \Phi_{j,z_j}[f(z_1,\ldots,z_j,\ldots,z_n)]=0\ \text{for all}\ z_k\in\bar{D}_k,\ k\neq i\}$$

where the subscript z_j in Φ_{j,z_j} indicates that the functional Φ_j is applied on the variable z_j. We shall use frequently such kind subscripts without any discussion.

We aim to find a complete description of the all continuous linear operators $M: H(\bar{D})\to H(\bar{D})$ with invariant subspaces $H_{\Phi_1},\ldots,H_{\Phi_n}$ and commuting with $\partial/\partial z_j$ in H_{Φ_j} for $j=1,\ldots,n$. This is made in section 1. In section 2 a connection of such kind operators with the coefficient multipliers of the multiple complex Dirichlet expansions is found, where a convolutional approach to the multiple Dirichlet expansions is developed. The results presented in the paper are generalizations of analogous results for double Dirichlet expansions established by the authors in [2] but now the more general case of multiple zeros of the entire functions defining the expansion is considered while the special case of simple zeros has been considered in [2].

SERDICA Bulgaricae mathematicae publicationes. Vol. 9, 1983, p. 172—188.

For brevity's sake we use denotations as

$$\int_{\zeta}^{z} f(\tau)d\tau \overset{\text{def}}{=} \int_{\zeta_1}^{z_1} \cdots \int_{\zeta_n}^{z_n} f(\tau_1, \ldots, \tau_n)d\tau_1 \ldots d\tau_n \text{ for } z, \zeta \in \mathbf{C}^n.$$

1. A representation of the commutant of $\partial/\partial z_1, \ldots, \partial/\partial z_n$ relative to the invariant subspaces $H_{\Phi_1}, \ldots, H_{\Phi_n}$. Let us introduce in $H(\bar{D}_j)$ the operations

$$f \overset{(z_j)}{*} g = \Phi_{j,\zeta_j}\{\int_{z_j}^{\zeta_j} f(z_j+\zeta_j-\tau_j)g(\tau_j)d\tau_j\} \text{ for } f,\ g \in H(\bar{D}_j) \tag{3}$$

$j=1, \ldots, n$. It is shown [3] that (3) is a continuous, bilinear, commutative and associative operation in $H(\bar{D}_j)$. Let us introduce also the entire functions

$$E_j(\zeta_j) \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{\Gamma_j} \gamma_j(\tau_j)e^{\zeta_j\tau_j}\, d\tau_j = \Phi_{j,\tau_j}\{e^{\zeta_j\tau_j}\}, \quad j=1, \ldots, n.$$

Lemma 1. a) *The resolvent $R^j_{\lambda_j}$ of d/dz_j relative to Φ_j defined in $H(\bar{D}_j)$ by the problem $dy/dz_j-\lambda_j y=f$, $\Phi_j(y)=0$ can be represented in the form*

$$R^j_{\lambda_j}f=\{-e^{\lambda_j z_j}/E_j(\lambda_j)\} \overset{(z_j)}{*} f, \quad f \in H(\bar{D}_j), \tag{4}$$

for each λ_j with $E_j(\lambda_j) \neq 0$ and

$$R^j_{\lambda_j}(f \overset{(z_j)}{*} g)=R^j_{\lambda_j}f \overset{(z_j)}{*} g \tag{5}$$

hold for all $f,\ g \in H(\bar{D}_j)$, $j=1, \ldots, n$.

b) *By formula (4) $R^j_{\lambda_j}$ make sense in $H(\bar{D})$ too for each $f \in H(\bar{D})$ and arbitrary fixed $z_k \in H(\bar{D}_k)$, $k \neq j$. Formula (4) defines an operator $R^j_{\lambda_j}$ mapping $H(\bar{D})$ onto H_{Φ_j} and*

$$(\partial/\partial z_j-\lambda_j)R^j_{\lambda_j}f=f,\ \Phi_j\{R^j_{\lambda_j}f\}=0 \text{ for each } f \in H(\bar{D}). \tag{6}$$

The basic tool in our approach is a convolution for all resolvents $R^1_{\lambda_1}, \ldots, R^n_{\lambda_n}$ in $H(\bar{D})$.

Definition 1. ([4]). *Let $M\colon X \to X$ be a linear operator in a linear space X. A bilinear commutative and associative operation $f*g$ in X is said to be a convolution of M in X iff*

$$M(f*g)=Mf*g=f*Mg \text{ hold for all } f, g \in X. \tag{7}$$

Every operator $M\colon X \to X$ satisfying (7) is said to be a multiplier of $f*g$. A nonzero element $f \in X$ is said to be an annihilator of $*$ iff $f*g=0$ for all $g \in X$.

Let $\gamma(z) \overset{\text{def}}{=} \gamma_1(z_1)\cdots\gamma_n(z_n)$, $\Gamma \overset{\text{def}}{=} \Gamma_1 \times \cdots \times \Gamma_n$ and let $\langle\zeta, z\rangle \overset{\text{def}}{=} \zeta_1 z_1 + \cdots + \zeta_n z_n$ for $\zeta,\ z \in \mathbf{C}^n$. Then

$$\Phi f \overset{\text{def}}{=} \frac{1}{(2\pi i)^n} \int_{\Gamma} \gamma(z) f(z) dz = \Phi_{1,z_1} \cdots \Phi_{n,z_n}\{ f(z_1, \ldots, z_n)\}$$

is a continuous linear functional in $H(\bar{D})$. Let us introduce also the entire function of the variables $\zeta_1, \ldots, \zeta_n$:

$$E(\zeta) \overset{\text{def}}{=} \frac{1}{(2\pi i)^n} \int_{\Gamma} \gamma(z) e^{\langle \zeta, z\rangle} dz = \Phi_z\{e^{\langle \zeta, z\rangle}\} = E_1(\zeta_1)\cdots E_n(\zeta_n).$$

Theorem 1. *The operation*

$$f * g \overset{\text{def}}{=} \Phi_\zeta\{\int_z^\zeta f(z+\zeta-\tau) g(\tau) d\tau\}; \quad f, g \in H(\bar{D}) \tag{8}$$

is a continuous convolution in $H(\bar{D})$ *for all resolvents* $R^1_{\lambda_1}, \ldots, R^n_{\lambda_n}$ *when* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ *is fixed with* $E(\lambda) \neq 0$, *and*

$$R_\lambda f \overset{\text{def}}{=} R^1_{\lambda_1} \cdots R^n_{\lambda_n} f = \{(-1)^n e^{\langle \lambda, z\rangle} / E(\lambda)\} * f \tag{9}$$

holds for all $f \in H(\bar{D})$. *The function* $e^{\langle \lambda, z\rangle}$ *is not divisor of zero of* $f*g$ *i. e.* $*$ *is without annihilators in* $H(\bar{D})$. *For* $j=1, \ldots, n$ *the equality*

$$\frac{\partial}{\partial z_j}(f*g) = \frac{\partial f}{\partial z_j} * g \text{ holds for } f \in H_{\Phi_j}, \quad g \in H(\bar{D}) \tag{10}$$

and the equality

$$\Phi_j\{f*g\} = 0 \text{ holds for all } f, g \in H(\bar{D}), \tag{11}$$

i. e. $*: H(\bar{D}) \times H(\bar{D}) \to H_{\Phi_1} \cap \cdots \cap H_{\Phi_n}$.

Proof. Evidently (8) is a continuous bilinear operation in $H(\bar{D})$. The proof of the theorem follows from a corollary of the well-known Runge approximation theorem [5, 53] stating that the polynomials of n-variables are dense in $H(\bar{D})$. We use the fact that the operation (8) splits into a product of the one-dimensional convolutions $\overset{(z_j)}{*}$ (3) for functions of the form

$$f(z_1, \ldots, z_n) = f_1(z_1) \cdots f_n(z_n), \tag{12}$$

i. e. $f*g = \prod_{j=1}^n f_j(z_j) \overset{(z_j)}{*} g_j(z_j)$. Using this "splitting property" the commutativity, the associativity, the convolutional properties $R^j_{\lambda_j}(f*g) = R^j_{\lambda_j} f * g$, $f, g \in H(\bar{D})$ then equalities (9)—(11) can be verified easily for functions of the form (12). From the bilinearity of $f*g$ it follows that these properties hold for polynomials. Then using a polynomial approximation these properties can be proved for arbitrary functions of $H(\bar{D})$.

It is clear that the operator R_λ is a right inverse of the operator $\boldsymbol{D}_\lambda = (\partial/\partial z_1 - \lambda_1) \cdots (\partial/\partial z_n - \lambda_n)$, i. e.

$$\boldsymbol{D}_\lambda R_\lambda f = f \quad \text{holds for all } f \in H(\bar{D}) \tag{13}$$

and for all fixed $\lambda \in \mathbf{C}^n$ with $E(\lambda) \neq 0$.

Lemma 2. *For each* $\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbf{C}^n$ *with* $E(\lambda)\neq 0$ *the function* $e^{\langle\lambda,z\rangle}/E(\lambda)$ *is a cyclic element in* $H(\bar{D})$ *relative to the set of multipliers* $\{R^1_{\lambda_1},\ldots,R^n_{\lambda_n}\}$, *i. e. all linear combinations of the functions* $(R^1_{\lambda_1})^{k_1}\cdots(R^n_{\lambda_n})^{k_n}$ $\{e^{\langle\lambda,z\rangle}/E(\lambda)\}$, $k_j=0,1,2,\ldots;\ j=1,\ldots,n$ *are dense in* $H(\bar{D})$.

