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ITERATIVE OPERATORY SPACES AND THE SYSTEM OF SCOTT AND DE BAKKER

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The notion of iterative operatory space (IOS) is introduced in [5], where the foundations of an intrinsic recursion theory on IOS are announced. Some simple examples of IOS are considered in [6] with applications to the ordinary recursion theory. A detailed exposition of these results is given in [7]. In the present paper a part of the IOS-theory is treated in details. A connection with the formal system of Scott and de Bakker from [1, 2] is established, which allows to reaffirm our results in the case of that system. The terminology and notations from [5, 6] are used here mostly without reference.

1. Iterative Operatory Spaces. Let $\mathscr{S}=\langle\mathscr{F}, I, \Pi, L, R\rangle$ be an operatory space (OS). If $\Gamma:\mathscr{F}\to\mathscr{F}$, then $\mu\theta.\Gamma(\theta)$ is written for the least solution of the inequality $\Gamma(\theta)\leqq\theta$, provided it exists. Certainly, $\mu\theta.\Gamma(\theta)$ is the least fixed point of Γ whenever Γ is monotonous.

An iterative operatory space is an OS with some μ-axiom satisfied, to write IOS $=$ OS $+\mu$A. We are going now to state four versions of such an axiom, namely μA_0, μA_1, μA_2 and μA_3. The axioms μA_2, μA_3 are introduced in [5, 7], respectively, while μA_0, μA_1 are new ones.

Axiom μA_0. *Two additional unary operations* $\langle\ \rangle$, $[\ ]$, *are supposed to be given, such that the following conditions are satisfied for all* φ, ψ, ψ' and τ *in* $\mathscr{F}$:

$$\text{(i)}\quad (\varphi L, \langle\varphi\rangle R)\leqq\langle\varphi\rangle,\quad R\psi\leqq\psi\psi'\&(\varphi L\psi, \tau\psi')\leqq\tau\Rightarrow\langle\varphi\rangle\psi\leqq\tau,$$

$$\text{(ii)}\quad (I, \varphi[\varphi]\leqq[\varphi],\quad (\psi, \varphi\tau)\leqq\tau\Rightarrow[\varphi]\psi\leqq\tau.$$

Taking $\psi=I$, $\psi'=R$, we get immediately that $\langle\varphi\rangle=\mu\theta.(\varphi L, \theta R)$ and $[\varphi]=\mu\theta.(I, \varphi\theta)$.

As a matter of fact (ii) is contained among the axioms of [4; 10]. The algebraic system considered in [4] is OS+(ii) (without $L\neq R$ supposed), and is in a certain sense equivalent to that in [10].

The concept of inductive mapping is introduced inductively as follows.

1. The mappings $\Gamma=\lambda\theta_1\ldots\theta_n.\theta_i$, $1\leqq i\leqq n$, and $\Gamma=\lambda\theta_1\ldots\theta_n.\psi$, $\psi\in\{I, L, R\}$, are inductive.

2. If Γ', Γ'': $\mathscr{F}^n\to\mathscr{F}$ are inductive, then so are $\Gamma=\lambda\theta_1\ldots\theta_n.\Gamma'(\theta_1,\ldots,\theta_n)\Gamma''(\theta_1,\ldots,\theta_n)$ and $\Gamma=\lambda\theta_1\ldots\theta_n.(\Gamma'(\theta_1\ldots,\theta_n), \Gamma''(\theta_1,\ldots,\theta_n))$.

3. If $\Gamma':\mathscr{F}^{n+1}\to\mathscr{F}$ is inductive and for all $\theta_1,\ldots,\theta_n$ the element $\mu\theta.\Gamma'(\theta_1,\ldots,\theta_n,\theta)$ exists, then $\Gamma=\lambda\theta_1\ldots\theta_n\ldots\mu\theta.\Gamma'(\theta_1,\ldots,\theta_n,\theta)$ is inductive.

The monotonicity of the inductive mappings follows easily by the definition.

In order to state μA_1, we assume that the element $\langle\varphi\rangle=\mu\theta.(\varphi L, \theta R)$ exists for all φ and call simple initial segments the sets of the form $\{\theta/\theta\chi\leqq\tau\}$ or $\{\theta/\langle\theta\rangle\leqq\langle I\rangle\tau\}$.

Axiom μA_1. *For any $n+1$-ary inductive mapping Γ and any $\theta_1, \ldots \theta_n$ the inequality $\Gamma(\theta_1, \ldots, \theta_n, \theta) \leq \theta$ has a solution, which is a member of any simple initial segment closed under the mapping $\lambda\theta . \Gamma(\theta_1, \ldots, \theta_n, \theta)$.*

The proofs of the Normal Formal Theorem and the First Recursion Theorem given below imply that it is sufficient to allow in μA_1 the mapping $\lambda\theta_1\theta . (\theta_1 L, \theta R)$, $\lambda\theta_1\theta . (, \theta_1\theta)$ and $\lambda\theta_1\theta . LR[\theta_1(I, \langle\theta\rangle)]$ only, where $[\varphi] = \mu\theta . (I, \varphi\theta)$ by definition. Therefore the axiom scheme μA_1 is equivalent to a first order axiom.

Axiom μA_2. *For all φ, ψ the inequality $\varphi(I, \theta\psi) \leq \theta$ has a solution, which is a member of any normal initial segment (Φ-initial segment in terms of* [5]), *closed under the mapping $\lambda\theta . \varphi(I, \theta\psi)$.*

It is sufficient for our purposes to take in μA_2 the mappings $\lambda\theta . (\varphi L, \theta R)$, $\lambda\theta . (I, \varphi\theta)$ instead of $\lambda\theta . \varphi(I, \theta\psi)$.

Axiom μA_3. *For any inductive mapping $\Gamma : \mathcal{F}^{n+1} \to \mathcal{F}$ and any $\theta_1, \ldots, \theta_n$ the inequality $\Gamma(\Theta_1, \ldots, \theta_n, \theta) \leq \theta$ has a solution, which is a member of any normal initial segment closed under the mapping $\lambda\theta . \Gamma(\theta_1, \ldots, \theta_n, \theta)$.*

It is obvious that μA_3 implies μA_2.

In any concrete case the μ-axiom used will be indicated by a corresponding number of asterisks. For instance, the proof of Proposition 1.30* given below makes use of μA_1.

Each of our μ-axioms provides in particular that for any φ there are element $\langle\varphi\rangle$ (translation of φ), and $[\varphi]$ (iteration of φ), such that $\langle\varphi\rangle = \mu\theta . (\varphi L, \theta R)$, $[\varphi] = \mu\theta . (I, \varphi\theta)$. For μA_1, μA_3 that is implied by the following statement.

Proposition 1.1*(1.1***). *Let θ_0 be the solution supposed in μA_1 (in μA_3). Then $\theta_0 = \mu\theta . \Gamma(\theta_1, \ldots, \theta_n, \theta)$.*

Proof. We have $\Gamma(\theta_1, \ldots, \theta_n, \theta_0) \leq \theta_0$. Let $\Gamma(\theta_1, \ldots, \theta_n, \tau) \leq \tau$. The set $\mathcal{E} = \{\theta / \theta \leq \tau\}$ is a simple initial segment (normal initial segment), and whenever $\theta \in \mathcal{E}$, then $\Gamma(\theta_1, \ldots, \theta_n, \theta) \leq \Gamma(\theta_1, \ldots, \theta_n, \tau) \leq \tau$, hence $\theta_0 \in \mathcal{E}$. Therefore $\theta_0 \leq \tau$, and we conclude that $\theta_0 = \mu\theta . \Gamma(\theta_1, \ldots, \theta_n, \theta)$. Notice that θ_0 is the least fixed point of $\lambda\theta . \Gamma(\theta_1, \ldots, \theta_n, \theta)$, since the monotonicity of the mapping concerned.

The following two statements give that μA_1 implies μA_0.

Proposition 1.2. *The axiom μA_1 implies* (i).

