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FOURIER OPERATIONAL CALCULUS

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With the help of the Fourier operational calculus, we give necessary and sufficient solvability conditions and construct periodic solutions of linear differential equations as well as solutions of special boundary value problems. Using functional analytical methods, we obtain a simple representation of generalized inverses of polynomial operators $p(d/dt)$. This simplifies recent results of G r o z d e v [3] and D i m o v s k i [2].

1. Introduction. Let $\mathscr{Z}$ denote the set of all integers. Let p be a monic complex polynomial of degree $n\geqq 2$

$$p(\lambda)=\sum_{k=0}^{n} a_k\lambda^k=\prod_{j=1}^{s}(\lambda-\lambda_j)^{n_j} \tag{1}$$

with pairwise different roots λ_j and let

$$p(\lambda)^{-1}=\sum_{j=1}^{s}\sum_{r=1}^{n_j}\alpha_{jr}(\lambda-\lambda_j)^{-r} \tag{2}$$

be the decomposition of $p(\lambda)^{-1}$ into vulgar fractions. Then let $K=\{k_1,\dots,k_m\}$ be the set of all $k\in\mathscr{Z}$ with $2\pi ik=\lambda_j$ for some $j=1,\dots,r$.

Let C_1 denote the Banach space of all (complex—valued) continuous 1-periodic functions f with norm $\|f\|_{C_1}=\sup_{t\in[0,1]}|f(t)|$. Further let L_1^p, $(1\leqq p<\infty)$ be the set of all classes of almost everywhere (a. e.) equal (complex-valued) 1-periodic functions f which are Lebesgue integrable to p-th power over $[0,1]$, with the norm $\|f\|_p=(\int_0^1|f(t)|^p dt)^{1/p}$. In the following, X denotes one of the spaces C_1 or L_1^p $(1\leq p<\infty)$ with the corresponding norm $\|.\|_X$.

For any two functions $f\in X$ and $g\in L_1^1$ the *periodic convolution* $f*g$ is defined (a. e) through the absolutely convergent integral

$$(f*g)(t)=\int_0^t f(t-\tau)\,g(\tau)\,d\tau+\int_t^1 f(1+t-\tau)\,g(\tau)\,d\tau.$$

Then $f*g\in X$ and

$$\|f*g\|_X\leqq\|f\|_X\|g\|_1. \tag{3}$$

Since $X\subseteqq L_1^1$, the periodic convolution is defined on X and is a commutative, associative and distributive operation on X [1, p. 10]. The r-times convolution $f*\cdots*f$ of $f\in L_1^1$ with itself is denoted by f^{*r} $(r=1,2,,\dots)$, where $f^{*1}=f$.

Now we consider the boundary value problem

(4) $$p(d/dt)\,x(t)=f(t),\ x^{(j)}(0)=x^{(j)}(1),\ (j=0,\ldots,n-1)$$

with a given function $f\in x$. For instance, a linear time-invariant feedback system with 1-periodic input and 1-periodic output x can be described by (4). Following result is well-known ([4, p. 216]; [2]):

Theorem. *Let $f\in X$. If $K=\emptyset$ that means $p(2\pi ik)\neq 0$ for all $k\in\mathscr{Z}$, then (4) has a unique solution $x(t)=(g*f)(t)$ with*

$$g(t)=\sum_{j=1}^{s}\sum_{r=1}^{n_j}\frac{a_{jr}}{(r-1)!}\,\frac{\partial^{r-1}}{\partial\lambda^{r-1}}\left(\frac{e^{\lambda t}}{1-e^{\lambda}}\right)\Big|_{\lambda=\lambda_j}.$$

In this paper we generalize this result and give necessary and sufficient solvability conditions of problem (4) in the case $K\neq\emptyset$.

2. Operators D **and** T. Let C_1^r $(r=1,2,\ldots)$ denote the set of all 1-periodic (complex-valued) functions which are r-times continuously differentiable. AC^1 $(=AC_1^0)$ is the set of all 1-periodic absolutely continuous functions. Correspondingly, AC_1^{r-1} $(r=1,2,\ldots)$ denotes the set of all functions f with $f^{(j)}\in AC_1$ $(j=0,\ldots,r-1)$. For $r=1,2,\ldots$, we set

$$W_X^r=\begin{cases}C_1^r & \text{if } X=C_1,\\ \{f\in L_1^p\,|\,f=\varphi \text{ a. e., } \varphi\in AC_1^{r-1},\ \varphi^{(r)}\in L_1^p\} & \text{if } X=L_1^p,\ (1\leq p<\infty).\end{cases}$$

Then the r-th derivative $D^r f$ of $f\in W_X^r$ is defined by

$$(D^r f)(t)=\begin{cases}f^{(r)}(t) & \text{if } X=C_1,\\ \varphi(r)(t) \text{ a. e.} & \text{if } X=L_1^p,\ (1\leq p<\infty),\end{cases}$$

where $\varphi\in AC_1^{r-1}$ with $\varphi^{(r)}\in L_1^p$ is such that $f(t)=\varphi(t)=$ a. e. [1, p. 34]. The densely defined linear operator D^r maps the domain Dom $D^r=W_X^r$ into X. Obviously, all eigenvalues $\mu_k=2\pi ik$ $(k\in\mathscr{Z})$ of D are simple and corresponding eigenfunctions are $x_k(t)=\exp 2\pi ikt$.

