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INTEGRALS OF GENERALIZED LAGUERRE POLYNOMIALS

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We evaluate integrals involving the generalized Laguerre polynomials $L_n^a(x)$ and $\ln(x)$. The results provide the modified moments of the weight function $x^\beta e^{-px}\ln x$ on $(0,\infty)$ with respect to the $L_n^a(x)$, and are suitable for computation. Results involving ψ-function are given and special cases are mentioned. Numerical tables are given for estimation and verification of some results.

1. In a recent paper [6], the authors have obtained the modified moments of the weight function $x^m(1-x)^m\ln(1/x(1-x))$ on $[0,1]$ with respect to the shifted Legendre polynomials $P_n^*(x)=P_n(2x-1)$. Some results involving $P_n^*(x)$ are also given by Blue [1] and Gautschi [4].

Recently, Gatteschi [3] has given some results involving $L_n^a(x)$. In the present paper we consider the integral $\int_0^\infty e^{-pt}t^\beta L_n^\alpha(t)dt$ and its partial derivatives with respect to β and α. The results of this paper generalize those of Gatteschi [3]. Incidently, we obtain the sum of a series involving the ψ-function, the logarithmic derivative of the gamma function [8]. Some results are computed for estimation and verification.

2. A Laguerre polynomial integrals. We start with the result [5, p. 844 (7)]

$$(2.1)\qquad K_{p,n}^{\alpha,\beta}=\int_0^\infty e^{-pt}t^\beta L_n^\alpha(t)dt=\frac{\Gamma(\beta+1)\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+1)p^{\beta+1}}\,{}_2F_1(-n,\beta+1;\alpha+1;\frac{1}{p})$$

$$=\frac{\Gamma(\beta+1)\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+1)p^{\beta+1}}\sum_{k=0}^{n}\frac{(-n)_k(\beta+1)_k p^{-k}}{(\alpha+1)_k\,k!},\quad \mathrm{Re}(\beta)>-1,\ \mathrm{Re}(p)>0,\ \alpha>-1.$$

Then

$$(2.2)\qquad \frac{\partial}{\partial\beta}K_{p,n}^{\alpha,\beta}=S_{p,n}^{\alpha,\beta}=\int_0^\infty e^{-pt}t^\beta\ln t_n^\alpha(t)dt=-\frac{\Gamma(\alpha+n+1)\Gamma(\beta+1)\ln p}{n!\,\Gamma(\alpha+1)p^{\beta+1}}\,{}_2F_1(-n,\beta+1;$$
$$\alpha+1;\frac{1}{p})+\frac{\Gamma(\alpha+n+1)}{n!\,p^{\beta+1}}\sum_{k=0}^{n}\frac{(-n)_k\Gamma/\beta+1+k)p^{-k}}{k!\,\Gamma(\alpha+1+k)}\psi(\beta+1+k)\quad \alpha,\ \mathrm{Re}(\beta)>-1,\ \mathrm{Re}(p)<0,$$

with $\psi(x)=\Gamma'(x)/\Gamma(x)$, the logarithmic derivative of the *gamma function* [8].

It is interesting to note that (2.2) provides us with the Laplace transform of $t^\beta\ln tL_n^\alpha(t)$ and, of course, the Mellin transform of $e^{-pt}\ln tL_n^\alpha(t)$, with the kernel t^β and $\beta=s-1$.

Now,

$$(2.3)\qquad S_{1,n}^{\alpha,\beta}=\frac{\Gamma(\alpha+n+1)}{n!}\sum_{k=0}^{n}\frac{(-n)_k\Gamma(\beta+1+k)}{k!\,\Gamma(\alpha+1+k)}\psi(\beta+1+k),\ \alpha,\ \mathrm{Re}(\beta)>-1.$$

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If we take (2.1) with $p=1$, which is the same as [5, p. 845 (11)]

$$(2.4)\qquad \int_0^\infty x^\beta e^{-x} L_n^\alpha(x)dx = \frac{\Gamma(\beta+1)\Gamma(\alpha+n-\beta)}{n!\,\Gamma(\alpha-\beta)}, \quad \mathrm{Re}(\beta)>-1,\ \alpha>-1,$$

and differentiate it with respect to β, we obtain

$$(2.5)\quad I_n^{\alpha,\beta} = \int_0^\infty x^\beta e^{-x} \ln x L_n^\alpha(x)dx = \frac{\Gamma(\alpha-\beta+n)\Gamma(\beta+1)}{n!\,\Gamma(\alpha-\beta)}[\psi(\beta+1)-\psi(\alpha-\beta+n)+\psi(\alpha-\beta)]$$

$$\alpha-\beta\neq 0,\ -1,\ -2, \ldots (1-n) \text{ if } n\geqq 1,\ \mathrm{Re}(\beta)>-1,\ \alpha>-1.$$

