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APPROXIMATION OF DIFFERENTIABLE FUNCTIONS BY MONOTONE POLYNOMIALS

K. B. SRIVASTAVA

In this paper, we have started the study of the monotone polynomials giving estimates of Timan Type for the functions whose first derivatives are continuous on $[-1, 1]$. We also announce a theorem on approximation of monotone functions which are $r \geq 0$ (finite) times continuously differentiable on $[-1, 1]$. Our simplest approach has been through trigonometric polynomials.

1. Introduction. In the past few years, attention has been paid to determine the rate of convergence for continuous functions in case of monotone approximation, i. e. approximation of monotone functions by monotone polynomials. In this regard the results of Shisha [6] and Roulier [5] are worth mentioning. Later, in their paper Zeller and Lorentz [4] elaborated that even the sharpened form of estimates (e. g. Timan type) can be obtained for continuous functions via trigonometric polynomials in this case.

Very recently Beaston [1] extended this result to the case when the monotone function to be approximated belongs to $C[-1/4, 1/4]$ and proved Jackson's theorem for differentiable functions.

Here, in this paper, our attempt is to show that if the function under consideration is differentiable in $[-1, 1]$ then we are able to define a monotone polynomial which reproduces Timan's theorem. The second part of this paper will be devoted to the study of the case when $f^{(r)} \in C[-1, 1]$. Our simplest approach is through trigonometric polynomials. An interesting feature of these polynomials is that they do not solely depend on the functions to be approximated as in the case of Beatson [1]. Instead, they now have actual structure.

Notation. Throughout his paper $C_1, C_2, \ldots,$ denote positive constants not depending upon x, n and f.

2. Before we prove our main result we require some auxiliary theorems on trigonometric polynomials which are as follows:

Theorem 1. *Let $f(y) \in C[-\pi, \pi]$ be an even and decreasing function, then we can find out a trigonometric polynomial $L_{nl}(f, y)$ such that*

(2.1)

$$L_{n1}(f, y) \text{ is decreasing and}$$

$$|L_{n1}^{(\nu)}(f, y) - f^{(\nu)}(y)| \leq C_\nu (1/n)^{1-\nu} \omega_{f'}(1/n), \tag{2.2}$$

where $\omega_{f'}(\cdot)$ is the modulus of continuity of f'.

Proof. Let us modify the traditional convolution operator

$$f * d\mu_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) d\mu_n(t-y) \tag{2.3}$$

by the following rule

$$f * d\mu_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \Big[\sum_{\nu=0}^{r} \frac{(y-t)^\nu}{\nu!} f^{(\nu)}(t) \Big] d\mu_n(y-t), \tag{2.4}$$

where $d\mu_n$ is an even Borel measure positive on $[-\pi, \pi]$. If we take

$$d\mu_n(t) = \Big(\frac{\sin nt/2}{\sin t/2}\Big)^{2r+4} dt, \tag{2.5}$$

then, for $r=1$, we obtain

$$L_{n1}(f, y) = \frac{1}{A_{n1}} \int_{-\pi}^{\pi} [f(t) + (y-t) f'(t)] \Big(\frac{\sin n/2(y-t)/2}{\sin (y-t)/2}\Big)^6 dt, \tag{2.6}$$

where A_{n1} is a normalizing constant given by

$$A_{n1} = \int_{-\pi}^{\pi} \Big(\frac{\sin nt/2}{\sin t/2}\Big)^6 dt = \int_{-\pi}^{\pi} \Big(\frac{\sin n(t-y)/2}{\sin (t-y)/2}\Big)^6 dt. \tag{2.7}$$

For $r=0$, (2.4) is the famous Jackson operator. Now to prove the second part of the theorem for $v=0$, we require the following useful equality

$$f(u) - \sum_{\nu=0}^{r} \frac{(u-v)^\nu}{\nu!} f^{(\nu)}(v) = 0(|u-v|^r \omega_{f^{(r)}}(|u-v|), \tag{2.8}$$

whose proof depends on the finite Taylor's expansion of the function and therefore can readily be verified.