Proof Let $R\psi \leq \psi\psi'$ and $(\varphi L\psi, \tau\psi') \leq \tau$. The set $\mathcal{E} = \{\theta / \theta\psi \leq \tau\}$ is a simple initial segment. If $\theta \in \mathcal{E}$, then $(\varphi L, \theta R)\psi = (\varphi L\psi, \theta R\psi) \leq (\varphi L\psi, \theta\psi\psi') \leq (\varphi L\psi, \tau\psi') \leq \tau$, hence $\langle\varphi\rangle \in \mathcal{E}$, and therefore $\langle\varphi\rangle\psi \leq \tau$.

Proposition 1.3. *The axiom μA_1 implies* (ii).

Proof. Let $(\psi, \varphi\tau) \leq \tau$. The set $\mathcal{E} = \{\theta / \theta\psi \leq \tau\}$ is a simple initial segment, and if $\theta \in \mathcal{E}$, then $(I, \varphi\theta)\psi = (\psi, \varphi\theta\psi) \leq (\psi, \varphi\tau) \leq \tau$. Hence $[\varphi] \in \mathcal{E}$, and thereby $[\varphi]\psi \leq \tau$.

Similarly μA_2 implies μA_0. At lasts, μA_3 implies μA_1. In fact if μA_3 is given, we prove that $\langle\theta\rangle \leq \langle I\rangle\tau$ iff $\forall n(\theta\bar{n} \leq \bar{n}\tau)$, where $\bar{n} = LR^n$ (using the normal initial segment $\{\theta / \forall n(\theta R^n \leq \langle I\rangle R^n\tau\}$ in the 'if'-part, and the equality $\bar{n}\langle\theta\rangle = \theta\bar{n}$ in the 'only if'-part), hence any simple initial segment is a normal initial segment as well.

It is worth mentioning that each of μA_2, μA_3 is essentially a second order axiom, i. e. not epuivalent to an effectively given set of first order axioms. That follows by the representabilty of the partial recursive functions (Proposition 1 [5]), similarly to a corresponding result in [9].

Now we are going to revise a part of the abstract recursion theory developed in [7], Chapter I, being interested especially in some key results (Normal

Form Theorem, First Recursion Theorem, Enumeration Theorem etc.). A number of proofs in [7] depend on μA_2, which is a second order axiom, and hence not appropriate for applications to the system of Scott and de Bakker. Fortunately, the weakest axiom μA_0 turns out to provide all the important statements ot the theory but the First Recursion Theorem, which can be proved by means of μA_1.

Proposition 1.4. *$L\langle\varphi\rangle=\varphi L$, $R\langle\varphi\rangle=\langle\varphi\rangle R$, $L[\varphi]=I$, and $R[\varphi]=\varphi[\varphi]$.*

Follows by the equalities $\langle\varphi\rangle=(\varphi L, \langle\varphi\rangle R)$, $[\varphi]=(I, \varphi[\varphi])$.

As a corollarly we get $\bar{n}\langle\varphi\rangle=\varphi\bar{n}$ for all n.

Proposition 1.5. $[\varphi]\psi=\mu\theta.(\psi, \varphi\theta)$.

Proof. We have $(\psi, \varphi[\varphi]\psi)=(I, \varphi[\varphi])\psi=[\varphi]\psi$, and whenever $(\psi, \varphi\tau)\leqq\tau$, then $[\varphi]\psi\leqq\tau$ by (ii). Hence $[\varphi]\psi=\mu\theta.(\psi, \varphi\theta)$.

Notice that (ii) and 1.5 are equivalent.

Proposition 1.6. $R[\varphi]\psi=\mu\theta.\varphi(\psi, \theta)$.

Proof. We have $\varphi(\psi, R[\varphi]\psi)=\varphi(I, \varphi[\varphi])\psi=\varphi[\varphi]\psi=R[\varphi]\psi$. If $\varphi(\psi, \tau)\leqq\tau$, then $(\psi, \varphi(\psi, \tau))\leqq(\psi, \tau)$, hence $[\varphi]\psi\leqq(\psi, \tau)$ by (ii), and therefore $R[\varphi]\psi\leqq\tau$.

Proposition 1.7. *$[\varphi]\psi=R[\sigma]$ with $\sigma=(\psi L, \varphi R)$.*

Proof. Making use of 1.5 and 1.6, we get $[\varphi]\psi=\mu\theta.(\psi, \varphi\theta)=\mu\theta.\sigma(I, \theta)$ $=R[\sigma]$.

Proposition 1.8. *$\varphi[\psi]=LR[\sigma]$ with $\sigma=(\varphi R^2, L, \psi R^2)$.*

Proof. Our usual convention is $(\varphi_1, \varphi_2, \ldots, \varphi_n)=(\varphi_1, (\varphi_2, \ldots, \varphi_n))$ for $n>2$; here $\sigma=(\varphi R^2, (L, \psi R^2))$.

We have

$(I, \psi R^2[\sigma])=(L, \psi R^2)[\sigma]=R\sigma[\sigma]=R^2[\sigma]$, hence $[\psi]\leqq R^2[\sigma]$.

Let $\tau=(I, \varphi[\psi], [\psi])$. Then $(I, \sigma\tau)=(I, \varphi[\psi], (I, \psi[\psi]))=\tau$, hence $[\sigma]\leqq\tau$, and therefore $R^2[\sigma]\leqq[\psi]$.

We get $[\psi]=R^2[\sigma]$, hence $\varphi[\psi]=\varphi R^2[\sigma]=L\sigma[\sigma]=LR[\sigma]$.

Proposition 1.9. *$[\varphi]\psi[\chi]=LR[\sigma]$ with $\sigma=((\psi R^2, \varphi LR), L, \chi R^2)$.*

Proof. We have $(I, \chi R^2[\sigma])=(L, \chi R^2)[\sigma]=R\sigma[\sigma]=R^2[\sigma]$, hence $[\chi]\leqq R^2[\sigma]$. Therefore, $(\psi[\chi], \varphi LR[\sigma])\leqq(\psi R^2[\sigma], \varphi LR[\sigma])=(\psi R^2, \varphi LR)[\sigma]=L\sigma[\sigma]=LR[\sigma]$, hence $[\varphi]\psi[\chi]\leqq LR[\sigma]$ by (ii).

Let $\tau=(I, [\varphi]\psi[\chi], [\chi])$. Then $(I, \sigma\tau=(I, (\psi[\chi], \varphi[\varphi]\psi[\chi], (I, \chi[\chi])])=(I, (I, \varphi[\varphi])\psi[\chi], [\chi])=\tau$, hence $[\sigma]\leqq\tau$, and $LR[\sigma]\leqq[\varphi]\psi[\chi]$. Thereby the proof is completed.

Proposition 1.10. *$[\varphi[\psi]]=LR[\sigma]$ with $\sigma=((L, \varphi R^2) LR, \psi R^2)$.*

Proof. We have $(LR[\sigma], \psi R^2[\sigma])=(LR, \psi R^2)[\sigma]=R^2[\sigma]$, hence $[\psi]LR[\sigma]$ $\leqq R^2[\sigma]$ by (ii). Therefore, $(I, \varphi[\psi]LR[\sigma])\leqq(I, \varphi R^2[\sigma])=(L, \varphi R^2)[\sigma]=L\sigma[\sigma]=LR[\sigma]$ implies $[\varphi[\psi]]\leqq LR[\sigma]$.

Let $\tau=(I, [\varphi[\psi]], [\psi][\varphi[\psi]])$. Then $(I, \sigma\tau)=(I, (I, \varphi[\psi][\varphi[\psi]]), [\varphi[\psi]], \psi[\psi]$ $[\varphi[\psi]])=(I, [\varphi[\psi]], (I\ \psi[\psi])[\varphi[\psi]])=\tau$, hence $[\sigma]\leqq\tau$, and therefore $LR[\sigma]\leqq[\varphi[\psi]]$.

Proposition 1.11. *Let $\varphi L\psi\leqq\chi L\tau$, $R\psi\leqq\psi\psi'$, and $\tau\psi'\leqq R\tau$ with a certain ψ'. Then $\langle\varphi\rangle\psi\leqq\langle\chi\rangle\tau$.*

Proof. We have $(\varphi L\psi, \langle\chi\rangle\tau\psi')\leqq(\chi L\tau, \langle\chi\rangle R\tau)=\langle\chi\rangle\tau$, hence $\langle\varphi\rangle\psi\leqq\langle\chi\rangle\tau$ by (i).