By virtue of the following property of the ψ-function

$$\psi(z+n) = \sum_{k=0}^{n-1} \frac{1}{z+k} + \psi(z),\ n=1, 2, 3, \ldots,$$

the result (2.5) can be expressed as

$$I_n^{\alpha,\beta} = \frac{(\alpha-\beta)_n\,\Gamma(\beta+1)}{n!}\,[\psi(\beta+1) - \sum_{k=0}^{n-1}(\frac{1}{\alpha-\beta+k})]\ \alpha-\beta\neq 0,\ -1,\ -2, \ldots,$$

$$\mathrm{Re}(\beta)>-1,\ \alpha>-1.$$

We also have

$$I_n^{\alpha;\,\alpha+\mu} = \frac{(-1)^n}{n!}\,\frac{\Gamma(\mu+1)\Gamma(\alpha+\mu+1)}{\Gamma(\mu-n+1)}\,[\psi(\mu+1)+\psi(\mu+\alpha+1)-\psi(\mu-n+1)],$$

$$\alpha,\ \alpha+\mu>-1,\ \mu\neq 0, 1, \ldots, n-1, \text{ if } n\geqq 1,$$

a result recently given by Gatteschi [3].

When $\beta-\alpha=m=0, +1, \ldots, n-1$, then by taking limit we have

$$I_n^{\alpha,\beta} = (-1)^{m-1}\Gamma(n-m)\Gamma(\beta+1)m!/n!,\ m=0, 1, \ldots, n-1,\ n\geqq 1,$$

and

$$(2.6)\qquad I_n^{\alpha,\alpha} = -\frac{\Gamma(\alpha+1)}{n},\quad n=1, 2, 3, \ldots (n\neq 0),\ \alpha>-1.$$

If we start from (2.4) with $n=0$, we can easily derive

$$(2.7)\qquad I_0^{\alpha,\alpha} = \Gamma'(\alpha+1) = \psi(\alpha+1)/\Gamma(\alpha+1),\quad \alpha>-1.$$

If we compare (2.3) with (2.5), we obtain

$$(2.8)\qquad \sum_{k=0}^{n} \frac{(-n)_k\,\Gamma(\beta+1+k)}{k!\,\Gamma(\alpha+1+k)}\,\psi(\beta+1+k)$$

$$= \frac{\Gamma(\alpha-\beta+n)\Gamma(\beta+1)}{\Gamma(\alpha-\beta)\Gamma(\alpha+n+1)}\,[\psi(\beta+1)-\psi(\alpha-\beta+n)]+\psi(\alpha-\beta)],$$

$$n=0, 1, 2, \ldots;\ \alpha,\ \mathrm{Re}(\beta)>-1 \text{ and } \beta-\alpha\neq 0, 1, 2, \ldots, n-1 \text{ if } n\geqq 1.$$

When $\beta-\alpha=m=0, 1, 2, \ldots, n-1$, we obtain

$$(2.9)\qquad \sum_{k=0}^{n} \frac{(-n)_k\,\Gamma(\alpha+m+k+1)}{k!\,\Gamma(\alpha+1+k)}\,\psi(\beta+1+k) = \frac{(-1)^{m-1}\,\Gamma(n-m)\,\Gamma(\alpha+m+1)\,m!}{\Gamma(\alpha+n+1)}.$$

Further $\alpha=\beta$ in (2.4) leads to $\int_0^\infty x^\alpha e^{-x}L_n^\alpha(x)dx=0,\ n\neq0$, which on differentiating partially with respect to α gives

$$\int_0^\infty x^\alpha e^{-x}\frac{\partial}{\partial\alpha}(L_n^\alpha(x))dx=-I_n^{\alpha,\alpha}=\frac{\Gamma(\alpha+1)}{n}\ \alpha>-1,\ n\neq0.$$