Proof. It is easy to see that the set of these linear combinations coincides with the set of the functions $e^{\langle\lambda,z\rangle}p(z)$ where $p(z)$ is an arbitrary polinomial of the variables $z_1,\ldots,z_n$.

Lemma 3. *Let* $\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbf{C}^n$ *be fixed and let* $E(\lambda)\neq 0$. *Let* j *with* $1\leq j\leq n$ *be fxed, too. Then an operator* $M\colon H(\bar{D})\to H(\bar{D})$ *has the properties* $M(H_{\Phi_j})\subset H_{\Phi_j}$ *and* $(\partial/\partial z_j)Mf=M(\partial/\partial z_j)f$ *for each* $f\in H_{\Phi_j}$ *iff* $MR^j_{\lambda_j}=R^j_{\lambda_j}M$ *in* $H(\bar{D})$.

Proof. Let $M(H_{\Phi_j})\subset H_{\Phi_j}$ and let $(\partial/\partial z_j)Mf=M(\partial/\partial z_j)f$ for each $f\in H_{\Phi_j}$. Let $g\in H(D)$. Then $R^j_{\lambda_j}g\in H_{\Phi_j}$ and hence $(\partial/\partial z_j-\lambda_j)MR^j_{\lambda_j}g=M(\partial/\partial z_j-\lambda_j)R^j_{\lambda_j}g=Mg$. Since $MR^j_{\lambda_j}g\in H_{\Phi_j}$ we get $R^j_{\lambda_j}Mg=R^j_{\lambda_j}(\partial/\partial z_j-\lambda_j)MR^j_{\lambda_j}g=MR^j_{\lambda_j}g-\Phi_j\{MR^j_{\lambda_j}g\}=MR^j_{\lambda_j}g$. Conversely, let $R^j_{\lambda_j}M=MR^j_{\lambda_j}$ in $H(\bar{D})$. If $f\in H_{\Phi_j}$ then there exists a $g\in H(\bar{D})$ such that $f=R^j_{\lambda_j}g$. Then $Mf-R^j_{\lambda_j}Mg\in H_{\Phi_j}$, i. e. $M(H_{\Phi_j})\subset H_{\Phi_j}$. With the same f we get $(\partial/\partial z_j-\lambda_j)Mf=(\partial/\partial z_j-\lambda_j)MR^j_{\lambda_j}g=(\partial/\partial z_j-\lambda_j)R^j_{\lambda_j}Mg=Mg=M(\partial/\partial z_j-\lambda_j)f$, hence $(\partial/\partial z_j)Mf=M(\partial/\partial z_j)f$.

Theorem 2. *Let* $M\colon H(\bar{D})\to H(\bar{D})$ *be a linear operator in* $H(\bar{D})$ *and let* $\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbf{C}^n$ *be an arbitrary fixed complex vector which is not zero of the entire function* $E(\zeta)$. *Then the following assertions are equivalent*:

a) *M is a continuous operator in* $H(\bar{D})$ *with invariant subspaces* $H_{\Phi_1},\ldots H_{\Phi_n}$ *commuting with* $\partial/\partial z_j$ *in* H_{Φ_j} *for* $j=1,\ldots,n$.

b) *M is a continuous linear operator in* $H(\bar{D})$ *commuting with all resolvents* $R^1_{\lambda_1},\ldots,R^n_{\lambda_n}$ *in* $H(\bar{D})$.

c) *M is a multiplier of the convolution* $f*g$.

d) *M is an operator of the form*

$$Mf=\mathbf{D}_\lambda[m_\lambda*f]\ \text{with}\ m_\lambda\overset{\text{def}}{=}M[(-1)^n e^{\langle\lambda,z\rangle}/E(\lambda)]\in H(\bar{D}).\tag{14}$$

e) *M is an operator of the form*

$$Mf=\rho_0 f+\sum_{k=1}^{n}\ \sum_{1\leq i_1<\ldots<i_n\leq n}\{\rho_{i_1,\ldots,i_k}(z_{i_1},\ldots,z_{i_k})\}\overset{(z_{i_1},\ldots,z_{i_k})}{*}\{f(z_1,\ldots,z_n)\}\tag{14'}$$

with $\rho_0\in\mathbf{C}$, $\rho_{i_1,\ldots,i_k}\in H(\bar{D}_{i_1}\times\cdots\times\bar{D}_{i_k})$, $k=1,\ldots,n$, *where*

$$f\overset{(z_{i_1},\ldots,z_{i_k})}{*}g\overset{\text{def}}{=}\Phi_{i_1,\zeta_1}\cdots\Phi_{i_k,\zeta_k}\{\int_{z_{i_1}}^{\zeta_{i_1}}\cdots\int_{z_{i_k}}^{\zeta_{i_k}}f(z_1,\ldots,z_{i_1}+\zeta_{i_1}-\tau_{i_1},$$

$$\ldots,z_{i_k}+\zeta_{i_k}-\tau_{i_k},\ldots,z_n)g(z_1,\ldots,\zeta_{i_1},\ldots,\zeta_{i_k},\ldots,z_n)d\zeta_{i_1}\ldots d\zeta_{i_k}$$

are auxiliary operations defined for $f,g\in H(\bar{D})$, $1\leq i_1<\cdots<i_k\leq n$.

Proof. a) $\Leftrightarrow$ b) follows from lemma 3. b) $\Rightarrow$ c). For each multiindex $(k)=(k_1,\ldots,k_n)$, $k_i=0, 1, 2,\ldots$ let us denote $R_\lambda^{(k)}=(R_{\lambda_1}^1)^{k_1}\cdots(R_{\lambda_n}^n)^{k_n}$. Then from the obvious identity $Mr_\lambda * r_\lambda = r_\lambda * Mr_\lambda$ where $r_\lambda(z)\overset{\text{def}}{=}(-1)^n e^{\langle\lambda, z\rangle}/E(\lambda)$ and from the commuting of M and $R_\lambda^{(k)}$ it follows that $MR_\lambda^{(p)}r_\lambda * R_\lambda^{(q)}r_\lambda = R_\lambda^{(p)}r_\lambda * R_\lambda^{(q)}Mr_\lambda$ for arbitrary multiindices (p) and (q). Then

$$Mf*g=f*Mg \tag{15}$$

holds for f, g which are linear combinations of $R_\lambda^{(k)}r_\lambda$ with various (k), hence (15) holds in $H(\bar{D})$ by lemma 2. According to (13) the complete multiplier relation (7) follows by application of $\boldsymbol{D}_\lambda$ to the extremities of the chain of identities $R_\lambda M(f*g)=r_\lambda * M(f*g)=Mr_\lambda*(f*g)=(Mr_\lambda*f)*g=(r_\lambda*Mf)*g=r_\lambda*(Mf*g)=R_\lambda(Mf*g)$. c) $\Rightarrow$ d). Let M be a multiplier of $f*g$. Using (9) we obtain that $R_\lambda Mf=r_\lambda*Mf=Mr_\lambda*f$ and we get (14) with $m_\lambda=Mr_\lambda$. d) $\Rightarrow$ e). (14′) is a developed form of (14) which can be obtained by immediate differentiation using (10). e) $\Rightarrow$ b). Since the operations $f \overset{(z_{i_1},\ldots,z_{i_n})}{*} g$ are continuous in $H(\bar{D})$, then (14′) defines a continuous linear operator in $H(\bar{D})$. Each of the terms in (14′) is an operator commuting with all the resolvents $R_{\lambda_1}^1,\ldots,R_{\lambda_n}^n$ which can be verified directly for functions of the form (12) since the "splitting property" holds for the operations $f \overset{(z_{i_1},\ldots,z_{i_n})}{*} g$ too. Then by polynomial approximation the multiplier relation (15) can be obtained in $H(\bar{D})$.

Now we shall describe all continuous convolutions in $H(\bar{D})$ for the resolvents $R_{\lambda_1}^1,\ldots,R_{\lambda_n}^n$.

Theorem 3. *Let* $\tilde{*}$ *be a bilinear operation in* $H(\bar{D})$ *and let* $\lambda\in\mathbf{C}^n$ *be fixed with* $E(\lambda)\neq 0$. *Then the following assertions are equivalent:*

a) $\tilde{*}$ *is a continuous convolution for* $R_{\lambda_1}^1,\ldots,R_{\lambda_n}^n$ *in* $H(\bar{D})$.

b) $\tilde{*}$ *is a continuous commutative and associative bilinear operation for which the subspaces* $H_{\Phi_1},\ldots,H_{\Phi_n}$ *are ideals in* $H(\bar{D})$ *and*

$$\frac{\partial}{\partial z_j}(f\tilde{*}g)=\frac{\partial f}{\partial z_j}\tilde{*}g \text{ holds for } f\in H_{\Phi_j},\ g\in H(\bar{D}) \tag{16}$$

for each $j=1,\ldots,n$.

c) *The "mixed generalized associativity relations"*

$$(f\tilde{*}g)*h=f*(g\tilde{*}h)=f\tilde{*}(g*h) \text{ hold for } f, g, h\in H(\bar{D}). \tag{17}$$

d) The operation $\tilde{*}$ *admits a representation of the form*

$$f\tilde{*}g=\boldsymbol{D}_\lambda^2[m_\lambda*f*g] \tag{18}$$

with $m_\lambda=\{e^{\langle\lambda, z\rangle}/E(\lambda)\}\tilde{*}\{e^{\langle\lambda, z\rangle}/E(\lambda)\}$.