Proposition 1.12. *Let $\varphi L\psi=\chi L\tau$, $R\psi=\psi\psi'$, and $\tau\psi'=R\tau$ with a certain ψ'. Then $\langle\varphi\rangle\psi=\langle\chi\rangle\tau$.*

Follows by 1.11.

Proposition 1.13. $\langle\varphi\rangle\langle\psi\rangle=\langle\varphi\psi\rangle$.

Proof. The equalities $\varphi L\langle\psi\rangle=\varphi\psi L$, $R\langle\psi\rangle=\langle\psi\rangle R$ imply $\langle\varphi\rangle\langle\psi\rangle=\langle\varphi\psi\rangle$ by 1.12, taking $\tau=I$, $\psi'=R$.

Proposition 1.14. $R[\langle\varphi\rangle]=[\langle\varphi\rangle]\varphi$.

Proof. Making use of 1.7, we get $[\langle\varphi\rangle]\varphi = R[(\varphi L, \langle\varphi\rangle R)] = R[\langle\varphi\rangle]$.

As a corollary we get $\bar{n}[\langle\varphi\rangle] = \varphi^n$ for all n.

The binary operation primitive recursion is introduced by means of the equality $\Delta(\varphi, \psi) = \langle\varphi\rangle[\langle\psi\rangle]$.

Proposition 1.15. $L\Delta(\varphi,\psi) = \varphi$, $R\Delta(\varphi, \psi) = \Delta(\varphi, \psi)\psi$.

Proof. We have

$$L\Delta(\varphi, \psi) = \varphi L[\langle\psi\rangle] = \varphi I = \varphi,$$

$$R\Delta(\varphi, \psi) = \langle\varphi\rangle R[\langle\psi\rangle] = \langle\varphi\rangle[\langle\psi\rangle]\psi = \Delta(\varphi, \psi)\psi.$$

Proposition 1.15 implies $\bar{n}\Delta(\varphi, \psi) = \varphi\psi^n$ for all n.

Proposition 1.16. *Let* $\varphi\chi = L\tau$, $\psi\chi = \chi\psi'$, *and* $\tau\psi' = R\tau$ *with a certain* ψ'. *Then* $\Delta(\varphi, \psi)\chi = \langle I\rangle\tau$ (hence $\Delta(\varphi, \psi)\chi = \tau$ *whenever* τ *is of the form* $\langle\tau_1\rangle\tau_2$).

Proof. The equalities

$$\varphi L[\langle\psi\rangle]\chi = \varphi\chi = L\tau, \quad R[\langle\psi\rangle]\chi = [\langle\psi\rangle]\psi\chi = [\langle\psi\rangle]\chi\psi', \quad \tau\psi' = R\tau$$

imply $\langle\varphi\rangle[\langle\psi\rangle]\chi = \langle I\rangle\tau$ by 1.12.

Proposition 1.17. $\langle\varphi\rangle = \Delta(\varphi L, R)$.

Follows by 1.16, with $\psi' = R$.

Proposition 1.18. $\Delta(\varphi, \psi) = \mu\theta\,.(\varphi, \theta\psi)$.

Proof. We have

$$(\varphi, \langle\varphi\rangle[\langle\psi\rangle]\psi) = (\varphi, \langle\varphi\rangle R[\langle\psi\rangle]) = (\varphi L, \langle\varphi\rangle R)[\langle\psi\rangle] = \langle\varphi\rangle[\langle\psi\rangle].$$

Let $(\varphi, \tau\psi) \leqq \tau$. Then $R[\langle\psi\rangle] = [\langle\psi\rangle]\psi$, $(\varphi L[\langle\psi\rangle], \tau\psi) = (\varphi, \tau\psi) \leqq \tau$ imply $\langle\varphi\rangle[\langle\psi\rangle] \leqq \tau$ by (i), which completes the proof.

Proposition 1.19. $\langle(\varphi, \psi)\rangle = C(\langle\varphi\rangle, \langle\psi\rangle)$ *with* $C = \Delta((L^2, LR), (RL, R^2))$

Proof. We have

$$(L^2, LR)(\langle\varphi\rangle, \langle\psi\rangle) = (\varphi, \psi)L = L\langle(\varphi, \psi)\rangle, \ (RL, R^2)(\langle\varphi\rangle, \langle\psi\rangle)$$
$$= (\langle\varphi\rangle, \langle\psi\rangle)R, \ \langle(\varphi, \psi)\rangle R = R\langle(\varphi, \psi)\rangle,$$

hence $C(\langle\varphi\rangle, \langle\psi\rangle) = \langle(\varphi, \psi)\rangle$ by 1.16.

It is worth mentioning that $\bar{n}C = (\bar{n}L, \bar{n}R)$ for all n.

Proposition 1.20. $\langle[\varphi]\rangle = C[\langle\varphi\rangle C]$.

Proof. We have

$$(L, \varphi(L^2, LR)[\langle\varphi\rangle C]) = (L, \varphi LC[\langle\varphi\rangle C]) = (L, LR[\langle\varphi\rangle C]) = (L^2, LR)[\langle\varphi\rangle C],$$

hence $[\varphi]L \leqq (L^2, LR)[\langle\varphi\rangle C]$ by (ii). Also

$$(R, \langle\varphi\rangle C(RL, R^2)[\langle\varphi\rangle C]) = (R, R\langle\varphi\rangle C[\langle\varphi\rangle C]) = (RL, R^2)[\langle\varphi\rangle C],$$

hence $[\langle\varphi\rangle C]R \leqq (RL, R^2)[\langle\varphi\rangle C]$ by (ii).

Taking $\tau = (I, \varphi[\varphi]L, R[\langle\varphi\rangle C]R)$, we get

$$(I, \langle\varphi\rangle C\tau) = (I, (\varphi L, \langle\varphi\rangle R)C\tau) = (I, \varphi(L^2, LR)\tau, \langle\varphi\rangle C(RL, R^2)\tau)$$

$$= (I, \varphi(L, \varphi[\varphi]L), \langle\varphi\rangle C(R, R[\langle\varphi\rangle C]R)) = (I, \varphi[\varphi]L, \langle\varphi\rangle C[\langle\varphi\rangle C]R) = \tau,$$

hence $[\langle\varphi\rangle C] \leqq \tau$, therefore $(L^2, LR)[\langle\varphi\rangle C] \leqq [\varphi]L$ and $(RL, R^2)[\langle\varphi\rangle C] \leqq [\langle\varphi\rangle C]R$.

Hence $(L^2, LR)[\langle\varphi\rangle C] = [\varphi]L$ and $(RL, R^2)[\langle\varphi\rangle C] = [\langle\varphi\rangle C]R$, which gives $C[\langle\varphi\rangle C] = \langle[\varphi]\rangle$ by 1.16.

Proposition 1.21. *Let* $\sigma = C([\langle\psi\rangle]L, \langle\varphi\rangle R^2)$. *Then* $\bar{1}[\sigma] = \mu\theta.(I, \varphi\theta\psi)$.

Proof. We have

$$(\psi, \sigma(\psi, R^2[\sigma])) = (\psi, C([\langle\psi\rangle]\psi, \langle\varphi\rangle R^3[\sigma])) = (\psi, C(R[\langle\psi\rangle], R\langle\varphi\rangle R^2[\sigma]))$$
$$= (\psi, R\sigma[\sigma]) = (\psi, R^2[\sigma]),$$

hence $[\sigma]\psi \leq (\psi, R^2[\sigma])$ by (ii). Therefore,

$$(I, \varphi\bar{1}[\sigma]\psi) \leqq (I, \varphi\bar{2}[\sigma]) = (L, \varphi\bar{2})[\sigma] = L\sigma[\sigma] = \bar{1}[\sigma].$$

Let $(I, \varphi\tau\psi) \leqq \tau$. Taking into account that

$$\langle I\rangle[\langle\psi\rangle] = (L, \langle I\rangle R)[\langle\psi\rangle] = (I, \langle\psi\rangle[\langle\psi\rangle]) = [\langle\psi\rangle],$$

we get

$$(I, \sigma(I, \Delta(\tau, \psi))) = (I, C([\langle\psi\rangle], \langle\varphi\rangle R\langle\tau\rangle[\langle\psi\rangle])) = (I, C([\langle\psi\rangle], \langle\varphi\rangle\langle\tau\rangle\langle\psi\rangle[\langle\psi\rangle]))$$
$$= (I, C(\langle I\rangle, \langle\varphi\tau\psi\rangle)[\langle\psi\rangle]) = (I, \langle(I, \varphi\tau\psi)\rangle[\langle\psi\rangle] \leqq (I, \langle\tau\rangle[\langle\psi\rangle]) = (I, \Delta(\tau, \psi)),$$

hence $[\sigma] \leqq (I, \Delta(\tau, \psi))$, and therefore $\bar{1}[\sigma] \leqq \bar{0}\Delta(\tau, \psi) = \tau$, which completes the proof.