Owing to (2.8) for $r=1$, we get from (2.6)

$$|L_{n1}(f, y) - f(y)| \leq \frac{C_1}{A_{n1}} \int_{-\pi}^{\pi} |t-y| \omega_{f'}(|t-y|) \Big(\frac{\sin n(t-y)/2}{\sin (t-y)/2}\Big)^6 dt \tag{2.9}$$

$$\leq \frac{C_1 \omega_{f'}(1/n)}{A_{n1}} \Big[n \int_{-\pi}^{\pi} u^2 \Big(\frac{\sin nu/2}{\sin 4/2}\Big)^6 du + \int_{-\pi}^{\pi} |u| \Big(\frac{\sin nu/2}{\sin u/2}\Big)^6 du\Big].$$

It is easy to establish that

$$\frac{1}{A_{n1}} \int_{-\pi}^{\pi} u^2 \Big(\frac{\sin nu/2}{\sin u/2}\Big)^6 du = o(1/n^2). \tag{2.10}$$

and

$$\frac{1}{A_{n1}} \int_{-\pi}^{\pi} |u| \Big(\frac{\sin nu/2}{\sin u/2}\Big)^6 du = o(1/n). \tag{2.11}$$

The proofs of these equalities are contained in [8] and are therefore omitted.

Combining (2.11) and (2.10) with (2.9), we at once have the proof of the theorem for $\nu=0$.

To prove the theorem for $\nu=1$, we differentiate (2.6) and write the identity

$$(2.12)\qquad -f'(y)+L'_{n1}(f,y)=\frac{1}{A_{n1}}\int_{-\pi}^{\pi}(f'(t)-f'(y))\Big(\frac{\sin n(t-y)/2}{\sin (t-y)/2}\Big)^6 dt$$

$$-\frac{1}{A_{n1}}\int_{-\pi}^{\pi}[f(y)-f(t)-(y-t)f'(t)]\frac{d}{dy}\Big(\frac{\sin n(t-y)/2}{\sin (t-y)/2}\Big)^6 dt=I_1+I_2.$$

Here we have used

$$(2.13)\qquad \int_{-\pi}^{\pi}\frac{d}{dy}\Big(\frac{\sin n(t-y)/2}{\sin (t-y)/2}\Big)^6 dt=0.$$

We immediately get the required form of I_1 with the help of a theorem of Jackson. To derive an estimation for I_2 in the desired form we use (2.8) for $r=1$ and then obtain

$$(2.14)\qquad |I_2|\leqq\frac{1}{A_{n1}}\int_{-\pi}^{\pi}|u|\,\omega_{f'}(|u|)\,\Big|\frac{d}{du}\Big(\frac{\sin nu/2}{\sin 4/2}\Big)^6\Big|du$$

$$\leqq\frac{\omega_{f'}(1/n)}{A_{n1}}\int_{-\pi}^{\pi}(nu^2+|u|)\Big|\frac{d}{du}\Big(\frac{\sin nu/2}{\sin u/2}\Big)^6\Big|du$$

$$\leqq\omega_{f'}(1/n)\Big\{\frac{n}{A_{n1}}\int_{-\pi}^{\pi}u^2\Big|\frac{d}{du}\Big(\frac{\sin nu/2}{\sin 4/2}\Big)^6\Big|du$$

$$+\frac{1}{A_{n1}}\int_{-\pi}^{\pi}|u|\Big|\frac{d}{du}\Big(\frac{\sin nu/2}{\sin u/2}\Big)^6\Big|du\Big\}.$$

Owing to the equalities (2.10) and (2.11) and Bernstein's inequality for the derivative of the polynomial, we now have

$$(2.15)\qquad \int_{-\pi}^{\pi}u^2\Big|\frac{d}{du}\Big(\frac{\sin nu/2}{\sin u/2}\Big)^6\Big|du\leqq 2\times 8\int_0^{\pi/2}u^2\Big|\frac{d}{du}\Big(\frac{\sin nu}{\sin u}\Big)^6\Big|du$$

$$\leqq 16\Big[u^2\Big(\frac{\sin nu}{\sin u}\Big)^6\Big]_0^{\pi/2}+32\int_0^{\pi/2}u\Big(\frac{\sin nu}{\sin u}\Big)^6du\leqq 16\,.\,\frac{\pi^2}{4}\,.\,n^4+32\,.\,C_2\,.\,n^4\leqq C_3\,.\,n^4$$

similarly, we can prove

$$(2.16)\qquad \int_{-\pi}^{\pi}|u|\Big|\frac{d}{du}\Big(\frac{\sin nu/2}{\sin u/2}\Big)^6\Big|du\leqq C_4n^5.$$