The operation $\tilde{*}$ *is without annihilators iff the element* m_λ *is a non-divisor of zero of the primary convolution* $f*g$.

Proof. a) $\Rightarrow$ c). The operator $T_g f\overset{\text{def}}{=}f\tilde{*}g$ is a continuous linear operator commuting with $R_{\lambda_1}^1,\ldots,R_{\lambda_n}^n$ and hence T_g is a multiplier of $*$ by theorem 2,

i. e. $(f\tilde{*}g)*h=f*(g\tilde{*}h)$ holds. The second equality in (17) follows easily from this. c) $\Rightarrow$ d). Using (17) we get $R_\lambda R_\lambda(f\tilde{*}g)=r_\lambda * r_\lambda * f\tilde{*}g=r_\lambda\tilde{*}r_\lambda * f*g=m_\lambda * f*g$ hence (18) holds. d) $\Rightarrow$ a). It is not difficult to prove directly that the continuous operator $T_g f\stackrel{\text{def}}{=}\boldsymbol{D}_\lambda^2[m_\lambda * f*g]$ commutes with all $R^1_{\lambda_1},\dots,R^n_{\lambda_n}$ in $H(\bar{D})$. Indeed let $A\stackrel{\text{def}}{=}(\partial/\partial z_2-\lambda_2)\cdots(\partial/\partial z_n-\lambda_n)$. Now $T_g R^1_{\lambda_1}f=\boldsymbol{D}_\lambda A[m_\lambda * f*g]$ and $R^1_{\lambda_1}T_g f=A\boldsymbol{D}_\lambda[m_\lambda * f*g]-\Phi\{A\boldsymbol{D}_\lambda[m_\lambda * f*g]\}$ since $R^1_{\lambda_1}(\partial/\partial z_1-\lambda_1)f=f-\Phi_1 f$ but $\Phi_1\{A\boldsymbol{D}_\lambda[m_\lambda * f*g]\}=\Phi_1\{A\boldsymbol{D}_\lambda[r_\lambda * r_\lambda * f\tilde{*}g]\}=A^2[R^2_{\lambda_2}\cdots R^n_{\lambda_n}]^2\Phi_1(\partial/\partial z_1-\lambda_1)(R^1_{\lambda_1})^2$ $(f\tilde{*}g)=\Phi_1 R^1_{\lambda_1}(f\tilde{*}g)=0$ hence $T_g R^1_{\lambda_1}f=R^1_{\lambda_1}T_g f$. Analogously $T_g R^j_{\lambda_j}=R^j_{\lambda_j}T_g$ for $j=2,\dots,n$. a) $\Rightarrow$ b). Let $f\in M_{\Phi_j}$. Then $f=R^j_{\lambda_j}h$ with $h\in H(\bar{D})$ and $\Phi_j(f\tilde{*}g)=\Phi_j R^j_{\lambda_j}(h*g)=0$, i. e. H_{Φ_j} is an ideal of $H(\bar{D})$ and $\partial/\partial z_j(f\tilde{*}g)=\partial/\partial z_j R^j_{\lambda_j}(h\tilde{*}g)=\lambda_j h\tilde{*}g=(\partial/\partial z_j R^j_{\lambda_j}h)\tilde{*}g=(\partial f/(\partial z_j)\tilde{*}g$. The continuity of $f\tilde{*}g$ follows from the representation (18) since a) implies d). b) $\Rightarrow$ a). Let $h=R^j_{\lambda_j}(f\tilde{*}g)-(R^j_{\lambda_j}f)\tilde{*}g$ for arbitrary fixed $f,g\in H(\bar{D})$. Then $(\partial/\partial z_j-\lambda_j)h=f\tilde{*}g-[(\partial/\partial z_j-\lambda_j)R^j_{\lambda_j}f]\tilde{*}g=0$ and $\Phi_j h=0$ since H_{Φ_j} is an ideal of $H(\bar{D})$ relative to $\tilde{*}$, hence we get $h=0$ because $E_j(\lambda_j)\neq 0$ and λ_j is not an eigenvalue of the previous problem.

2. A convolutional approach to multiple complex Dirichlet expansions. Now let $E_j(\zeta_j)$, $j=1,\dots,n$ be entire function of exponential type (i. e. of order 1 and of normal type) with infinite sequence of different zeros $\{\lambda^j_k\}_{k=0}^\infty$ with multiplicities $\{m^j_k\}_{k=0}^\infty$. Let $\gamma_j(z_j)$ be the Borel transform of $E_j(\zeta_j)$ and let D_j, $j=1,\dots,n$ be a finite convex domain in the complex z_j-plane such that $\bar{D}_j$ contains all singularities of the Borel transform γ_j. Let us assume for sake of convenience that $0\in\bar{D}_j$ for all j. Let $D\stackrel{\text{def}}{=}D_1\times\cdots\times D_n$ and let

$$\Phi_j f=\frac{1}{2\pi i}\int_{\Gamma_j}f(z_j)\gamma_j(z_j)dz_j,\quad f\in H(\bar{D}_j) \tag{19}$$

be a continuous linear functional in $H(\bar{D}_j)$ defined by means the Borel transform γ_j. It is known [7, 24] that

$$E_j(\zeta_j)=\Phi_{j,z_j}\{e^{\zeta_j z_j}\}=\frac{1}{2\pi i}\int_{\Gamma_j}\gamma_j(z_j)e^{\zeta_j z_j}dz_j \tag{20}$$

for an arbitrary contour Γ_j enclosing $\bar{D}_j$. The problem for expanding of a function $f\in H(\bar{D})$ in a multiple Dirichlet series of the form $\sum_{k_1,\dots,k_n=0}^\infty d_{k_1,\dots,k_n}(z)$ $\cdot e^{\lambda^1_{k_1}z_1+\cdots+\lambda^n_{k_n}z_n}$, where $d_{k_1,\dots,k_n}(z)$ are polynomials have been studied by V. P. Gromov [6] and by A. F. Leontiev [7], [8]. In [2] the authors have applied a convolutional approach for description of their coefficient multipliers in the case $n=2$ when $E_1(\zeta_1)$ and $E_2(\zeta_2)$ have simple zeros only, i. e. when $m^1_{k_1}=m^2_{k_2}=1$. Now we generalize these results for multiple zeros of $E_1(\zeta_1),\dots,E_n(\zeta_n)$ too.

As in previous section we use the compact denotations

$$\gamma(z) \stackrel{\text{def}}{=} \gamma_1(z_1)\cdots\gamma_n(z_n),\ \Gamma \stackrel{\text{def}}{=} \Gamma_1\times\cdots\Gamma_n \text{ and}$$

$$E(\zeta) \stackrel{\text{def}}{=} \frac{1}{(2\pi i)^n}\int_\Gamma \gamma(z)e^{\langle\zeta,\, z\rangle}\,dz = E_1(\zeta_1)\cdots E_n(\zeta_n) = \Phi_z\{e^{\langle\zeta,\, z\rangle}\}$$

where

$$\Phi f \stackrel{\text{def}}{=} \frac{1}{(2\pi i)^n}\int_\Gamma \gamma(z)\,f(z)dz = \Phi_{1,z_1}\ldots\Phi_{n,z_n}\{f\}.$$

Let also $\lambda_{(k)} \stackrel{\text{def}}{=} (\lambda^1_{k_1}, \ldots, \lambda^n_{k_n})$ for each multiindex $(k)=(k_1, \ldots, k_n)$, $k_j=0, 1, 2, \ldots$. We shall use the denotation $z^{(k)}=z_1^{k_1}\ldots z_n^{k_n}$ too.

Now let $\lambda=(\lambda_1, \ldots, \lambda_n)\in \mathbf{C}^n$ be a non-zero of the entire function $E(\zeta)$, i. e. let $\lambda_j \neq \lambda^j_{k_j}$ for all $j=1, \ldots, n$, $k_j=0, 1, 2, \ldots$ Let us consider the resolvents $R^j_{\lambda_j}$ of d/dz_j in $H(\bar{D}_j)$ relative to Φ_j, extended in $H(\bar{D})$ by (4). It is clear that $\{e^{\lambda^j_k z_j}\}_{k=0}^{\infty}$ is the eigenfunctionsystem of the spectral problem $dy/dz_j=\mu y$, $\Phi_j(y)=0$ for $y\in H(\bar{D}_j)$ corresponding to the eigenvalue $\mu=\lambda^j_k$. However if λ^j_k is a multiple zero of $E_j(\zeta_j)$, i. e. if $m^j_k>1$ then there is a system of functions $\{z_j e^{\lambda^j_k z_j}, \ldots, z_j^{m^j_k-1} e^{\lambda^j_k z_j}\}$ *associated* with the eigenfunction $e^{\lambda^j_k z_j}$ and corresponding to λ^j_k. Let $S^j_k=\{e^{\lambda^j_k z_j}, z_j e^{\lambda^j_k z_j}, \ldots, z_j^{m^j_{k_j}-1} e^{\lambda^j_k z_j}\}$ be the system of all generalized eigenfunctions corresponding to λ^j_k and let

$$S_{(k)} \stackrel{\text{def}}{=} \{z^{(s)}e^{\langle\lambda(k),\, z\rangle}:\quad 0\leq s_j\leq m^j_{k_j}-1,\ 1\leq j\leq n\} \tag{21}$$

be the system of tensorial products of the functions from $S^j_{k_j}$, $i=1, \ldots, n$ for arbitrary fixed multiindex $(k)=(k_1, \ldots, k_n)$. According to Theorem 1 the operators $R^1_{\lambda_1}, \ldots, R^n_{\lambda_n}$ have a convolution without annihilators $f*g$ in $H(\bar{D})$ representing $R_\lambda=R^1_{\lambda_1}\ldots R^n_{\lambda_n}$ by (9).