Proposition 1.22. *Let* $\varphi_1 = \varphi(\psi L, \chi R)$, $\theta_0 = \mu\theta.(I, \varphi_1\theta\rho)$ *and* $\theta_1 = \varphi_1\theta_0$ (θ_0 *exists by* 1.21). *Then* $\theta_1 = \mu\theta.\varphi(\psi, \chi\theta\rho)$.

Proof. We have $\varphi(\psi, \chi\theta_1\rho) = \varphi_1(I, \varphi_1\theta_0\rho) = \varphi_1\theta_0 = \theta_1$.

If $\varphi(\psi, \chi\tau\rho) \leqq \tau$, then

$$(I, \varphi_1(I, \tau\rho)\rho) = (I, \varphi(\psi, \chi\tau\rho)\rho) \leqq (I, \tau\rho),$$

hence $\theta_0 \leqq (I, \tau\rho)$. Therefore $\theta_1 \leqq \varphi_1 (I\ \tau\rho) \leqq \tau$, and the proof is completed.

Proposition 1.23. *Let* $D = \Delta(L^2, \langle R\rangle R)$, $\rho = C([I]L, R)$ *and* $\Gamma = \lambda\theta.D\langle\theta\rangle\rho$. *Then* $\bar{n}\rho[\Gamma(\varphi)] = [\bar{n}\varphi]$ *for all* φ *and* n.

Proof. Let $\sigma = \Gamma(\varphi)$ and $\tau = \Gamma(R(\varphi)$. We shall prove at first that $L\rho[\sigma] = [L\varphi]$, $R\rho[\sigma] = \rho[\tau]$.

It follows easily that $\sigma = (L\sigma, R\sigma)$, $L\sigma = L\varphi(L, LR)$, $R\sigma = \tau(L, R^2)$. We have

$$(I, L\varphi(L, LR)[\sigma]) = (I, L\sigma[\sigma]) = (L, LR)[\sigma],$$

hence $[L\varphi] \leqq (L, LR)[\sigma]$. Also

$$(I, \tau(L, R^2)[\sigma]) = (I, R\sigma[\sigma]) = (L, R^2)[\sigma]$$

implies $[\tau] \leqq (L, R^2)[\sigma]$. Then

$$(I, \sigma(I, L\varphi[L\varphi], \tau[\tau])) = (I, L\varphi(I, L\varphi[L\varphi]), \tau(I, \tau[\tau])) = (I, L\varphi[L\varphi], \tau[\tau])$$

gives $[\sigma] \leqq (I, L\varphi[L\varphi], \tau[\tau])$, hence

$$L\rho[\sigma] = (L, LR)[\sigma] = [L\varphi],\ R\rho[\sigma] = \rho(L, R^2)[\sigma] = \rho[\tau].$$

We have now $L\rho[\Gamma(\varphi)] = [L\varphi]$ and $R\rho[\Gamma(\varphi)] = \rho[\Gamma(R\varphi)]$ for all φ, hence $\bar{n}\rho[\Gamma(\varphi)] = L\rho[\Gamma(R^n\varphi)] = [\bar{n}\varphi]$ for all φ, n. Thereby the proof is completed.

Proposition 1.24. *Let* $\rho = \Delta(L, R^2)$, $P = \Delta(\rho R, \rho)$, $A = (R, RL)$, $\sigma = \Delta(L^2, A)$ *and* $Q = \mu\theta.\sigma(\theta R, L)$. *Then* $\langle\langle\varphi\rangle\rangle = P\langle\varphi\rangle Q$ *for all* φ.

Comments. While the elements P, Q considered in [5; 7] correspond to the Cantor's pairing function, here P, Q correspond to the pairing function

$J=\lambda mn.(2m+1)2^n$ (i.e. $\bar{m}\,\bar{n}P=\overline{J(m,n)},\ \overline{J(m,n)}\,Q=\bar{m}\,\bar{n}$ for all m, n). The element P is primitive recursive, and Q is recursive by 1.21, 1.22.

Proof. We have

$$L\sigma=L^2,\quad R^2\sigma=\sigma A^2,\quad LA^2=RL,$$

hence $\rho\sigma=\langle I\rangle L$ by 1.16. Therefore, $\rho R\sigma=\rho\sigma A=\langle I\rangle LA=\langle I\rangle R$. The equality, $Q=\sigma(QR, L)$ multiplied to the left by ρR and ρ gives $\rho RQ=\langle I\rangle L$ and $\rho Q=QR$ respectively. Besides,

$$L\langle\varphi\rangle=\varphi L=\varphi L\rho=L\langle\varphi\rangle\rho\qquad R^2\langle\varphi\rangle=\langle\varphi\rangle R^2,\quad \langle\varphi\rangle\rho R^2=\langle\varphi\rangle R\rho=R\langle\varphi\rangle\rho,$$

and therefore $\rho\langle\varphi\rangle=\langle\varphi\rangle\rho$ by 1.16. We get

$$\rho R\langle\varphi\rangle Q=\langle\varphi\rangle\rho RQ=\langle\varphi\rangle L=L\langle\langle\varphi\rangle\rangle,\quad \rho\langle\varphi\rangle Q=\langle\varphi\rangle\rho Q=\langle\varphi\rangle QR,\quad \langle\langle\varphi\rangle\rangle R=R\langle\langle\varphi\rangle\rangle$$

hence $P\langle\varphi\rangle Q=\langle\langle\varphi\rangle\rangle$ by 1.16 and the proof is completed.

The statements 1.2.16, 1.2.17 [7] compose the Representation Theorem (Proposition 1 [5]), concerning primitive recursiveness and recursiveness respectively. Though the proof of 1.2.17 [7] depends on μA_2, the axiom μA_0 is sufficient whenever a representability without the requirement $f(s_1, \ldots, s_n)\uparrow \Rightarrow \bar{s}_1\ldots\bar{s}_n\varphi=O$ is considered. This fact implies (similarly to a corresponding result in [9]), that the theory of IOS with a μ-axiom μA_0 is essentially undecidable.

An element φ is simple in a subset $\mathscr{B}$ of $\mathscr{F}$ iff φ can be constructed from members of $\{L, R\}\cup\mathscr{B}$ by means of the operations multiplication, Π, $\langle\,\rangle$.

The following statement is called Normal Form Theorem for the elements recursive in $\mathscr{B}$.

Proposition 1.25. *If φ is recursive in $\mathscr{B}$, then $\varphi=\bar{1}[\sigma]$ with a certain σ simple in $\mathscr{B}$.*

Proof. Using an induction on the construction of φ, we shall prove at first that $\varphi=\sigma'[\sigma'']$ with σ', σ'' simple in $\mathscr{B}$. Taking the element A as initial, the element R and the operation Π could be omitted in the definition of recursiveness, since $R=LA$ and $(\varphi, \psi)=A[\varphi L^2][\psi L]$ for all φ, ψ (the latter equality originates to [4]).

If $\varphi\in\{L, A\}\cup\mathscr{B}$, then $\varphi=R[\varphi L]$.