The *finite Fourier transform* of $f\in X$ is defined by

$$\hat f(k)=\int_0^1 f(t)\,e^{-2\pi ikt}\,dt,\ (k\in\mathscr{Z}).$$

For any $f,g\in X$ and all $k\in\mathscr{Z}$ we have ([1, p. 168])

(5) $$f(t)*e^{2\pi ikt}=\hat f(k)\,e^{2\pi ikt}$$

and by the convolution theorem

(6) $$[f*g]\ (k)=\hat f(k)\,\hat g(k).$$

Note that the periodic convolution has divisors of zero. For any $k,m\in\mathscr{Z}$ we obtain by (5) $e^{2\pi ikt}*e^{2\pi imt}=\delta_{km}\,e^{2\pi ikt}$, where δ_{km} denotes the Kronecker symbol.

The linear operator $T:X\to X$ defined by

$$(Tf)(t)=\left(\frac{1}{2}-t\right)*f(t)=\int_0^t f(\tau)\,dr-\int_0^1\left(t-\tau+\frac{1}{2}\right)f(\tau)\,d\tau,\ (t\in[0,1])$$

is bounded by (3) and fulfils

$$T(f*g)=(Tf)*g=f*(Tg) \tag{7}$$

for any $f, g \in X$ by the properties of the periodic convolution. Furthermore, $T1=0$ and $(Tf)^\frown(0)=1*(Tf)=(T1)*f=0$ by (5) and (7). If B_r $(r=1, 2, \dots)$ denotes the r-th Bernoulli polynomial, then

$$T^r\left(\frac{1}{2}-t\right)=T^r(-B_1(t))=-\frac{1}{(r+1)!}B_{r+1}(t) \quad (t\in[0,1]) \tag{8}$$

that means

$$(-B_1(t))^{*r}=-\frac{1}{r!}B_r(t) \quad (t\in[0,1]). \tag{9}$$

By (6) this implies

$$\left[-\frac{1}{r!}B_r(t)\right]^\frown(k)=\begin{cases}0 & \text{for } k=0,\\ (2\pi ik)^{-r} & \text{for } k\in\mathscr{Z},\ k\neq 0.\end{cases} \tag{10}$$

Using (7) and (8) we obtain

$$(T^rf)(t)=\left(-\frac{1}{r!}(B_r(t)\right)*f(t) \quad (t\in[0,1]) \tag{11}$$

for all $f\in X$. By (6) and (10) it follows that

$$(T^rf)^\frown(k)=\begin{cases}0 & \text{for } k=0.\\ (2\pi ik)^{-r}\widehat{f}(k) & \text{for } k\in\mathscr{Z},\ k\neq 0.\end{cases} \tag{12}$$

Note that $T^rf\in$ Dom D^r for all $f\in X$. By a theorem [1, p. 173] the statement $T^rf\in$ Dom D^r is equivalent to $T^rf\in W[X;(2\pi ik)^r]$, where $W[X;(2\pi ik)^r]$ is the set of all functions $h\in X$ for which there exists $g\in X$ such that $(2\pi ik)^r\widehat{h}(k)=\widehat{g}(k)$ for all $k\in\mathscr{Z}$. By (12) we have $(2\pi ik)^r(T^rf)^\frown(k)=\widehat{g}(k)$ for every $k\in\mathscr{Z}$ with $g=f-\widehat{f}(0)\in X$. Furthermore

$$\begin{aligned}&(D^rT^rf)(t)=f(t)-\widehat{f}(0) \quad \text{for all } f\in X,\\ &(T^rD^rf)(t)=f(t)-\widehat{f}(0) \quad \text{for all } f\in \text{Dom } D^r.\end{aligned} \tag{13}$$

Then it follows that

$$\begin{aligned}T^rD^rT^rf&=T^rf \quad \text{for all } f\in X,\\ D^rT^rD^rf&=D^rf \quad \text{for all } f\in \text{Dom } D^r.\end{aligned}$$

Hence T^r is a bounded generalized inverse of D^r. If $X=L_1^2$, then T^r is the orthogonal generalized inverse of D^r. Let I denote the identity. Since $I-T^rD^r$ is a projector of Dom D^r onto the kernel Ker D^r, we obtain Ker $D^r=$ Lin $\{1\}$ by (11). Since D^rT^r is a projector of X onto the closure of the image Im D^r, it follows by (11) that $\overline{\text{Im } D^r}=$ Im D^r and that the cokernel Coker D^r is Lin $\{1\}$.