For brevity's sake in the next we use a multiindex denotations. The multiindices $(p)=(p_1, \ldots, p_n)$ are considered as n-dimensional vectors with usual operations and the partial order relation: $(p)\leq(q)$ iff $p_j\leq q_j$ for all $j=1,\ldots,n$. If $(p)\leq(q)$ and $(p)\neq(q)$ we use the denotation $(p)<(q)$. We use also the compact denotations $(0)=(0, \ldots, 0)$, $(1)=(1, \ldots, 1)$, $m_{(k)}=(m^1_{k_1}, \ldots, m^n_{k_n})$ and $\nu_{(k)}=(m^1_{k_1}-1, \ldots, m^n_{k_n}-1)=m_{(k)}-(1)$. In the next the symbol $\Sigma^{(q)}_{(s)=(p)}$ denotes a summation over all the multiindices (s) with $(p)\leq(s)\leq(q)$, i. e. over all (s) with $p_j\leq s_j\leq q_j$, $j=1, \ldots, n$, *i.* e. $\Sigma^{(q)}_{(s)=p}=\Sigma^{q_1}_{s_1=p_1}\ldots\Sigma^{q_n}_{s_n=p_n}$, $(k)!=k_1!\ldots!k_n!$

D e f i n i t i o n 2. *Formal Dirichlet expansion of a function $f\in H(\bar{D})$ relative to the system $S=\bigcup_{(k)} S_{(k)}$ of Dirichlet polynomials* (21) *is said to be the correspondence*

(22) $$f \sim \sum_{(k)} P_{(k)} f,$$

where the Gromov-Leontiev projection

(23) $$P_{(k)} f = \sum_{(s)=(0)}^{\nu(k)} \overset{\nu(k)-(s)}{a_{(k)}} (f)\, z^{(s)} e^{\langle \lambda_{(k)}, z \rangle}/(s)!$$

on the space $H_{(k)}$ *spanned on* $S_{(k)}$ *is defined by*

(23′) $$P_{(k)} f = \frac{1}{(2\pi i)^n} \int_{c_{(k)}} \Phi_\zeta \{ \int_0^\zeta f(\zeta - x) e^{\langle \tau, x \rangle} dx \} \frac{e^{\langle \tau, z \rangle}}{E(\tau)} d\tau.$$

Here $c_{(k)} \overset{\text{def}}{=} c^1_{k_1} \times \cdots \times c^n_{k_n}$ *and* c^i_p *is a contour enclosing only* λ^j_p *of the zeros of* $E_j(\zeta_j)$, $j = 1, \ldots, n$, *i. e. it does not enclose other zeros* λ^k_q *with* $(k, q) \neq (j, p)$.

We note that for the sake of convenience by the next considerations the indices of the coefficient functionals $\overset{(s)}{a_{(k)}}(f)$, $(0) \leqq (s) \leqq \nu_{(k)}$ in formula (23′) are taken decreasing when the degrees of the corresponding Dirichlet polynomials $z^{(s)} e^{\langle \lambda_{(k)}, z \rangle}$ increase.

Theorem 4. a) *The projections* $P_{(k)}$ *are multipliers of* $f * g$ *and they can be represented by*

(24) $$P_{(k)} f = f * \varphi_{(k)} \text{ for } f \in H(\bar{D}), \text{ where}$$

(25) $$\varphi_{(k)}(z) \overset{\text{def}}{=} \frac{1}{(2\pi i)^n} \int_{c_{(k)}} \frac{e^{\langle \tau, z \rangle}}{E(\tau)} d\tau$$

is a function of the space $H_{(k)}$.

b) *The function* $\varphi_{(k)}$ *"splits":*

(26) $$\varphi_{(k)}(z) = \prod_{j=1}^n \varphi^j_{k_j}(z_j) \text{ where } \varphi^j_{k_j}(z_j) = \frac{1}{2\pi i} \int_{c^i_{k_i}} \frac{e^{\tau_j z_j}}{E_j(\tau_j)} d\tau_j \text{ and}$$

(27) $$\varphi_{(k)} * \varphi_{(p)} = \begin{cases} 0 & \text{for } (k) \neq (p), \\ \varphi_{(k)} & \text{for } (k) = (p), \end{cases}$$

i. e. the projections $P_{(k)}$ *form an orthogonal system.*

c) *The projection* $P_{(k)}$ *is the unique continuous projection mapping* $H(\bar{D})$ *onto* $H_{(k)}$ *having invariant subspaces* $H_{\Phi_1}, \ldots, H_{\Phi_n}$ *and commuting with* $\partial/\partial z_j$ *in* H_{Φ_j} *for* $j = 1, \ldots, n$.

Proof. It is enough to prove (24) for functions of the form $f(z) = f_1(z_1) \ldots f_n(z_n)$ only. Since now $f * \varphi_{(k)} = \prod_{j=1}^n f_j \overset{(z_j)}{*} \varphi^j_{k_j}$ and $P_{(k)} f = \prod_{j=1}^n P^j_{k_j} f_j$, where

$$P^j_{k_j} f_j \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{c^j_{k_j}} \Phi_{j,\zeta_j} \{ \int_0^{\zeta_j} f_j(\zeta_j - x_j) e^{\tau_j x_j} dx_j \} \frac{e^{\tau_j z_j}}{E_j(\tau_j)} d\tau_j,$$

it is clear that (24) is enough to be proved in the case $n = 1$ only. But now $(k) = k$, $\langle \tau, z \rangle = \tau z$ and $\int_z^\zeta$ is the usual one-dimensional integral and we have

$$f*\varphi_k=\Phi_\zeta\{\int_z^\zeta[\frac{1}{2\pi i}\int_{c_k}\frac{e^{\tau(z+\zeta-\sigma)}}{E(\tau)}d\tau]f(\sigma)\,d\sigma\}$$

$$=\frac{1}{2\pi i}\int_{c_k}\Phi_\zeta\{\int_z^\zeta e^{\tau(z+\zeta-\sigma)}f(\sigma)d\sigma\}\frac{d\tau}{E(\tau)}=\frac{1}{2\pi i}\int_{c_k}F(\tau,\,z)d\tau,$$

where $F(\tau,\,z)=\frac{e^{\tau z}}{E(\tau)}\Phi_\zeta\{\int_0^\zeta e^{\tau(\zeta-\sigma)}f(\sigma)\,d\sigma\}-\int_0^z e^{\tau(z-\sigma)}f(\sigma)d\sigma.$

Hence (24) follows since $\int_0^z e^{\tau(z-\sigma)}f(\sigma)d\sigma$ is an entire function of τ. The belonging of $\varphi_{(k)}$ to $H_{(k)}$ follows from the identity $\boldsymbol{D}^{(s)}\{\varphi_{(k)}e^{-\langle\lambda_{(k)},\,z\rangle}\}=0$ for all $(s)>\nu_{(k)}$, hence $\varphi_{(k)}e^{-\langle\lambda_{(k)},\,z\rangle}$ is a polynomial of the form $\Sigma_{(s)=(0)}^{\nu(k)}\alpha_{(k)}^{\nu_{(k)}-(s)}z^{(s)}$. Also since $\varphi_{(k)}*\varphi_{(p)}=\Pi_{j=1}^n\varphi_{k_j}^j\overset{(z_j)}{*}\varphi_{p_j}^j$ it is enough to calculate $\varphi_{(k)}*\varphi_{(p)}$ in the case $n=1$ and to establish b) in this case. Now let $n=1$ and let c_k and c_p' be contours enclosing only λ_k and λ_p of the zeros respectively and let $c_k\cap c_p'=\emptyset$. It is possible $\lambda_k=\lambda_p$ too. Using that $E(\zeta)=\Phi_z\{e^{\zeta z}\}$ after elementary calculations we obtain:

$$\varphi_k*\varphi_p=\Phi_\xi\{\int_z^\xi[\frac{1}{2\pi i}\int_{c_k}\frac{e^{\tau(z+\xi-\zeta)}}{E(\tau)}d\tau\,\frac{1}{2\pi i}\int_{c_p'}\frac{e^{\sigma\zeta}}{E(\sigma)}d\sigma]d\zeta\}$$

$$=\frac{1}{(2\pi i)^2}\int_{c_k}d\tau\int_{c_p'}\frac{d\sigma}{E(\tau)E(\sigma)}\Phi_\xi\{\int_z^\xi e^{\tau(z+\xi-\zeta)}e^{\sigma\zeta}d\zeta\}$$

$$=\frac{1}{(2\pi i)^2}\int_{c_k}d\tau\int_{c_p'}\frac{1}{E(\tau)E(\sigma)}\Phi^\xi\{\frac{e^{\sigma z}E(\tau)-e^{\tau z}E(\sigma)}{\tau-\sigma}\}d\sigma$$

$$=\frac{1}{(2\pi i)^2}\int_{c_p'}\frac{e^{\sigma z}}{E(\sigma)}d\sigma\int_{c_k}\frac{d\tau}{\tau-\sigma}-\frac{1}{(2\pi i)^2}\int_{c_k}\frac{e^{\tau z}}{E(\tau)}d\tau\int_{c_p'}\frac{d\tau}{\tau-\sigma}.$$

Now let $\lambda_k\neq\lambda_p$ and let c_k does not enclose c_p' in its inside and conversely. Then $\int_{c_k}\frac{d\tau}{\tau-\sigma}=0$ for $\sigma\in c_p'$ and $\int_{c_p'}\frac{d\sigma}{\tau-\sigma}=0$ for $\tau\in c_k$ hence $\varphi_k*\varphi_p=0$. If $\lambda_k=\lambda_p$ let us take c_k enclosing c_k' in its inside. Then $\int_{c_k'}\frac{d\sigma}{\tau-\sigma}=0$ too but $\int_{c_k}\frac{d\tau}{\tau-\sigma}=2\pi i$ for $\tau\in c_k'$ hence $\varphi_k*\varphi_k=\varphi_k$ and b) is established. Let now $Q:H(\bar{D})\to H_{(k)}$ be a continuous projection on $H_{(k)}$ with invariant subspaces $H_{\Phi_1},\dots,H_{\Phi_n}$ commuting with $\partial/\partial z_j$ in H_{Φ_j} for $j=1,\dots,n$. Then Q is a multiplier of $f*g$ by Theorem 2 hence Q commutes with $P_{(k)}$ and $Q=P_{(k)}$.