On account of (2.16) and (2.15), we obtain $|I_2|\leqq C_5\omega_{f'}(1/n)$. Here we have used that

$$(2.17)\qquad -A_{nr}=o(n^{2r+3}),\qquad r=0,\,1,\,2,\,\ldots,$$

which has been determined in [8]. To complete the proof of the theorem, we establish either (2.10) or (2.11). To this end, we use the technique of [8] and obtain

$$\frac{1}{A_{n1}}\int_{-\pi}^{\pi}u^2\Big(\frac{\sin nu/2}{\sin u/2}\Big)^6du=\frac{16}{A_{n1}}\Big[\int_0^{\pi/2n}u^2\Big(\frac{\sin nu}{\sin u}\Big)^6du+\int_{\pi/2n}^{\pi/2}u^2\Big(\frac{\sin nu}{\sin u}\Big)^6du\Big].$$

Because of the inequalities $|\sin nt| \leq n|\sin t|$ and $\sin t \geq 2t/\pi$, $0 \leq t \leq \pi/2$, we get,

$$\frac{1}{A_{n1}} \int_{-\pi}^{\pi} u^2 \left(\frac{\sin nu/2}{\sin u/2}\right)^6 du \leq \frac{10\times 16}{11\pi n^5} [n^6 \int_0^{\pi/2n} u^2 du + \frac{n^6}{6u} \int_{\pi/2n}^{\pi/2} du$$

$$\leq \frac{160}{11\pi n^5} [\frac{n^6\pi^3}{2un^3} + \frac{\pi^6}{bu} \frac{\pi}{2}] = o(n^{-2}).$$

To prove the first part of the theorem, it is sufficient to consider the function

$$(2.18) \qquad g(f, y) = \begin{cases} f(t_k) + (y - t_k) f'(t_k), & t_k \leq y \leq t_k + 1, \\ b_k(y), & t_k = k\pi/n, \\ g(y), & -\pi \leq y \leq 0. \end{cases}$$

Then the function g is even and

$$g(f, y) - f(y)| \leq (y - t_k)\, \omega_{f'}(y - t_k) \leq \pi/n\, \omega_{f'}(\pi/n),$$

$$|g'(f, y) - f'(y)| \leq |f'(y) - f'(t_k)| \leq \omega_{f'}(y - t_k) \leq \omega_{f'}(\pi/n).$$

Therefore, we have

$$|L_{n1}^{(\nu)}(g, y) - f^{(\nu)}(y)| \leq |L_{n1}^{(\nu)}(f, y) - f^{(\nu)}(y)| + \| L_{n1}(g, y) - L_{n1}(f, y) \|$$

$$\leq (C_5/n)^{1-\nu}_{\omega_{f'}}(1/n) + |g^{(\nu)}(f, y) - f^{(\nu)}(y)| = o[(1/n)^{1-\nu} \omega_{f'}(1/n)].$$

Here we have made use of Theorem 1(b) and (2.19). Now we have made use of Theorem 1 (b) and (2.19).

Now we have to show that $L_{n1}(g, y)$ is decreasing. Let d_k, $k = \overline{0, n-1}$, be defined by $b_k(y) = d_k + \ldots + d_{n-1}$, $k = \overline{0, n-1}$ then, because of the definition of $b_k(y)$, we have $d_k \geq 0$, and

$$L_{n1}(g, y) = \frac{1}{A_{n1}} \sum_{k=0}^{n-1} d_k \int_{-\pi(k+1/n}^{\pi(k+1)/n} \mu_n(y-t) dt$$

$$= \frac{1}{A_{n1}} \sum_{k=1}^{n} d_{k-1} \int_{-t_k}^{t_k} \left(\frac{\sin n(y-t)/2}{\sin (y-t)/2}\right)^6 dt.$$

If the operator $L_{n1}(g, y)$ has to be decreasing, then it is required to show that the functions

$$\varphi_k(y) = \int_{-t_k}^{t_k} \left(\frac{\sin n(y-t)/2}{\sin (y-t)/2}\right)^6 dt = \int_{y-t_k}^{y+t_k} \left(\frac{\sin nt/2}{\sin t/2}\right)^6 dt$$

are decreasing on $(0, \pi)$. But

$$\varphi_k'(y) = \frac{\sin^6 n(y+t_k)/2}{\sin (y+t_k)/2} - \frac{\sin^6 n(y-t_k)/2}{\sin^6 (y-t_k)/2}$$

$$= \sin^6 n(y+t_k)/2 [1/\sin^6 (y+t_k)/2 - 1/\sin^6 (y-t_k)/2] \leq 0,$$

since $\sin(a+\beta) \geq |(\alpha-\beta)|$.