Now, differentiation of (2.4) partially with respect to α, gives

$$J_n^{\alpha,\beta}=\int_0^\infty x^\beta e^{-x}\left(\frac{\partial}{\partial\alpha}L_n^\alpha(x)\right)dx=\frac{\Gamma(\beta+1)\,\Gamma(\alpha+n-\beta)}{n!\,\Gamma(\alpha-\beta)}\,[\psi(\alpha-\beta+n)-\psi(\alpha-\beta)],$$

$$\beta-\alpha=m\neq0,\ 1,\ 2,\ \ldots,\ n-1,\ n\geqq1,\ \alpha>-1,\ \mathrm{Re}(\beta)>-1.$$

For $m=0,1,2,\ldots,n-1,\ n>1$, we have $J_n^{\alpha;\alpha+m}=(-1)^m\,\Gamma(\alpha+m+1)\,\Gamma(n-m)\,m!/n!,\ \alpha>-1,\ \mathrm{Re}(\beta)>-1$, and $J_n^{\alpha,\alpha}=\Gamma(\alpha+1)/n,\ n\neq0,\ \alpha>-1$.

3. Special Cases. $S_{p,0}^{\alpha,\beta}=[\Gamma(\beta+1)/p^{\beta+1}]\,[\psi(\beta+1)-\ln p]$, independent of α, of course $(L_0^\alpha=1)\ \mathrm{Re}(p)>0,\ \mathrm{Re}(\beta)>-1$, which is a known result [2, p. 315 (9)].

For $\alpha=\pm1/2$, $L_n^\alpha(x)$ reduces to the Hermite polynomials [7, p. 81]

$$L_n^{1/2}(x)=\frac{(-1)^n}{2^{2n}n!}H_{2n}(\sqrt{x}),\quad L_n^{-1/2}(x)=\frac{(-1)^n}{2^{2n+1}n!}\frac{H_{2n+1}(\sqrt{x})}{\sqrt{x}},$$

which lead to

$$S_{p,n}^{-1/2,\beta}=\frac{(-1)^n}{2^{2n}n!}\int_0^\infty e^{-pt}t^\beta\ln t\,H_{2n}(\sqrt{t})dt=-\frac{\Gamma(n+1/2)\Gamma(\beta+1)\ln p}{n!\sqrt{\pi}\,p^{\beta+1}}\,{}_2F_1(-n,\beta+1;1/2;1/p)$$

$$+\frac{\Gamma(n+1/2)}{n!\,p^{\beta+1}}\sum_{k=0}^n\frac{(-n)_k\Gamma(\beta+1+k)p^{-k}}{k!\,\Gamma(k+1/2)}\psi(\beta+1+k),\ \beta>-1,\ \mathrm{Re}(p)>0$$

and

$$S_{p,n}^{1/2,\beta}=\frac{(-1)^n}{2^{2n+1}n!}\int_0^\infty e^{-pt}t^{\beta-1/2}\ln t\,H_{2n+1}(\sqrt{t})\,dt$$

$$=-\frac{2\Gamma(n+3/2)\ \Gamma(\beta+1)\ln p}{n!\ \sqrt{\pi}\,p^{\beta+1}}\,{}_2F_1\left(-n,\beta+1;\ \frac{3}{2};\ \frac{1}{p}\right)$$

$$+\frac{\Gamma(n+3/2)}{n!\ p^{\beta+1}}\sum_{k=0}^n\frac{(-n)_k\Gamma(\beta+1+k)p^{-k}}{k!\ \Gamma(k+3/2)}\psi(\beta+1+k),$$

$$S_{p,0}^{0,0}=-[\gamma+\ln p]/p,\quad \mathrm{Re}(p)>0,$$

where γ is Euler-Mascheroni constant.

$$S_{p,n}^{0,\beta}=-\frac{\Gamma(\beta+1)\ln p}{p^{\beta+1}}\,{}_2F_1\left(-n,\beta+1;1;\ \frac{1}{p}\right)+p^{-\beta-1}\sum_{k=0}^n\frac{(-n)_k\,\Gamma(\beta+1+k)p^{-k}}{(k!)^2}\pi(\beta+1+k)$$

$$\mathrm{Re}(\beta)>-1,\ \mathrm{Re}(p)>0.$$

Further, we observe that

$$I_0^{\alpha,\beta}=\Gamma(\beta+1)\psi(\beta+1),\ \alpha>-1,\ \mathrm{Re}(\beta)>-1$$

independent of α, as $L_0^\alpha(x)=1$.