Now we shall prove a uniquess theorem for the multiple complex Dirichlet expansions.

Lemma 4. *Let $g(z)\in H(\bar{D})$ and let $\{g(z)\}\overset{(z_j)}{*}\{\varphi_k^j(z_j)\}=0$ for all $k=0, 1, 2, \ldots,$ for arbitrary fixed $z_j\in \bar{D}_j$, $j\neq i$ and for all $z_j\in \bar{D}_j$. Then $g\equiv 0$ in $\bar{D}$. Here $\overset{(z_j)}{*}$ denotes the operation* (3) *acting only on the variable z_j.*

Proof. From Theorem 3 in the case $n=1$ it follows that $f\overset{(z_j)}{*}\{\varphi_k^j(z_j)\}$ for $f\in H(\bar{D}_j)$ defines the Gromov — Leontiev projection $P^j_{k_j}$ in $H(\bar{D}_j)$ relative to the one-variable system $\{e^{\lambda^j_{k_j} z_i}, z_i e^{\lambda^i_{k_j} z_i}, \ldots, z_j^{m^i_k-1} e^{\lambda^j_{k} z_j}\}$ in the case of the variable z_j. Now from Leontiev uniqueness theorem [6, 255] for the one-dimensional case it follows that $g(z_1, \ldots, z_j, \ldots, z_n)=0$ holds for all $z_k\in \bar{D}_k$ and arbitrary fixed $z_k\in \bar{D}_k$, $j\neq k$.

Theorem 5 (Uniqueness theorem). *Let $f\in H(\bar{D})$ and let $P_{(k)}f=f*\varphi_{(k)}=0$ for all $(k)\geqq(0)$. Then $f=0$, i. e. the projections $P_{(k)}$ form a total multiplier projection system.*

Proof. Using (26) we have

$$f*\varphi_{(k)}=\{[(f\overset{(z_n)}{*}\varphi^n_{k_n})\overset{(z_{n-1})}{*}\varphi^{n-1}_{k_{n-1}}]*\cdots\}\overset{(z_1)}{*}\varphi^1_{k_1}=0$$

for all $(k)\geqq(0)$ and applying successive Lemma 4 we get $f\equiv 0$.

It is clear that $\dim H_{(k)}=m^1_{k_1}\ldots m^n_{k_n}$ for each (k). Let now $\boldsymbol{D}\overset{\text{def}}{=}\frac{\partial}{\partial z_1}\cdots\frac{\partial}{\partial z_n}$ (i. e. $\boldsymbol{D}=\boldsymbol{D}_\lambda$ for $\lambda=0$). Let also $\boldsymbol{D}^{(k)}\overset{\text{def}}{=}\frac{\partial^{k_1+\cdots+k_n}}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}$.

Now if $\lambda^j_{k_j}$, $k_j=0, 1, 2, \ldots,$ $j=1, \ldots, n$ are simple zeros of the all entire functions $E_i(\zeta_i)$, $j=1, \ldots, n$, then $\dim H_{(k)}=1$ for all (k), $\varphi_{(k)}=e^{\langle\lambda_{(k)}, z\rangle}/\boldsymbol{D}E(\lambda_{(k)})$ and $P_{(k)}$ can be represented in the form

$$P_{(k)}f=\overset{(0)}{a}_{(k)}(f)e^{\langle\lambda_{(k)}, z\rangle}, \tag{28}$$

where

$$\overset{(0)}{a}_{(k)}(f)=\frac{1}{\boldsymbol{D}E(\lambda_{(k)})}\Phi_\zeta\{\int_0^\zeta f(\zeta-\tau)e^{\langle\lambda_{(k)}, \tau\rangle}d\tau\} \tag{29}$$

are multiplicative linear functionals relative to $f*g$, precisely

$$\overset{(0)}{a}_{(k)}(f*g)=\boldsymbol{D}E(\lambda_{(k)})\overset{(0)}{a}_{(k)}(f)\overset{(0)}{a}_{(k)}(g) \tag{30}$$

holds for all $f, g\in H(\bar{D})$ and each $(k)\geqq(0)$.

In the general case for arbitrary multiplicities of the zeros of $E_j(\zeta_j)$, $j=1, \ldots, n$ the coefficient functionals $\overset{(s)}{a}_{(k)}(f)$ (see (23)) of the projection $P_{(k)}$ can be expressed by

$$\overset{(s)}{a}_{(k)}(f)=\frac{1}{(2\pi i)^n}\int_{c_k}\Phi_\zeta\{\int_0^\zeta f(\zeta-x)e^{\langle\tau, x\rangle}dx\}\frac{(\tau-\lambda_{(k)})^{(\nu(k)-(s))}}{E(\tau)}d\tau, \tag{31}$$

where $(s)! \overset{\text{def}}{=} s_1! \dots s_n!$ and $(\tau-\lambda_{(k)})^{(s)}=(\tau_1-\lambda^1_{k_1}{}^{s_1})\dots(\tau_n-\lambda^n_{k_n})^{s_n}$. This follows easily from Taylor formula for polynomials. Indeed as in Theorem 4 it can be proved that $q_{(k)}(z)=(P_{(k)}f)e^{-\langle\lambda_{(k)},\, z\rangle}$ is a polynomial and $\overset{(s)}{a}_{(k)}(f)$, $(0)\leqq(s)\leqq\nu_{(k)}$ are its coefficients. Then

$$\overset{(s)}{a}_{(k)}(f)=\{\boldsymbol{D}^{(\nu(k)-(s))}q_{(k)}(z)\}\Big|_{z=0}$$

$$=\frac{\boldsymbol{D}^{(\nu(k)-(s))}}{(2\pi i)^n}\int_{c_{(k)}}\Phi_\zeta\{\int_0^\zeta f(\zeta-x)\,e^{\langle\tau,\, x\rangle}\,dx\}\,\frac{e^{\langle\tau-\lambda_{(k)},\, z\rangle}}{E(\tau)}\,d\tau\Big|_{z=0}$$

$$=\frac{1}{(2\pi i)^n}\int_{c_{(k)}}\Phi_\zeta\{\int_0^\zeta f(\zeta-x)e^{\langle\tau,\, x\rangle}\,dx\}\frac{(\tau-\lambda_{(k)})^{(\nu(k)-(s))}\,e^{\langle\tau-\lambda_{(k)},\, z\rangle}\,d\tau}{E(\tau)}\Bigg|_{z=0}.$$

Analogously

$$\varphi_{(k)}=\sum_{(s)=(0)}^{\nu(k)}\overset{\nu(k)-(s)}{\alpha}_{(k)}\,z^{(s)}\,e^{\langle\lambda_{(k)},\, z\rangle}/(s)! \tag{32}$$

with $\quad\overset{(s)}{\alpha}_{(k)}=\dfrac{1}{(2\pi i)^n}\displaystyle\int_{c_{(k)}}\dfrac{(\tau-\lambda_{(k)})^{(\nu(k)-(s))}}{E(\tau)}\,d\tau$ for $(0)\leqq(s)\leqq\nu_{(k)}$.

We note that $\overset{(s)}{a}_{(k)}(f)$ are multiplicative for $(s)=(0)$ too but they are non-multiplicative for $(s)\neq(0)$. Precisely

$$\overset{(0)}{a}_{(k)}(f*g)=\overset{(0)}{\beta}_{(k)}\,\overset{(0)}{a}_{(k)}(f)\,\overset{(0)}{a}_{(k)}(g), \tag{33}$$

$$\overset{(0)}{a}_{(k)}(f*g)=\sum_{(q)=(0)}^{(s)}\overset{(s)-(q)}{\beta}_{(k)}\sum_{(j)=(0)}^{(q)}\overset{(q)-(j)}{a}_{(k)}(f)\,\overset{(j)}{a}_{(k)}(g)\ \text{ for } (0)<(s)\leqq\nu_{(k)}, \tag{33'}$$

where $\overset{(s)}{\beta}_{(k)}\overset{\text{def}}{=}\boldsymbol{D}^{((s)+m(k))}\,E(\lambda_{(k)})/((s)+m_{(k)})!$, $(0)\leqq(s)\leqq\nu_{(k)}$ are the coefficients of the multiple power series representing the entire function $E(\zeta)=E_1(\zeta_1)\dots E_n(\zeta_n)$ $=(\zeta-\lambda_{(k)})^{(m(k))}\Sigma_{(s)\geqq(0)}\overset{(s)}{\beta}_{(k)}(\zeta-\lambda_{(k)})^{(s)}$ around $\lambda_{(k)}$.