Let $\varphi=\sigma'[\sigma'']$, $\psi=\tau'[\tau'']$. Then

$$\varphi\psi=\sigma'\bar{1}[((\tau' R^2, \sigma''\bar{1}), \bar{0}, \tau''R^2)]$$

by 1.9. We have $\langle\varphi\rangle=\langle\sigma'\rangle C[\langle\sigma''\rangle C]$ by 1.13, 1.20. Substituting $\langle(L^2, LR)\rangle[\langle RL, R^2\rangle\rangle]$ for C, and making use of 1.10, 1.9, we get $\langle\varphi\rangle=\sigma_1'[\sigma_1'']$ with certain σ_1' σ_1'' simple in σ', σ''. At last,

$$[\varphi]=\bar{1}[((\bar{0}, \sigma'R^2), \bar{1}, \sigma''R^2)]$$

by 1.10.

Now let $\varphi=\sigma'[\sigma'']$ with σ', σ'' simple in $\mathscr{B}$.

Then $\varphi=\bar{1}[(\sigma'R^2, \bar{0}, \sigma''R^2)]$ by 1. 8, which completes the proof.

The following statement is called Normal Form Theorem for the unary mappings recursive in $\mathscr{B}$.

Proposition 1.26. *If* Γ *is an unary mapping recursive in* $\mathscr{B}$, *then* $\Gamma=\lambda\theta.\overline{1}[\varphi(I,\langle\theta\rangle)]$ *with a certain* φ *simple in* $\mathscr{B}$.

Proof. We shall prove at first that $\Gamma=\lambda\theta.\psi[\chi(I,\langle\theta\rangle\rho)]$ with certain ψ, χ, ρ recursive in $\mathscr{B}$, using an induction on the construction of Γ. Again the element A will be taken as initial instead of R, and the case of the operation Π will be omitted.

If $\Gamma=\lambda\theta.\psi$, $\psi\in\{L,A\}\cup\mathscr{B}$, then $\Gamma(\theta)=R[\psi L]=R[\psi L^2(I,\langle\theta\rangle)]$.

If $\Gamma=\lambda\theta.\theta$, then $\Gamma(\theta)=R[\theta L]=R[LR(I,\langle\theta\rangle)]$.

Let $\Gamma'=\lambda\theta.\psi'[\chi'(I,\langle\theta\rangle\rho')]$, $\Gamma''=\lambda\theta.\psi''[\chi''(I,\langle\theta\rangle\rho'')]$.

If $\Gamma=\lambda\theta.\Gamma'(\theta)\Gamma''(\theta)$, then 1.9 gives $\Gamma(\theta)=\psi'\overline{1}[\sigma]$, where

$$\sigma=((\psi''R^2,\chi'(I,\langle\theta\rangle\rho')\overline{1}),\overline{0},\chi''(I,\langle\theta\rangle\rho'')R^2)=\chi_1(I,\langle\theta\rangle\rho_1,\langle\theta\rangle\rho_2)=\chi_1(I,\langle\theta\rangle L\rho_3,\langle\theta\rangle LR\rho_3)$$

$$=\chi_1(I,\overline{0}\langle\langle\theta\rangle\rangle\rho_3,\ \overline{1}\langle\langle\theta\rangle\rangle\rho_3)=\chi_2(I,\langle\langle\theta\rangle\rangle\rho_3)=\chi_2(I,P\langle\theta\rangle Q\rho_3)=\chi(I,\langle\theta\rangle\rho)$$

with certain χ_1, χ_2, χ, ρ_1, ρ_2, ρ_3, ρ recursive in χ', ρ', ψ'', χ'', ρ''.

If $\Gamma=\lambda\theta.\langle\Gamma'(\theta)\rangle$, then 1.13, 1.20, 1.19 give $\Gamma(\theta)=\langle\psi'\rangle C[\sigma]$, where

$\sigma=\langle\chi'(I,\langle\theta\rangle\rho')\rangle C=\langle\chi'\rangle C(\langle I\rangle,\ \langle\langle\theta\rangle\rangle\langle\rho'\rangle)C=\chi_1(I,\ \langle\langle\theta\rangle\rangle\rho_1)$ with certain χ_1, ρ_1 recursive in χ', ρ', and we continue as in the previous case.

If $\Gamma=\lambda\theta.[\Gamma'(\theta)]$, then 1.10 gives $\Gamma(\theta)=\overline{1}[\sigma]$, where

$$\sigma=((\overline{0},\psi'R^2),\ \overline{1},\chi'(I,\langle\theta\rangle\rho')R^2)=\chi(I,\langle\theta\rangle\rho)$$

with certain χ, ρ recursive in ψ', χ', ρ'.

Now let $\Gamma=\lambda\theta.\psi[\chi(I,\langle\theta\rangle\rho)]$, and ψ, χ, ρ be recursive in $\mathscr{B}$. Proposition 1.8 implies $\Gamma(\theta)=\overline{1}[\sigma]$, where

$$\sigma=(\psi R^2,\overline{0},\chi(I,\langle\theta\rangle\rho)R^2)=\chi_1(I,\langle\theta\rangle\rho_1)$$

with χ_1, ρ_1 recursive in $\mathscr{B}$. We get $\sigma=\chi_1(L,\langle\theta\rangle R)[\rho_1 L]=\overline{1}[\sigma_1]$ by 1.8, where

$$\sigma_1=(\chi_1(\overline{2},\langle\theta\rangle R^3),\ \overline{0},\ \rho_1\overline{2})=\chi_2(I,\langle\theta\rangle R^3)$$

with χ_1 recursive in $\mathscr{B}$. The Normal Form Theorem 1.25 gives that there is an element φ_1 simple in $\mathscr{B}$, such that $\chi_2=\overline{1}[\varphi_1]$, hence

$$\sigma_1=\overline{1}[\varphi_1](L,\langle\theta\rangle R)[R^3L]=\overline{1}\,\overline{1}[\sigma_2]$$

by 1.9, where

$$\sigma_2=(((\overline{2},\ \langle\theta\rangle R^3),\ \varphi_1\overline{1}),\ \overline{0},R^3\,\overline{2})=\varphi_2(I,\langle\theta\rangle)$$

with φ_2 simple in $\mathscr{B}$, making use of the equality $\langle\theta\rangle R=R\langle\theta\rangle$. Then $\sigma=\overline{1}\,\overline{1}[\sigma_3]$ by 1.10, where

$$\sigma_3=((\overline{0},\overline{1}\,\overline{3}),\overline{1},\varphi_2(R^2,\langle\theta\rangle R^2))=\varphi_3(I,\langle\theta\rangle)$$

with φ_3 simple in $\mathscr{B}$. Applying 1.10 once more, we get $\Gamma(\theta)=\overline{1}\,\overline{1}[\varphi_4(I,\langle\theta\rangle)]$ with φ_4 simple in $\mathscr{B}$. At last, 1.8 gives $\Gamma(\theta)=\overline{1}[\sigma_4]$, where

$$\sigma_4=(\overline{1}\,\overline{3},\overline{0},\varphi_4(R^2,\langle\theta\rangle R^2))=\varphi(I,\langle\theta\rangle)$$

with φ simple in $\mathscr{B}$. Thereby the proof is completed.

A concept of n-ary mapping over $\mathscr{F}$ recursive in $\mathscr{B}$ can be introduced too, by taking as initial the mappings $\lambda\theta_1 \ldots \theta_n . \psi$, $\psi \in \{L, R\} \cup \mathscr{B}$, and $\lambda\theta_1 \ldots \theta_n . \theta_i$, $1 \leq i \leq n$, instead of $\lambda\theta . \psi$, $\lambda\theta . \theta$.

The following Normal Form Theorem for n-ary mappings recursive in $\mathscr{B}$ can be proved by modifying the proof of 1.26.

Proposition 1.27. *If* Γ *is a n-ary mapping recursive in* $\mathscr{B}$, $n \geq 1$, *then* $\Gamma = \lambda\theta_1 \ldots \theta_n . \bar{1}[\varphi(I, \langle\theta_1\rangle, \ldots, \langle\theta_n\rangle)]$ *with a certain* φ *simple in* $\mathscr{B}$.

Another version with

$$\Gamma = \lambda\theta_1 \ldots \theta_n . \bar{1}[\varphi(I, \langle(\theta_1, \ldots, \theta_n)\rangle)]$$

can be obtained as an easy corollary to 1.26.

Proposition 1.28. *The element* $\mu\theta . \varphi[\psi(I, \theta\chi)]$ *exists for all* φ, ψ, χ, *and is recursive in* φ, ψ, χ.