Thus the polynomial operator $L_{n1}(g, y)$ is decreasing and this furnishes the proof of the theorem (a).

3. In this section we announce our main result and show how this depends upon the theorem proved in the last section.

Theorem 2. *If $F(x)$ is an increasing function differentiable on $[-1, 1]$, then there exists a sequence of polynomials $P_{n1}(F, x)$ increasing on $[-1, 1]$ such that*

$$(3.1) \qquad |F^{(\nu)}(x) - P_{n1}^{(\nu)}(F, x)| \leq C[\Delta_n(x)]^{1-\nu}\omega_{F'}(\Delta_n(x))$$

$\nu = 0, 1$, *where C_ν is a constant not depending on x, n and*

$$(3.2) \qquad \Delta_n(x) = ((\sqrt{1-x^2})/n + 1/n^2).$$

This reproduces a result of A. F. Timan [9] in our case of monotone approximation.

Proof. The function $F(\cos y) = f(y)$ is even and decreasing on $[-\pi, \pi]$. We define the algebraic polynomial $P_{n1}(F, x)$ as follows:

$$(3.3) \qquad P_{n1}(F, x) = L_{n1}(f, y) = \frac{1}{A_{n1}} \int_{-\pi}^{\pi} [f(t) + (y-t) f'(t)] \left(\frac{\sin n(t-y)/2}{\sin (t-y)/2}\right)^6 dt.$$

Then, for the function $F(\cos y)$, we established similar to Theorem 1:

$$(3.4) \qquad |f^{(\nu)}(y) - L_{n-1}^{(\nu)}(f, y)| \leq C[\Delta_n(x)]^{1-\nu}\omega_{F'}(\Delta_n(x)), \quad \nu = 0, 1.$$

For $\nu = 0$, the brief sketch of the proof is as follows:

$$(3.5) \qquad |f(y) - L_{n1}(f, y)|$$
$$\leq \frac{C_l}{A_{n1}} \int_{-\pi}^{\pi} |\cos t - \cos y|\, \omega_{F'}(|\cos t - \cos y|) \left(\frac{\sin n(t-y)/2}{\text{sni}\,(t-y)/2}\right)^6 dt$$
$$\leq \frac{C_1}{A_{n1}} \left[\frac{1}{\Delta_n(x)} \int_{-\pi}^{\pi} (\cos t - \cos y)^2 \left(\frac{\sin n(t-y)/2}{\sin (t-y)/2}\right)^6 dt\right.$$
$$\left. + \int_{-\pi}^{\pi} |\cos t - \cos y| \left(\frac{\sin n(t-y)/2}{\sin (t-y)/2}\right)^6 dt\right] \omega_{F'}(\Delta_n(x)).$$

It remains to show that

$$(3.6) \qquad \frac{1}{A_{n1}\Delta_n(x)} \int_{-\pi}^{\pi} (\cos t - \cos y)^2 \left(\frac{\sin n(t-y)/2}{\sin (t-y)/2}\right)^6 dt = o(\Delta_n(x)$$

and

$$(3.7) \qquad \frac{1}{A_{n1}} \int_{-\pi}^{\pi} |\cos t - \cos y| \left(\frac{\sin n(t-y)/2}{\sin (t-y)/2}\right)^6 dt = 0(\Delta_n(x)).$$