For the proof the "splitting property" of $\overset{(s)}{a}_{(k)}(f)$ for functions of the form $f(z)=f_1(z_1)\dots f_n(z_n)$ can be used to reduce the general case to the case $n=1$. To the last case a general approach [9; 10; 11] can be used.

We note that there is a basis in $H_{(k)}$ more convenient for studying of the convolutional properties of the Dirichlet series relative to the convolution $f*g$ in contrast to the usual basis $S_{(k)}$ (21). Indeed let $\boldsymbol{D}_\lambda^{(s)}\overset{\text{def}}{=}(\partial/\partial z_1-\lambda_1)^{s_1}\dots(\partial/\partial z_n-\lambda_n)^{s_n}$ for $\lambda\in\mathbf{C}^n$. Then $\overset{(s)}{\varphi}_{(k)}\overset{\text{def}}{=}\boldsymbol{D}_\lambda^{(\nu(k)-(s))}\varphi_{(k)}$ with $(0)\leqq(s)\leqq\nu_{(k)}$ form a basis of $H_{(k)}$ which can be expressed from (25) by

$$\overset{(s)}{\varphi}_{(k)}=\frac{1}{(2\pi i)^n}\int_{c_{(k)}}\frac{(\tau-\lambda_{(k)})^{(\nu(k)-(s))}e^{\langle\tau,\, z\rangle}}{E(\tau)}\,d\tau.$$

Obviously $\overset{\nu(k)}{\varphi} = \varphi_{(k)}$. Namely, $\{\overset{(s)}{\varphi}_{(k)}: (0)\leqq(s)\leqq\nu_{(k)}\}$ is the most convenient basis to study the inner convolutional structure of $H_{(k)}$ relative to $f*g$ since $P_{(k)}$ has the representation

$$P_{(k)}f = \sum_{(s)=(0)}^{\nu(k)} \overset{\nu(k)-(s)}{C_{(k)}}(f)\, \overset{(s)}{\varphi}_{(k)}, \tag{34}$$

where the coefficient functionals $\overset{(s)}{C}_{(k)}(f)$, $(0)\leqq(s)\leqq\nu_{(k)}$ have the most simple multiplicative behaviour relative to $f*g$, namely

$$\overset{(0)}{C}_{(k)}(f*g) = \overset{(0)}{C}_{(k)}(f)\, \overset{(0)}{C}_{(k)}(g), \tag{35}$$

$$\overset{(s)}{C}_{(k)}(f*g) = \sum_{(j)=(0)}^{(s)} \overset{(s)-(j)}{C_{(k)}}(f)\, \overset{(j)}{C}_{(k)}(g) \quad \text{for } (0)<(s)\leqq\nu_{(k)}. \tag{35'}$$

The new coefficient functionals $\overset{(s)}{C}_{(k)}(f)$ are connected with the formulas:

$$\overset{(s)}{a}_{(k)}(f) = \sum_{(j)=(0)}^{(s)} \overset{(s)-(j)}{\alpha_{(k)}}\, \overset{(j)}{C}_{(k)}(f) \text{ with } \overset{(s)}{\alpha}_{(k)} \text{ defined after (32)}, \tag{36}$$

$$\overset{(s)}{C}_{(k)}(f) = \sum_{(j)=(0)}^{(s)} \overset{(s)-(j)}{\beta_{(k)}}\, \overset{(j)}{a}_{(k)}(f) \text{ with } \overset{(s)}{\beta}_{(k)} \text{ defined after (33)}. \tag{36'}$$

From the "splitting property" of the projection $P_{(k)}$ and the fact that $\overset{(s)}{\varphi}_{(k)}$ is a function of the form (12) it follows easily that the new functionals $\overset{(s)}{C}_{(k)}(f)$ satisfy the "splitting property" too, and the proof of (35) and (36) can be reduced to the case $n=1$ too. Especially the proof of (35) for $n=1$ follows from a general approach developed in [9]. The connection (36) between both systems of functionals when $n=1$ follows also from this general approach which can be applied since when $n=1$, $f*g$ is a convolution for *one operator with simple point spectrum* (d/dz_1 considered in H_{Φ_1} or equivalently its resolvent $R^1_{\lambda_1}$) and the projections $P_{(k)}$ define its generalized eigenfunction expansion. See also [10, 11].

Theorem 7. *Let M be a linear operator in $H(\bar{D})$ and let $\lambda\in\mathbf{C}^n$ be fixed such that $E(\lambda)\neq 0$. Then the following assertions are equivalent:*

a) *M commutes with $\partial/\partial z_1, \ldots, \partial/\partial z_n$ in $H_{(k)}$ and with $P_{(k)}$ in $H(\bar{D})$ for each $(k)\geqq(0)$.*

b) *M is a multiplier of $f*g$.*

c) *M is a continuous linear operator having invariant subspaces $H_{\Phi_1}, \ldots, H_{\Phi_n}$ and commuting with $\partial/\partial z_j$ in H_{Φ_j} for $j=1, \ldots, n$.*

d) *M commutes with all resolvents $R^1_{\lambda_1}, \ldots, R^n_{\lambda_n}$.*

e) *M admits a representation of the form* (14) *or* (14') *with $m_\lambda\in H(\bar{D})$.*

Proof. It follows from Theorem 2 that b)$\Leftrightarrow$c)$\Leftrightarrow$d)$\Leftrightarrow$e) for the present choice of the functionals $\Phi_1, \ldots, \Phi_n$. a) $\Rightarrow$ b). Using (10) and the fact that $\{\overset{(s)}{\varphi}_{(k)}: (0)\leqq(s)\leqq\nu_{(k)}\}$ is a basis in $H_{(k)}$, from the obvious equality $M[\varphi_{(k)}*\varphi_{(k)}]$

$=M\varphi_{(k)}=M\varphi_{(k)}*\varphi_{(k)}$ we get $M(f*g)=Mf*g$ for $f, g\in H_{(k)}\subset H_{\Phi_1}\cap\cdots\cap H_{\Phi_n}$. Now using that $P_{(k)}$ is a *homomorphism*, i. e. $P_{(k)}(f*g)=P_{(k)}f*P_{(k)}g$ we have $P_{(k)}[M(f*g)-Mf*g]=0$ for each (k) and arbitrary fixed $f, g\in H(\bar{D})$. Hence $M(f*g)=Mf*g$ holds for $f, g\in H(\bar{D})$ by Theorem 6. b) $\Rightarrow$ a). Let M be a multiplier of $f*g$. Then M commutes with $P_{(k)}$ in $H(\bar{D})$. Now since b) implies c) and since $H_{(k)}\subset H_{\Phi_1}\cap\cdots\cap H_{\Phi_n}$ for each (k) we get that b) implies a).

Remark. We note that a) implies the continuity of M.

Now we shall characterize other convolutions of the Dirichlet expansions.

Theorem 8. *Let* $\tilde{*}$ *be a bilinear operation in* $H(\bar{D})$ *and let* $\lambda=(\lambda_1, \ldots, \lambda_n)\in\mathbf{C}^n$ *be such that* $E(\lambda)\neq 0$. *Then the following assertions are equivalent:*

a) $\tilde{*}$ *is a convolition for all* $\partial/\partial z_1, \ldots, \partial/\partial z_n$ *in* $H_{(k)}$ *and for* $P_{(k)}$ *in* $H(\bar{D})$ *or all* $(k)\geq(0)$.

b) *The "generalized associativity relations"* (17) *hold for* $f, g, h\in H(\bar{D})$.

c) $\tilde{*}$ *admits a representation of the form* (18) *with* $m_\lambda=\{e^{\langle\lambda, z\rangle}/E(\lambda)\}\tilde{*}\{e^{\langle\lambda, z\rangle}/E(\lambda)\}\in H(\bar{D})$ *for arbitrary fixed* $\lambda\in\mathbf{C}^n$ *with* $E(\lambda)\neq 0$.

d) $\tilde{*}$ *is a continuous, commutative and associative operation for which the subspaces* $H_{\Phi_1}, \ldots, H_{\Phi_n}$ *are ideals of* $H(\bar{D})$ *and* $\partial/\partial z_j(f\tilde{*}g)=(\partial f/\partial z_j)\tilde{*}g$ *holds for* $f\in H_{\Phi_j}$, $g\in H(\bar{D})$, $j=1, \ldots, n$.

e) $\tilde{*}$ *is a continuous convolution in* $H(\bar{D})$ *for all resolvents* $R^1_{\lambda_1}, \ldots, R^n_{\lambda_n}$.

Proof. The equivalence relations b) $\Leftrightarrow$ c) $\Leftrightarrow$ d) $\Leftrightarrow$ e) hold from Theorem 3 for the present choice of the functionals $\Phi_1, \ldots, \Phi_n$. Then b) implies a) since both b) and d) implies a). a) $\Rightarrow$ b). Since (17) is evident for $f=g=h=\varphi_{(k)}$, as in previous theorem it can be proved that (17) holds for $f, g, h\in H_{(k)}$ for each (k) and by the totality of $\{P_{(k)}\}$ we obtain that (17) holds in $H(\bar{D})$.