Proof. Similarly to 1.10 the equality $[\varphi[(\psi, \chi)]] = R^2\bar{1}[\sigma]$ with $\sigma = ((\psi\rho R^2\bar{1}, \bar{0}, \varphi\rho), \chi\rho)$ and $\rho = (\bar{1}\,\bar{1}, \bar{0}\,\bar{1}, R^2)$ can be established (by verifying that $[(\psi, \chi)]R^2\bar{1}[\sigma] \leq \rho[\sigma]$, $[\varphi[(\psi, \chi)]] \leq R^2\bar{1}[\sigma]$ and $[\sigma] \leq (L, (\psi[(\psi, \chi)], I, I), \chi[(\psi, \chi)])[\varphi[(\psi, \chi)]])$.

Let $\varphi, \psi, \chi \in \mathscr{F}$ and

$$\sigma = \Delta((\chi\bar{1}\,\bar{1}, R^2\bar{1}, \bar{0}, \psi(\bar{1}\,\bar{1}, \varphi R^2\bar{2})), (\bar{0}\,\bar{1}, R^2)).$$

Making use of 1.7 and the equality stated above, we get

$$\varphi[\psi(I, \varphi R^2\bar{1}[\sigma]\chi)] = \varphi[\psi(L, \varphi R^2\bar{2})[(\chi L, \sigma R)]] = \varphi R^2\bar{1}[((\chi\bar{1}\,\bar{1}, R^2\bar{1}, \bar{0}, \psi(\bar{1}\,\bar{1}, \varphi R^2\bar{2}))$$
$$\sigma(\bar{0}\,\bar{1}, R^2))] = \varphi R^2\bar{1}[\sigma].$$

Let $\varphi[\psi(I, \theta_1\chi)] \leq \theta_1$. Writing ρ for $[\psi(I, \theta_1\chi)]$ and τ for $(I, \Delta((\chi\rho, \rho, \rho), \chi\rho))$, we have

$$(\chi\bar{1}\,\bar{1}, R^2\bar{1}, \bar{0}, \psi(\bar{1}\,\bar{1}, \varphi R^2\bar{2}))\tau = (\chi\rho, \rho, I, \psi(\rho, \varphi\rho\chi\rho)) = (\chi\rho, \rho, I, \psi(I, \varphi\rho\chi)\rho)$$
$$\leq (\chi\rho, \rho, I, \psi(I, \theta_1\chi)\rho) = (\chi\rho, \rho, \rho) = LR\tau, \ (\bar{0}\,\bar{1}, R^2)\tau = (\chi\rho, R\tau\chi\rho)$$
$$= \tau\chi\rho, \quad R\tau\chi\rho = R^2\tau,$$

which gives

$$\langle(\chi\bar{1}\,\bar{1}, R^2\bar{1}, \bar{0}, \psi(\bar{1}\,\bar{1}, \varphi R^2\bar{2}))\rangle[\langle(\bar{0}\,\bar{1}, R^2)\rangle]\tau \leq R\tau$$

by an easy application of 1.11, making use of 1.14. Therefore $\sigma\tau \leq R\tau$, hence $(I, \sigma\tau) \leq \tau$ and we get $[\sigma] \leq \tau$. At last,

$$\varphi R^2\bar{1}[\sigma] \leq \varphi R^2\bar{1}\tau = \varphi\rho \leq \theta_1.$$

We conclude that $\varphi R^2\bar{1}[\sigma] = \mu\theta . \varphi[\psi(I, \theta\chi)]$, which completes the proof.

Now the stronger axiom μA_1 is to be applied essentially.

Proposition 1.29.* *The element* $\theta_0 = \mu\theta . \varphi[\psi(I, \langle\theta\rangle)]$ *is recursive in* φ, ψ.

Proof. Let $\varphi' = \langle\varphi\rangle C$, $\psi' = \langle\psi\rangle C(CL, PR)$ and $\chi' = QC$. The element $\theta' = \mu\theta . \varphi'[\psi'(I, \theta\chi')]$ is recursive in φ', ψ', χ' by 1.28, hence θ' is recursive in φ, ψ. Making use of 1.13, 1.20, 1.19 and 1.24, we get

$$\langle\varphi[\psi(I,\ \langle\theta\rangle)]\rangle=\langle\varphi\rangle C[\langle\psi\rangle C(\langle I\rangle,\ \langle\langle\theta\rangle\rangle)C]=\langle\varphi\rangle C[\langle\psi\rangle C(C,\ P\langle\theta\rangle QC)]$$
$$=\varphi'[\psi'(I,\ \langle\theta\rangle\chi')]$$

for all θ. In particular,

$$\langle\theta_0\rangle=\langle\varphi[\psi(I,\ \langle\theta_0\rangle)]\rangle=\varphi'[\psi'(I,\ \langle\theta_0\rangle\chi')],$$

hence $\theta'\leq\langle\theta_0\rangle$.

The set $\mathscr{E}=\{\theta/\langle\theta\rangle\leqq\theta'\}=\{\theta/\langle\theta\rangle\leq\langle I\rangle\theta'\}$ is a simple initial segment. If $\theta\in\mathscr{E}$, then

$$\langle\varphi[\psi(I,\ \langle\theta\rangle)]\rangle=\varphi'[\psi'(I,\ \langle\theta\rangle\chi')]\leqq\varphi'[\psi'(I,\ \theta'\chi')]=\theta',$$

hence $\langle\theta_0\rangle\leqq\theta'$ by μA_1. Therefore $\langle\theta_0\rangle=\theta'$ and hence $\theta_0=L\theta'(I,\ I)$, which completes the proof.

The following statement is called First Recursion Theorem. It is an analog to Theorem XXVI in [8], and to the First Recursion Theorem in [9].

P r o p o s i t i o n 1.30.* *If* Γ *is an unary mapping recursive in* $\mathscr{B}$, *then the element* $\mu\theta\,.\,\Gamma(\theta)$ *is recursive in* $\mathscr{B}$.

Follows by 1.26, 1.29*.

A parametrized version of 1.30* can be established as well, since the proof of 1.29* is uniform with respect to φ, ψ.

P r o p o s i t i o n 1.31.* *If* Γ *is a* $(n+1)$-*ary mapping recursive in* $\mathscr{B}$, *then the mapping* $\lambda\theta_1\ldots\theta_n\,.\,\mu\theta\,.\,\Gamma(\theta_1,\ldots,\theta_n,\ \theta)$ *is recursive in* $\mathscr{B}$.

As a corollary we get that the class of all the inductive mappings meets that of all the recursive (in $\varnothing$) ones.

It is of interest whether the axioms μA_1 (more precisely, its part used above) and μA_2 could be weakened, preserving the First Recursion Theorem. The axiom μA_0 turns out to provide the following fixed point result: Any unary mapping recursive in $\mathscr{B}$ has a fixed point recursive in $\mathscr{B}$.

The following Enumeration Theorem takes place.

P r o p o s i t i o n 1.32.* *Let* $\mathscr{B}$ *be finite. Then there is an element* σ *recursive in* $\mathscr{B}$, *such that whenever* σ *is recursive in* $\mathscr{B}$, *then* $\varphi=\bar{n}\sigma$ *with a certain* n.

P r o o f. Let $\mathscr{B}=\{\psi_0,\ldots,\psi_m\}$ and $\chi=(A,\ \psi_0,\ldots,\ \psi_m)$. Then all the members of $\mathscr{F}$ recursive in $\mathscr{B}$ can be constructed from L and χ by means of the operations multiplication, translation and iteration.

Let $H=\Delta(\langle L\rangle,\ \langle R\rangle)$ and $\rho=C([I]L,\ R)$. The mapping

$$\Gamma=\lambda\theta\,.\,Q((L,\ \chi,\ L),\ Q\langle\theta\rangle\theta,\ H\langle\theta\rangle,\ \rho[D\langle\theta\rangle\rho],\ L)$$

is recursive in $\mathscr{B}$, hence the element $\sigma=\mu\theta\,.\,\Gamma(\theta)$ is recursive in $\mathscr{B}$ by 1.30*. Let J be the function considered in the comments to 1.24. Using the equality $\sigma=\Gamma(\sigma)$, we get

$$\overline{J(0,0)}\sigma=\overline{J(0,0)}\Gamma(\sigma)=\bar{0}(L,\ \chi,\ L)=L,$$

and similarly $\overline{J(1,0)}\sigma=\chi$.