We see that

$$(3.8) \qquad \frac{1}{A_{n1}\Delta_n(x)} \int_{-\pi}^{\pi} (\cos t - \cos y)^2 \left(\frac{\sin n(t-y)/2}{\sin (t-y)/2}\right)^6 dt$$

$$\leqq \frac{4}{A_{n1}\Delta_n(x)} \int_{-\pi}^{\pi} [\sin^2 y \sin^2(t-y)/2 + 2 \sin y \sin^3 |t-y|/2$$

$$+ \sin^4(t-y)/2](\frac{\sin n(t-y)/2}{\sin (t-y)/2})^6 dt$$

$$\leqq \frac{4n}{A_{n1}} \int_{-\pi}^{\pi} \sin y \,.\, \sin^2(t-y)/2(\frac{\sin n(t-y)/2}{\sin (t-y)/2})^6 dt$$

$$+ \frac{4n^2}{A_{n1}} \int_{-\pi}^{\pi} \sin^4(t-y)/2\,(\frac{\sin n(t-y)/2}{\sin (t-y)/2})^6 dt$$

$$+ \frac{8n^2 \sin y}{A_{n1}} \int_{-\pi}^{\pi} \sin^3 |t-y|/2(\frac{\sin n(t-y)/2}{\sin (t-y)/2})^6 dt$$

$$+ 4n \sin y \,.\, I_2 + 8n^2 \sin y \,.\, I_3 + 4n^2 I_4,$$

where

(3.9) $$I_r = \frac{1}{A_{n1}} \int_{-\pi}^{\pi} \sin^r |t-y|/2(\frac{\sin n(t-y)/2}{\sin (t-y)/2})^6 dt.$$

Similarly, we can find that

(3.10) $$-\frac{1}{A_{n1}} \int_{-\pi}^{\pi} \cos t - \cos y(\frac{\sin n(t-y)/2}{\sin (t-y)/2})^6 dt \leqq 2 \sin y \,.\, I_1 + 2I_2.$$

Using the lemma that follows we at once get

$$|F(x) - P_{n1}(F, x)| = o[\Delta_n(x)\omega_{F'}(\Delta_n(x))].$$

Lemma. *For $t,\ y \in [-\pi, \pi]$, we have*

(3.11) $$I_r = o(1/n^r), \quad r = \overline{0,\ 4}.$$

Proof. Let $r = 1,\ 3$, then, we have

(3.12) $$I_r \leqq \frac{1}{11n^5} \int_{-\pi}^{\pi} \left|\frac{\sin n(t-y)/2}{\sin 1/2(t-y)}\right|^{6-r} dt$$

$$\leqq \frac{n}{11n^5} \int_{-\pi}^{\pi} (\frac{\sin nu/2}{\sin u/2})^{6-r-1} du \ \ (|\sin nt| \leqq n |\sin t|) \leqq \frac{1 \,.\, A_n(1-r)/2}{11n^4} = o(1/n^r).$$

Here we have made use of

(3.13) $$A_{n(1-r)/2} = o(n^{4-r}),$$

which has been established in [8].

To prove the theorem for $\nu = 1$, we again use the method of Theorem 1, techniques employed in [7]. We now have one factor sign in the denominator of the intermediate estimates involved. For this, we use the method of change of variable of the integral. Thus we have proved the Theorem 2.

In order to show that $P_{n1}(F, x)$ is increasing, we take help of Theorem 1(a).

4. In this section we describe the main theorem of this paper which is as follows:

Theorem 3. *Let $F(x)$ be an increasing function differentiable $r \geqq 0$ times and $F^{(r)}(x) \in C[-1, 1]$, then there exists an increasing polynomial $P_{nr}(F, x)$ such that*

$$(4.1) \qquad |F(x)-P_{nr}(F, x)| \leqq C_r |\Delta_n(x)|^r \omega_{F^{(r)}}(\Delta_n(x)).$$

Proof. The proof of the theorem depends upon a lemma given in [8] which is as follows

$$\int_{-\pi}^{\pi} |\cos t - \cos y|^r \omega_{F'}(|\cos t - \cos y|) \left(\frac{\sin n(t-y)/2}{\sin (t-y)/2}\right)^{2r+4} dt$$

$$= o[\Delta_n(x)]^r \omega_{F'}(\Delta_n(x)),$$

where $\Delta_n(x)$ is given by (3.2). The proof of the theorem is done by mathematical induction on r. The theorem has already been proved to be true for $r=1$.

This completes the proof of the theorem.

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29, Birbal Sahni Hall
University of Lucknow
Lucknow-7 (U. P.) India

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