Remark. We note that if the convolution $\tilde{*}$ is without annihilators, i. e. if the representing element r_λ in (17) is a nondivizor of zero for $*$ then $\tilde{*}$ and $*$ have one of the same set of multipliers and the new convolution $\tilde{*}$ can be used for representation of multipliers instead of $*$.

Definition 3. *An operator* M: $H(\bar{D})\rightarrow H(\bar{D})$ *is said to be a coefficient multiplier of the formal Dirichlet expansion* $f\sim\Sigma_{(k)}P_{(k)}f$ *iff there is a numerical multiple sequence* $\mu_{(k)}$ *such that for arbitrary* (k)

$$P_{(k)}Mf=\mu_{(k)}P_{(k)}(f) \text{ holds for all } f\in H(\bar{D}), \tag{37}$$

i. e. there is a sequence $\{\mu_{(k)}\}$ *such that* M *maps the expansion* $f\sim\Sigma_{(k)}P_{(k)}f$ *to the expansion* $Mf\sim\Sigma_{(k)}\mu_{(k)}P_{(k)}f$.

Remark. If $E_j(\zeta_j)$, $j=1, \ldots, n$ have simple zeros only then M is a coefficient multiplier iff it satisfies the relation

$$\overset{(0)}{a}_{(k)}(Mf)=\mu_{(k)}\overset{(0)}{a}_{(k)}(f) \text{ for all } f\in H(\bar{D}) \text{ and arbitrary fixed } (k). \tag{38}$$

Now an operation $\tilde{*}$ is said to be a *coefficient convolution of the simple Dirichlet expansion* $f\sim\Sigma_{(k)}\overset{(0)}{a}_{(k)}(f)e^{\langle\lambda_{(k)}, z\rangle}$ iff there is a number sequence $\{\mu_{(k)}\}$ such that

(39) $\overset{(0)}{a}_{(k)}(f \tilde{*} g)=\mu_{(k)}\overset{(0)}{a}_{(k)}(f)\overset{(0)}{a}_{(k)}(g)$ holds for all $f, g \in H(\bar{D})$

and arbitrary fixed $(k)\geqq(0)$.

The next theorem gives a complete description of such kind coefficient multipliers and coefficient convolutions.

Theorem 9. *Let $E_j(\zeta_j)$, $j=1,\dots,n$ have simple zeros only and let $\lambda\in\mathbf{C}^n$ be fixed such that $E(\lambda)\neq 0$. Then:*

a) *An operator M: $H(\bar{D})$ is a coefficient multiplier of the simple formal Dirichlet expansion iff it is a multiplier of $f*g$ or equivalently iff M can be represented by* (14) *or by* (14′) *with $m_\lambda=M\{(-1)^n e^{\langle\lambda,z\rangle}/E(\lambda)\}\in H(\bar{D})$.*

b) *A bilinear operation $\tilde{*}$: $H(\bar{D})\times H(\bar{D})\to H(\bar{D})$ is a coefficient convolution of the simple formal Dirichlet expansion iff $\tilde{*}$ can be represented by* (18) *with*

$$m_\lambda=\{e^{\langle\lambda,z\rangle}/E(\lambda)\}\tilde{*}\{e^{\langle\lambda,z\rangle}\ E(\lambda)\}\in H(\bar{D}).$$

Proof. a) Using (30) we obtain that $\overset{(0)}{a}_{(k)}[M(f*g)-Mf*g]=0$ for each (k). Hence by the totality we get $M(f*g)=Mf*g$. Conversely if M is a multiplier of $f*g$ then M can be represented of the form $Mf=\boldsymbol{D}_\lambda[m_\lambda*f]$ with $m_\lambda\in H(\bar{D})$, and since $P_{(k)}$ is a multiplier of $f*g$ and $m_\lambda*f\in H_{\Phi_1}\cap\dots\cap H_{\Phi_n}$ then $P_{(k)}\boldsymbol{D}_\lambda[m_\lambda*f]=\boldsymbol{D}_\lambda P_{(k)}[m_\lambda*f]$ follows from Theorem 2a). Now by (28) we get $\overset{(0)}{a}_{(k)}(Mf)e^{\langle\lambda_{(k)},z\rangle}=P_{(k)}Mf=P_{(k)}\boldsymbol{D}_\lambda[m_\lambda*f]=\boldsymbol{D}_\lambda P_{(k)}(m_\lambda*f)=\boldsymbol{D}_\lambda[P_{(k)}m_\lambda\}*P_{(k)}f]$
$=\boldsymbol{D}_\lambda[\overset{(0)}{a}_{(k)}(m_\lambda)e^{\langle\lambda_{(k)},z\rangle}*\overset{(0)}{a}_{(k)}(f)e^{\langle\lambda_{(k)},z\rangle}]=\overset{(0)}{a}_{(k)}(m_\lambda)\overset{(0)}{a}_{(k)}(f)\boldsymbol{D}E(\lambda_{(k)})\boldsymbol{D}_\lambda e^{\langle\lambda_{(k)},z\rangle}$
$=\overset{(0)}{a}_{(k)}(m_\lambda)\overset{(0)}{a}_{(k)}(f)\boldsymbol{D}E(\lambda_{(k)})(\lambda_{(k)}-\lambda)^{(1)}e^{\langle\lambda_{(k)},z\rangle}$, hence $\overset{(0)}{a}_{(k)}(Mf)=\mu_{(k)}\overset{(0)}{a}_{(k)}(f)$ with $\mu_{(k)}=\overset{(0)}{a}_{(k)}(m_\lambda)\boldsymbol{D}E(\lambda_{(k)})(\lambda_{(k)}-\lambda)^{(1)}$.

b) If $\tilde{*}$ is a convolution for the simple Dirichlet exspansion then from (39) it follows that the "mixed generalized associativity relations" (17) hold for $*$ and $\tilde{*}$. Indeed if $\rho=(f\tilde{*}g)*h)-f*(g\tilde{*}h)$ it follows from (30) and (39) that $\overset{(0)}{a}_{(k)}(\rho)=0$ for all $(k)\geqq(0)$, hence $\rho=0$. Conversely let $\tilde{*}$ be an operation of the form (18) i. e. $f\tilde{*}g=\boldsymbol{D}_\lambda^2[m_\lambda*f*g]$. Using that $f*g\in H_{\Phi_1}\cap\dots\cap H_{\Phi_n}$ and (10) we obtain $f\tilde{*}g=\boldsymbol{D}_\lambda[m_\lambda*\boldsymbol{D}_\lambda(f*g)]$. Now $\overset{(0)}{a}_{(k)}(f\tilde{*}g)e^{\langle\lambda_{(k)},z\rangle}=P_{(k)}(f\tilde{*}g)$

$=\boldsymbol{D}_\lambda P_{(k)}[m_\lambda*\boldsymbol{D}_\lambda(f*g)]=\boldsymbol{D}_\lambda[P_{(k)}m_\lambda*\boldsymbol{D}_\lambda P_{(k)}(f*g)]$

$=\boldsymbol{D}_\lambda\{[\overset{(0)}{a}_{(k)}(m_\lambda)e^{\langle\lambda_{(k)},z\rangle}]*\boldsymbol{D}_\lambda[\overset{(0)}{a}_{(k)}(f*g)e^{\langle\lambda_{(k)},z\rangle}]\}$

$=\overset{(0)}{a}_{(k)}(m_\lambda)\boldsymbol{D}E(\lambda_{(k)})\overset{(0)}{a}_{(k)}(f)\overset{(0)}{a}_{(k)}(g)\boldsymbol{D}_\lambda^2[e^{\langle\lambda_{(k)},z\rangle}*e^{\langle\lambda_{(k)},z\rangle}]$

$=\overset{(0)}{a}_{(k)}(m_\lambda)\boldsymbol{D}E(\lambda_{(k)})^2\overset{(0)}{a}_{(k)}(f)\overset{(0)}{a}_{(k)}(g)\boldsymbol{D}_\lambda^2[e^{\langle\lambda_{(k)},z\rangle}]$

$=\overset{(0)}{a}_{(k)}(m_\lambda)\boldsymbol{D}E(\lambda_{(k)})^2(\lambda_{(k)}-\lambda)^{(2)}\overset{(0)}{a}_{(k)}(f)\overset{(0)}{a}_{(k)}(g)e^{\langle\lambda_{(k)},z\rangle}$. Hence $\overset{(0)}{a}_{(k)}(f\tilde{*}g)=\mu_{(k)}\overset{(0)}{a}_{(k)}(f)$ $\overset{(0)}{a}_{(k)}(g)$ with $\mu_{(k)}=\overset{(0)}{a}_{(k)}(m_\lambda)\boldsymbol{D}E(\lambda_{(k)})^2(\lambda_{(k)}-\lambda)^{(2)}$.

In the case of arbitrary multiplicities of the zeros of the entire functions $E_j(\zeta_j)$, $j=1, 2, \ldots, n$ a complete description of the coefficient multiplier gives the next

Theorem 10. *Let* $\lambda \in \mathbf{C}^n$ *be fixed such that* $E(\lambda) \neq 0$. *Then an operator* $M: H(\bar{D}) \to H(\bar{D})$ *is a coefficient multiplier of the formal Dirichlet expansion* (22) – (22′) *iff it admits a representation of the form* (14) *with* $m_\lambda \sim \Sigma_{(k)} \mu_{(k)} R_\lambda \varphi_{(k)}$ *(here* $R_\lambda = R^1_{\lambda_1} \ldots R^n_{\lambda_n}$, *see* (11), *i. e.* $Mf = \boldsymbol{D}_\lambda[m_\lambda * f]$ *with* $m_\lambda \in H(\bar{D})$ *satisfying* $m_\lambda \sim \Sigma_{(k)} \mu_{(k)} P_{(k)} \{(-1)^n e^{\langle \lambda, z \rangle} / E(\lambda)\}$.