Let $\varphi=\bar{k}\sigma$, $\psi=\bar{l}\sigma$. Then

$$\overline{J(J(k,l),1)}\sigma=\bar{k}\,\bar{l}\langle\sigma\rangle\sigma=\bar{k}\sigma\bar{l}\,\sigma=\varphi\psi,$$

$\overline{J(k,2)}\sigma=\bar{k}H\langle\sigma\rangle=\langle\bar{k}\sigma\rangle=\langle\varphi\rangle$, since $\bar{n}H=\langle L\rangle\langle R\rangle^n=\langle\bar{n}\rangle$ for all n. At last, $\overline{J(k,3)}\sigma=$

$$\bar{k}\rho[D\langle\sigma\rangle\rho]=[\bar{k}\sigma]=[\varphi]$$

by making use of 1.23. Thereby the proof is completed.

The universal element σ is constructed uniformly with respect to χ, and that allows an Enumeration Theorem for mappings to be established as well. We shall consider the case of unary mappings (The case of n-ary mappings can be reduced to that of unary ones.)

Proposition 1.33.* *Let $\mathcal{B}$ be finite. Then there is an unary mapping Σ recursive in $\mathcal{B}$, such that whenever Γ is an unary mapping recursive in $\mathcal{B}$, then $\Gamma=\lambda\theta.\bar{n}\Sigma(\theta)$ with a certain n.*

It is worth mentioning that the above Enumeration Theorems could be proved by means of the μ-axiom μA_0 only. The proof needs some auxiliary lemmas, so we shall restrict ourselves to an outline of its idea. An element is constructed at first, which is recursive in $\mathcal{B}$ and universal for a special class or elements simple in $\mathcal{B}$. After that Proposition 1.23 and an improved version of the Normal Form Theorem 1.25 (1.26, respectively) are to be applied.

Analogs to some other standard statements, such as the Second Recursion Theorem, Roger's Theorem, etc., are established in [7] too. However, our excursion in the theory of IOS has come to an end, due to the limited size of the paper.

2. A formal system for IOS. Toward some applications to the system of Scott and de Bakker we restate in a ceriain formal system the axioms of IOS with a μ-axiom μA_1.

Constants I, L, R are considered. Letters θ, θ_0, $\theta_1, \ldots$ stand for variables. Terms are constructed by means of the operations $\varphi\psi$, (φ, ψ) and $(\mu\theta.\varphi)$, the third being a variable binding μ-operator, and φ, ψ standing for terms. When possible $\mu\theta.\varphi$ is written for $(\mu\theta.\varphi)$. Notations $\langle\varphi\rangle$, $[\varphi]$ are used for $\mu\theta.(\varphi L, \theta R)$ and $\mu\theta.(I, \varphi\theta)$ respectively, with a certain θ not free in φ. Atomic formulas Φ, Ψ are either equalities $\varphi=\psi$ or inequalities $\varphi\leq\psi$, and formulas are conjunctions of atomic formulas (written as finite lists $\bar{\Phi}$, $\bar{\Psi}$). Theorems of the system are of the form $\bar{\Phi}\vdash\bar{\Psi}$.

The nonlogical axioms and rules of the systems include in the first place the axioms of a partially ordered semigroup with an unit. Secondly those of OS are supposed:

$$\theta_1\leq\theta_2,\ \theta_3\leq\theta_4\vdash(\theta_1,\ \theta_3)\leq(\theta_2,\ \theta_4)$$

$$\vdash(\theta_1,\ \theta_2)\theta_3=(\theta_1\theta_3,\ \theta_2\theta_3)$$

$$\vdash L(\theta_1,\ \theta_2)=\theta_1,\ R(\theta_1,\ \theta_2)=\theta_2.$$

Corresponding to μA_1 are given at last the axiom

$$\vdash\varphi(\mu\theta.\varphi/\theta)\leq\mu\theta.\varphi$$

and the rule

$$(\mu)\qquad \frac{\bar{\Phi},\ \Psi\vdash\Psi(\varphi/\theta)}{\bar{\Phi}\vdash\Psi(\mu\theta.\varphi/\theta)}$$

with Ψ being either $\theta\chi\leq\tau$ or $\langle\theta\rangle\leq\langle I\rangle\tau$ and θ not free in $\bar{\Phi}$, χ, τ.

Any theorem of this system is a theorem of the theory of IOS with a μ-axiom μA_1. Amongst the statements of IOS those considered above are deducible in the system anyway.

3. Applications to the System of Scott and de Bakker. The notations used below are those from [1], except that $\theta, \theta_0, \theta_1, \ldots$ denoted function variables, $I, O, \leq$ and $(\mu\theta . \varphi)$ are written for $E, \Omega, \subseteq, \mu\theta . [\varphi]$ respectively, and; is always omitted. The set of all the free variables in φ is denoted by $\mathrm{Fr}(\varphi)$.

Consider the system of Scott and de Bakker with additional constants L, R, K and a predicate constant p_0, such that

$$\vdash L(p_0 \rightarrow K\theta_1, K\theta_2) = \theta_1, \ R(p_0 \rightarrow K\theta_1, K\theta_2) = \theta_2$$

(An interpretation of L, R, K as f_1, f_2, f_3, and p_0 as a predicate true on M_1, false on M_2 and undefined otherwise, is given in [6], example 1). The assumption just stated is an algebraic analog to that of Böhm and Jacopini from [3], and can be replaced by the stronger one from [1, p. 46]. We write now (φ, ψ) for $(p_0 \rightarrow K\varphi, K\psi)$, and $\tilde{p}$ for $(p \rightarrow L, R)$.

In order to conclude that all the axioms and rules of our formal system for IOS are deducible now, we have to verify only that the rule (μ) is a special case of the Scott's μ-induction rule. It is sufficient for that purpose to show that $\vdash \Psi(O/\theta)$ is dedulible for all Ψ allowed in (μ). If Ψ is $\theta\chi \leq \tau$, then $\vdash O\chi = O$, $O \leq \tau$ gives $\vdash O\chi \leq \tau$. If Ψ is $\langle\theta\rangle \leq \langle I\rangle\tau$, we get $\vdash \langle O\rangle \leq \langle I\rangle O$ by the Scott's rule, and therefore $\vdash \langle O\rangle \leq \langle I\rangle\tau$, since $\vdash \langle I\rangle O \leq \langle I\rangle\tau$ is deducible.

It should he mentioned however that the rule (μ) is essentially weaker than the Scott's rule. Allowing constants in the language of our system for IOS, we have that the condition (*) [5] provides the validity of (μ), but not that of the Scott's rule. The condition (**) provides the validity of the Scott's rule, and so does the condition (**) [7]. Roughly speaking, while the Scott's rule requires the continuity of the operations, the rule (μ) requires their monotonicity only. To the point, (*) [5] with the assumptions $L\varphi = \sup L\mathscr{H}$ and $R\varphi = \sup R\mathscr{H}$ omitted still provides the validity of μA_1 and (μ). Similarly the more general condition (*) [7] can be weakened. Let us mention too, that $\forall\varphi(O \leq \varphi)$ implies $LO \leq L(O, O) = O$, and similarly $RO \leq O$, hence the requirement $LO = RO = O$ in the conditions (**) [5] and (**) [7] may be omitted (noticed by N. Georgieva). Besides that, both $\forall\varphi(O \leq \varphi)$ and $\forall\varphi(O\varphi = O)$ could be replaced by $OL \leq R$.

Let us return to the system of Scott and de Bakker. A term is said to be an IOS-term if I, L, R, function variables, and operations $\varphi\psi$, (φ, ψ) and μ-operator only are used when constructing. All the results of our system for IOS hold now at once, provided IOS-terms are considered. Certainly, not all terms are IOS-terms. This difficulty can be avoided in a way similar to that in [3]. At first, an auxiliary statement can be proved by making use of $\vdash (p \rightarrow \varphi, \psi) = \tilde{p}(\varphi, \psi)$.