If $f\in$ Dom D^r and $g\in L_1^1$, then by (5), (7) and (11)

(14) $$f * g = (T^r D^r f + \hat{f}(0)) * \hat{g} = \hat{f}(0)\hat{g}(0) + T^r(D^r f * g) \in \mathrm{Dom}\ D^r$$

and by (6), (11) and $(D^r f)^\frown(0) = 0$

(15) $$D^r(f * g) = D^r T^r(D^r f * g) = (D^r f * g) - (D^r f * g)^\frown(0)$$
$$= (D^r f * g) - (D^r f)^\frown(0)\hat{g}(0) = (D^r f) * g.$$

Lemma 1. *Let f be a r-times continuously differentiable function. Then for $t \in [0, 1]$*

$$h(t) = f(t) - \sum_{k=0}^{r} (f^{(k)}(1) - f^{(k)}(0)) \frac{1}{(k+1)!} B_{k+1}(t) \in \mathrm{Dom}\ D^r.$$

Proof. Obviously, h is r-times continuously differentiable. Using $B'_{k+1}(t) = (k+1)B_k(t)$, $B_{k+1}(1) = B_{k+1}(0)$ $(k = 1, 2, \ldots)$, we obtain

$$h^{(j)}(t) = f^{(j)}(t) - \sum_{k=j}^{r} (f^{(k)}(1) - f^{(k)}(0)) \frac{1}{(k-j+1)!} B_{k-j+1}(t)$$

and thus

$$h^{(j)}(1) - h^{(j)}(0) = 0 \quad \text{for } j = 0, 1, \ldots, r.$$

3. Operators $(D-\lambda)^r$. For any complex number λ the *exponential element* e_λ is defined by

(16) $$e_\lambda(t) = \begin{cases} \dfrac{e^{\lambda t}}{1-e^\lambda} & \text{if } \lambda \neq 2\pi ik \text{ for all } k \in \mathscr{Z}, \\ \left(\dfrac{1}{2} - t\right) e^{2\pi ikt} & \text{if } \lambda = 2\pi ik \text{ for some } k \in \mathscr{Z}, \end{cases}$$

for every $t \in [0, 1]$. This element e_λ has the characteristic property

(17) $$\hat{e}_\lambda(k) = \begin{cases} (2\pi ik - \lambda)^{-1} & \text{if } \lambda \neq 2\pi ik, \\ 0 & \text{if } \lambda = 2\pi ik \end{cases}$$

for all $k \in \mathscr{Z}$. It is easy to prove that for $r = 2, 3, \ldots$

(18) $$e_\lambda^{*r}(t) = \begin{cases} \dfrac{\partial^{r-1}}{\partial \lambda^{r-1}} \left(\dfrac{e^{\lambda t}}{1-e^\lambda} \right) & \text{if } \lambda \neq 2\pi ik \quad \text{for all } k \in \mathscr{Z}, \\ -\dfrac{1}{r!} B_r(t) e^{2\pi ikt} & \text{if } \lambda = 2\pi ik \quad \text{for some } k \in \mathscr{Z} \end{cases}$$

for $t \in [0, 1]$. By (6) and (17) it follows that

(19) $$(e_\lambda^{*r})^\frown(k) = \begin{cases} (2\pi ik - \lambda)^{-r} & \text{if } \lambda \neq 2\pi ik, \\ 0 & \text{if } \lambda = 2\pi ik \end{cases}$$

for all $k \in \mathscr{Z}$. Note that (9) and (10) are special cases of (18) and (19). Especially we obtain $e_0(t) = 1/2 - t$ for $t \in [0, 1]$. Furthermore we remark that $e_\lambda \notin \mathrm{Dom}\ D$.