$$I_n^{0,\beta}=\frac{(-\beta)_n\Gamma(\beta+1)}{n!}[\psi(\beta+1)-\sum_{k=0}^{n-1}\frac{1}{-\beta+k}],\quad \mathrm{Re}(\beta)>-1,$$

$$I_n^{\alpha,0}=\frac{(\alpha)_n}{n!}[\psi(\alpha)-\psi(\alpha+n)-\gamma],\quad \alpha>-1,$$

$$I_n^{\pm 1/2,\beta}=\frac{(\pm 1/2-\beta)_n\Gamma(\beta+1)}{n!}[\psi(\beta+1)-\sum_{k=0}^{n-1}\frac{1}{(\pm 1/2-\beta+k)}],\quad \mathrm{Re}(\beta)>-1,$$

$$J_n^{0,\beta}=\frac{(-\beta)_n\Gamma(\beta+1)}{n!}\sum_{k=0}^{n-1}(\frac{1}{-\beta+k}),\quad \mathrm{Re}(\beta)>-1.$$

By repeatedly differentiating (2.1) with respect to β, one can evaluate integrals of the type $\int_0^\infty e^{-pt}\,t^\beta(\ln t)^m L_n^\alpha(t)dt$.

4. Numerical Evaluations. Table 1 deals with the verification of the results (2.6) and (2.7) for some selected values of α and n. In this table v_1 represents the integral $\int_0^B x^\alpha \ln x\, e^{-x} L_n^\alpha(x)dx$, evaluated by an appeal to the Simpson's method. Various tests with the computer, lead us to take $B=50$, which adapts well to ∞ due to the presence of the exponential factor in the integrand, whereas h is taken as $B/2000$. v_2 represents the value of $-\Gamma(a+1)/n$, $n\neq 0$ and $\psi(\alpha+1)\Gamma(\alpha+1)$ for $n=0$. As expected the values of v_1 and v_2 are in complete agreement except small differences generated due to the approximate method of integration.

Table 2 gives us a numerical verification of our result (2.8), for selected values of α, β and n. In this case v_3 represents the numerical values of the

Table 1

α	n	V_1	V_2
1.5	0	.934728	.934735
2.8	0	5.62216	5.62216
5.0	0	204.734	204.734
1.0	1	—.999902	—1.00000
2.5	1	—3.32335	—3.32335
5.0	1	—120.000	—120.000
1.0	2	—.499852	—.500000
2.5	2	—1.66167	—1.66168
5.0	2	—60.0000	—60.0000
1.0	3	—.333134	—3.33333
2.5	3	—1 10778	—1.10778
5.0	3	—40.0000	—40.0000
1.0	4	—.249749	—.250000
1.5	4	—.332391	—.332335
2.5	4	—.830836	—.830838
5.0	4	—30.0000	—30.0000
1.0	5	—.199697	—.200000
2.5	5	—.664668	—.664670
5.0	5	—25.0000	—24.0000
5.0	8	—15.0000	—15.0000
5.0	10	—12.0001	—12.0000

Table 2

α	β	n	V_3	V_4
3.0	2.0	1	—.643464D—02	—.643464D—02
5.5	3.2	2	.240590D—02	.240590D—02
5.0	2.5	3	.633579D—03	.633579D—03
8.2	4.5	4	.351225D—04	.351225D—04
10.0	6.2	5	.443417D—05	.443417D—05
8.5	2.0	6	.654440D—06	.654439D—06
12.0	5.0	8	.540216D—08	.540216D—08
6.5	3.5	10	—.697273D—05	—.697273D—05

Table 3

α	n	V_5	V_6
4.0	1	—.200000	—.200000
5.5	2	—.205128D—01	—.205128D—01
3.0	3	—.166667D—01	—.166667D—01
6.4	5	—.346445D—03	—.346445D—03
8.0	6	—.555001D—04	—.555001D—04
3.5	8	—.428402D—03	—.428402D—03
9.8	10	—.619294D—06	—.619294D—06

series and v_4 gives the corresponding values of the right hand side of the expression.

Similarly, Table 3 verifies our result (2.9) for $m=0$ and some selected values of α and n. v_5 represents the numerical value of the series, whereas v_6 that of the other side of the expression.

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