Proposition 3.1. *For any term φ with predicate variables $p_1, \ldots p_n$ there is an IOS-term ψ and function variables $\theta_1, \ldots, \theta_{n+1}$, such that*

$$\vdash \varphi = \psi(\tilde{p}_1, \ldots, \tilde{p}_n, \ K/\theta_1, \ldots, \theta_{n+1}).$$

Using 3.1, the results from section 1 can be reestablished for the system of Scott and de Bakker enriched with the additional assumption stated above. In a different way, the proofs of those results could be repeated directly in the present system.

A term φ is called simple iff I is not used when constructing φ, and the μ-operator is applied only as $\langle\,\rangle$. A term φ is canonical iff the μ-operator is applied only as $\langle\,\rangle$ and $[\,]$ when constructing.

Correspondingly to 1.27 we get a Normal Form Theorem.

P r o p o s i t i o n 3.2. *For any canonical term* φ *and any* $p_1, \ldots, p_m$, $\theta_1, \ldots, \theta_n$ *there is a simple term* ψ, *such that*

$$\mathrm{Fr}(\psi) \subseteq \mathrm{Fr}(\varphi)/\{\tilde{p}_1, \ldots, p_m, \theta_1, \ldots, \theta_n\}$$

and

$$\vdash \varphi = \bar{1}[\psi(I, \langle \tilde{p}^1 \rangle, \ldots, \langle \tilde{p}_m \rangle, \langle \theta_1 \rangle, \ldots, \langle \theta_n \rangle)]$$

(We may have $m=0$ or $n=0$, meaning $\vdash \varphi = \bar{1}[\psi]$ whenever $m=n=0$.)

The First Recursion Theorem states now as follows.

P r o p o s i t i o n 3.3. *For any canonical* φ *and any* θ *there is a canonical* ψ, *such that* $\mathrm{Fr}(\psi) \subseteq \mathrm{Fr}(\varphi)/\{\theta\}$ *and* $\vdash \mu\theta . \mu\theta . \varphi = \psi$.

As an immediate corollary we get the following statement.

P r o p o s i t i o n 3.4. *For any term* φ *there is a canonical term* ψ, *such that* $\mathrm{Fr} \subseteq \mathrm{Fr}(\varphi)$ *and* $\vdash \varphi = \psi$.

In Computer Science terms, the proofs of the Normal Form Theorem and the First Recursion Theorem give an algorithm, which transforms equivalently recursive program schemes into program schemes of a certain standard type, with no recursion allowed.

Correspondingly to 1.33* we have an Enumeration Theorem.

P r o p o s i t i o n 3.5. *For any* $p_1, \ldots, p_m$, $\theta_1, \ldots, \theta_n$ *there is a cavonical term* σ, *such that* $\mathrm{Fr}(\sigma) = \{p_1, \ldots, p_m, \theta_1, \ldots, \theta_n\}$, *and whenever* φ *is a canonical term and* $\mathrm{Fr}(\varphi) \subseteq \{p_1, \ldots, p_m, \theta_1, \ldots, \theta_n\}$, *then* $\vdash \varphi = \bar{k}\sigma$ *with a certain* k.

Taking into consideration 3.4, we see that the term φ in propositions 3.2, 3.5 needs not be a canonical one.

It is worth mentionning that the regular terms studied in [1] can be characterized now by means of terms with the μ-operator used as [ ] only, to call them prime canonical terms. It is immediate that all the prime canonical terms are regular.

P r o p o s i t i o n 3.6. *Let* φ *be regular in* $\theta_1 \ldots, \theta_n$ *and* φ *be regular. Then there is a prime canonical term* ψ, *such that* $\mathrm{Fr}(\psi) \subseteq \mathrm{Fr}(\varphi) \setminus \{\theta_1, \ldots, \theta_n\}$ *and* $\vdash \varphi = \psi(I, \theta_1, \ldots, \theta_n)$. *In particular, whenever* φ *is regular, then* $\vdash \varphi = \psi$ *with a certain prime canonical* ψ.

(Analog to a well-known result from [3], generalized in [1, 10, 4].)

The proof uses an induction on the construction of φ. We shall outline the most interesting step. Namely, let $\vdash \varphi = \psi(I, \theta_1, \ldots, \theta_n)$, and $\varphi_1 = \mu\theta_n . \varphi$. If $n > 1$, we take $\chi = \psi(\bar{0}L, \ldots, \overline{n-2}L, R^{n-1}L, R)$ and get $\vdash \varphi\chi((I, \theta_1, \ldots, \theta_{n-1})\,\theta)$, which gives $\vdash \varphi_1 = R[\chi](I, \theta_1, \ldots, \theta_{n-1})$ according to 1.6. If $n=1$, then $\vdash \varphi_1 = R[\psi]$ according to 1.6 again.

Proposition 3.6 allows a Normal Form Theorem for regular terms to be obtained, corresponding to an improved version of the normal form result established in [4].

P r o p o s i t i o n 3.7. *For any regular term* φ, *any* θ *and* n *not less than the number of the free occurrkences of* θ *in* φ, *there is a term* ψ *with no* I *and* μ*-operator used when constructing, such that* $\mathrm{Fr}(\psi] \subseteq \mathrm{Fr}(\varphi) \setminus \{0\}$ *and*

$$\vdash \varphi = \bar{1}[\psi(I, \theta\bar{4}, \ldots, \theta\overline{n+3})]$$

In particular we have $\vdash \varphi = \bar{1}[\psi]$ *whenever* $\theta \notin \mathrm{Fr}(\varphi)$.

The proof follows a part of the proof oj 1.26, and makes use of a certain analog to 3.1.

A version of 3.7 for a greater number of variables could be given as well.

The equalities $p*\theta = R[(p \to \theta R, L)]$ and $[\theta] = (\neg p_{0*}(K\theta))K = L(p_{0*}(K(R, \theta L)))K$ are easily deducible, so the important role of iteration is well known. The operation translation seems to be new one. Though introduced by the μ-operator, it is of an elementary nature. In the case of function-like interpretations the operation could be realized by making use of an additional counting register. In particular, our canonical terms may be interpreted as unary structured program schemes with two counting registers, and therefore Propositions 3.2 (combined with 3.7), 3.4, 3.5 establish some properties of such program schemes.

The present version of the system of Scott and de Bakker may be enriched further on. For instance, predicate terms may be allowed instead of predicate variables, taking $\pi = P_i$, $\pi = \neg\pi^1$, $\pi = (\pi_1 \,\&\, \pi_2)$, $\pi = (\bot = \psi)$; more complicated operators from Dynamic Logic may be used as well. New axioms may be added to the system. Properly reformulated, the results established above will still take place (by modifying 3.1), provided the μ-operator $\mu\theta . \varphi$ is applied only if θ occurs free in no predicate subterm of φ.

The system of Scott and de Bakker was proposed as a framework for studying programs. Iterative operatory spaces were introduced to provide an unified axiomatical setting for Recursion Theory. And they happened to be closely connected!

The axiomatical system ot IOS has no standard model; on the contrary, one of its mots remarkable features ii its abundance in classes of models. The members of $\mathcal{F}$ may be interpreted as partial functions of various kinds (including sequence functions, stack functions, multiple valued functions, probabilistic functions, functions with finite type arguments, set functions, etc.), as well as functionals and operators of higher order, etc. Therefore the system of Scott and de Bakker can be considered a specific extension of IOS, orientated towards the partial single valued functions and predicates. Other extensions may be of interest too, having other concrete interpretations in mind. For instance, partial multiple valued functions (i. e. relations) and partial multiple valued predicates can be treated by a certain modification of the system considered in this section. That system allows interpretations by means of multiple valued functions. Notice that only a few of the axioms from [1] were used in our considerations. In particular, the second axiom for conditionals was not used. However, that is the axiom which fails when interpreting predicate variables as miltiple valued predicates $p: M - \to \{0, 1, \{0, 1\}\}$ (for a detailed study of the McCarthy's equivalences see [9]. So that axiom could be weakened by substituting $\geq$ for $=$, preserving the results established above. Some specific axiomme could be added too, e. g. a constant U may be supposed, such that U is sup $\{L, R\}$.

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