Proof. Let M be a coefficient multiplier of the formal Dirichlet expansion. Then it is clear that $P_{(k)}[M(f*g) - Mf*g] = 0$ for all $(k) \geqq (0)$ by (37), hence M is a multiplier of $f*g$ and it can be represented as $Mf = \boldsymbol{D}_\lambda[m_\lambda * f]$ with $m_\lambda = Mr_\lambda$, $r_\lambda = (-1)^n e^{\langle \lambda, z \rangle} / E(\lambda)$. Then $P_{(k)} m_\lambda = \mu_{(k)} P_{(k)} r_\lambda = \mu_{(k)} r_\lambda * \varphi_{(k)} = \mu_{(k)} R_\lambda \varphi_{(k)}$.

Conversely let $Mf = \boldsymbol{D}_\lambda[m_\lambda * f)]$ with $m_\lambda \in H(\bar{D})$ satisfying $m_\lambda \sim \Sigma_{(k)} \mu_{(k)} R_\lambda \varphi_{(k)}$. Then $P_{(k)} m_\lambda = \mu_{(k)} R_\lambda \varphi_{(k)}$ and since $m_\lambda * f \in H_{\Phi_1} \cap \cdots \cap H_{\Phi_n}$ we have $P_{(k)} Mf = P_{(k)} \boldsymbol{D}_\lambda [m_\lambda * f] = \boldsymbol{D}_\lambda [P_{(k)} m_\lambda * f] = \boldsymbol{D}_\lambda [\mu_{(k)} R_\lambda \varphi_{(k)} * f] = \mu_{(k)} \boldsymbol{D}_\lambda R_\lambda [\varphi_{(k)} * f] = \mu_{(k)} P_{(k)} f$.

It is clear that Theorem 9 a) is a special case of Theorem 1.

It remains to consider the problem for finding of the *coefficient convolutions* of the general Dirichlet expansion (22) when $E_j(\zeta_j)$, $j=1, \ldots, n$ have multiple zeros.

Let $B_{(k)} = \{\overset{(s)}{\psi}_{(k)}: (0) \leqq (s) \leqq \nu_{(k)}\}$ be a basis in $H_{(k)}$ which is a tensorial product of the basises $B^j_{k_j} = \{\psi^j_{k_j, s_j}\}^{m^j_{k_j}-1}_{s_j=0}$ (chosen such that deg $\{\psi^j_{k_j, s_j} . e^{-\lambda^j_{k_j} z_j}\} = s_j$) of the subspace $H^j_{k_j}$ spanned on $\{e^{\lambda^j_{k_j} z_j}, \ldots, z^{m^j_{k_j}-1} e^{\lambda^j_{k_j} z_j}\}$, i. e. the "splitting property" $\overset{(s)}{\psi}_{(k)}(z) = \Pi^n_{j=1} \psi^j_{k_j, s_j}(z_j)$ holds. Let also $P_{(k)} f = \Sigma^{\nu(k)}_{(s)=(0)} \overset{(s)}{h}_{(k)}(f) \overset{(s)}{\psi}_{(k)}$ be the representation of $P_{(k)}$ relative to $B_{(k)}$. It can be proved that $\overset{(s)}{h}_{(k)}(f)$: $(0) \leqq (s) \leqq \nu_{(k)}$ satisfy the "splitting property" as $\overset{(s)}{a}_{(k)}(f)$ or $\overset{(s)}{C}_{(k)}(f)$ too for functions of the form (12).

Definition 4. *A bilinear operation* $\tilde{*}$ *in* $H(\bar{D})$ *is said to be a coefficient convolution of the formal Dirichlet expansion* (22) *relative to the basis* $B_{(k)}$ *iff there exists a multiple sequence* $\overset{(s)}{\mu}_{(k)}$: $(0) \leqq (s) \leqq \nu_{(k)}$, $(k) \geqq (0)$ *such that*

$$\overset{(s)}{h}_{(k)}(f \tilde{*} g) = \sum_{(p)=(0)}^{(s)} \overset{(s)-(p)}{\mu}_{(k)} \sum_{(j)=(0)}^{(p)} \overset{(p)-(j)}{h}_{(k)}(f) \overset{(j)}{h}_{(k)}(g) \tag{40}$$

holds for $f, g \in H(\bar{D})$ *and each* $(k) \geqq (0)$.

As in 9 it can be proved that for arbitrary (k) there are numbers $\{\overset{(s)}{\delta}_{(k)}$ $\overset{(s)}{\varepsilon}_{(k)}: (0) \leqq (s) \leqq \nu_{(k)}\}$ with $\overset{(0)}{\delta}_{(k)} \overset{(0)}{\varepsilon}_{(k)} = 1$, $\Sigma^{(s)}_{(j)=(0)} \overset{(s)-(j)}{\delta}_{(k)} \overset{(j)}{\varepsilon}_{(k)} = 0$ for $(0) \leqq (s) \leqq \nu_{(k)}$ and

$$\overset{(s)}{\psi}_{(k)} = \sum_{(j)=(0)}^{(s)} \overset{(s)-(j)}{\delta}_{(k)} \overset{(j)}{\varphi}_{(k)}, \quad \overset{(s)}{h}_{(k)}(f) = \sum_{(j)=(0)}^{(s)} \overset{(s)-(j)}{\varepsilon}_{(k)} \overset{(j)}{C}_{(k)}(f)$$

$$\overset{(s)}{\varphi}_{(k)} = \sum_{(j)=(0)}^{(s)} \overset{(s)-(j)}{\varepsilon}_{(k)} \overset{(j)}{\psi}_{(k)}, \quad \overset{(s)}{C}_{(k)}(f) = \sum_{(j)=(0)}^{(s)} \overset{(s)-(j)}{\delta}_{(k)} \overset{(j)}{h}_{(k)}(f) \text{ for } (0)\leqq(s)\leqq\nu_{(k)}$$

(compare (32), (36)). It can be proved by these equalities that the notion of coefficient convolution does not depend on the choice of such kind basis $B_{(k)}$. So it is enough to use the dual basises $\{(\overset{(s)}{\varphi}_{(k)}, \overset{\nu(k)-(s)}{C}_{(k)})\}$, or $\{(z^{(s)}e^{\langle\lambda_{(k)}, z\rangle}, \overset{\nu_{(k)}-(s)}{a}_{(k)})\}$, $(0)\leqq(s)\leqq\nu_{(k)}$. For example

$$\overset{(s)}{C}_{(k)}(f\tilde{*}g) = \sum_{(p)=(0)}^{(s)} \overset{(s)-(p)}{\mu}_{(k)} \sum_{(j)=(0)}^{(p)} \overset{(p)-(j)}{C}_{(k)}(f)\,\overset{(j)}{C}_{(k)}(g) \tag{40'}$$

holds for $f, g \in H(\bar{D})$ and each $(k)\geqq(0)$.

T h e o r e m 11. *An operation* $\tilde{*}$ *in* $H(\bar{D})$ *is a coefficient convolution of the formal Dirichlet expansion iff* $\tilde{*}$ *satisfies at least one of the equivalent assertions a) — e) in Theorem 8, i. e. iff* $\tilde{*}$ *can be represented by* $f\tilde{*}g=\boldsymbol{D}^2_\lambda[m_\lambda * f * g]$ *with* $m_\lambda \in H(\bar{D})$. *The coefficients* $\overset{(s)}{\mu}_{(k)}$ *in* (40') *are the coefficients in* $\varphi_{(k)}\tilde{*}\varphi_{(k)} = \Sigma_{(s)=(0)}^{\nu(k)} \overset{\nu(k)-(s)}{\mu}_{(k)} \overset{(s)}{\varphi}_{(k)}$. *The operation* $\tilde{*}$ *is without annihilators in* $H(\bar{D})$ *iff* $\overset{(0)}{\mu}_{(k)} \neq 0$ *for all* $(k)\geqq(0)$. *Every coefficient convolution is a continuous operation in* $H(\bar{D})$.

P r o o f. We note that a) in Theorem 8 implies (40'). Indeed $\overset{(s)}{\varphi}_{(k)} = \boldsymbol{D}_{\lambda_{(k)}}^{(\nu(k)-(s))}\varphi_{(k)}$ implies $\overset{(s)}{\varphi}_{(k)} * \overset{(j)}{\varphi}_{(k)} = \boldsymbol{D}_{\lambda_{(k)}}^{(2\nu(k)-(s)-(j))}\varphi_{(k)}$ and (40') can be proved in $H_{(k)}$ and hence in $H(\bar{D})$ using the totality and $\overset{(s)}{C}_{(k)}(f)=\overset{(s)}{C}_{(k)}(P_{(k)}f)$. Conversely (40') implies b) in Theorem 8. Indeed (40') implies that $\overset{(s)}{C}_{(k)}[(f\tilde{*}g)*h - f*(g\tilde{*}h)]=0$ for all $(0)\leqq(s)\leqq\nu_{(k)}$, $(k)\geqq(0)$ hence $(f\tilde{*}g)*h=f*(g\tilde{*}h)$.

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Received 27. 5. 1981