The densely defined linear operator $(D-\lambda)^r$ maps the domain $\mathrm{Dom}\ D^r$ nto X. Obviously, we have

$$\operatorname{Ker}\,(D-\lambda)^r=\begin{cases}\operatorname{Lin}\,\{e^{2\pi ikt}\} & \text{if } \lambda=2\pi ik \text{ for some } k\in\mathscr{Z}.\\ \{0\} & \text{otherwise.}\end{cases}$$

The linear operator $T_\lambda: X\to X$ defined by

$$T_\lambda f=e_\lambda * f \quad \text{for all } f\in X \tag{20}$$

is bounded by (3) and fulfils $T_\lambda(f*g)=(T_\lambda f)*g=f*(T_\lambda g)$ for all $f, g\in X$. Note that $T_0=T$. Furthermore we obtain by (5), (6), (17) and (20)

$$T_{2\pi ik}\, e^{2\pi ikt}=0 \tag{21}$$

and $(T_{2\pi ik}f)^{\wedge}(k)=0$ for any $f\in X$ and all $k\in\mathscr{Z}$. Using (18) and (20) we get

$$T_\lambda^r f=e_\lambda^{*r} * f \quad \text{for all } f\in X \text{ and } r=1, 2, \ldots \tag{22}$$

with $T_\lambda^1=T_\lambda$. By (20), T_λ commutes with T_μ for all complex numbers λ, μ.

Theorem 2. *Let λ be a complex number and $r=1, 2, \ldots$. Then $T_\lambda^r f\in \operatorname{Dom} D^r$ for every $f\in X$. In the case $\lambda\neq 2\pi ik$ for all $k\in\mathscr{Z}$, the operator T_λ^r is a bounded inverse of $(D-\lambda)^r$. In the case $\lambda=2\pi ik$ for some $k\in\mathscr{Z}$, the operator T_λ^r is a bounded generalized inverse of $(D-\lambda)^r$.*

Proof. We show that $T_\lambda^r f\in\operatorname{Dom} D^r$,

$$((D-\lambda)^r T_\lambda^r f)(t)=\begin{cases} f(t) & \text{if } \lambda\neq 2\pi ik \text{ for all } k\in\mathscr{Z},\\ f(t)-\hat f(k)\, e^{2\pi ikt} & \text{if } \lambda=2\pi ik,\ k\in\mathscr{Z}\end{cases} \tag{23}$$

or all $f\in X$ and

$$(T_\lambda^r(D-\lambda)^r f)(t)=\begin{cases} f(t) & \text{if } \lambda\neq 2\pi ik \text{ for all } k\in\mathscr{Z},\\ f(t)-\hat f(k)\, e^{2\pi ikt} & \text{if } \lambda=2\pi ik,\ k\in\mathscr{Z}\end{cases} \tag{24}$$

for all $f\in\operatorname{Dom} D^r$. Hence by (21) if follows that $T_\lambda^r(D-\lambda)^r T_\lambda^r f=T_\lambda^r f$ for all $f\in X$ and $(D-\lambda)^r T_\lambda^r(D-\lambda)^r f=(D-\lambda)^r f$ for all $f\in\operatorname{Dom} D^r$.

It is only necessary to prove (23) and (24) for $r=1$. By Lemma 1, it follows that $h(t)=e_\lambda(t)+B_1(t)+\frac{\lambda}{2}B_2(t)\in\operatorname{Dom} D$ $(t\in[0,1])$. Hence

$$\begin{aligned}(T_\lambda f)(t)&=(h*f)(t)-((B_1+\tfrac{\lambda}{2}B_2)*f)(t)\\ &=(h*f)(t)+(Tf)(t)+\lambda(T^2f)(t)\in\operatorname{Dom} D\end{aligned}$$

by (14) for arbitrary $f\in X$. Further it follows by (13) and (15)

$$\begin{aligned}(DT_\lambda f)(t)&=((Dh)*f)(t)+(DTf)(t)+\lambda(DT^2f)(t)\\ &=((Dh)*f)(t)+f(t)-\hat f(0)+\lambda(Tf)(t).\end{aligned}$$

By (16) we have

$$(Dh)(t)=\begin{cases}\lambda e_\lambda(t)+1+\lambda B_1(t) & \text{if } \lambda\neq 2\pi ik \text{ for all } k\in\mathscr{Z},\\ \lambda e_\lambda(t)+1+\lambda B_1(t)-e^{2\pi ikt} & \text{if } \lambda=2\pi ik,\ k\in\mathscr{Z}.\end{cases}$$

Then we obtain for $\lambda\neq 2\pi ik$ for all $k\in\mathscr{Z}$

$$(DT_\lambda f)(t)=\lambda(T_\lambda f)(t)+f(t)$$

by (5), (11) and (20). In the case $\lambda=2\pi ik$, $k\in\mathscr{Z}$ we get

$$(DT_\lambda f)(t)=\lambda(T_\lambda f)(t)+f(t)-\hat{f}(k)\,e^{2\pi ikt}$$

by (5), (11) and (20). Hence we obtain (23) for $r=1$ in both cases. For any $f\in \operatorname{Dom} D$ we have by (15), (20)

$$T_\lambda(D-\lambda)f=e_\lambda*(D-\lambda)f=(D-\lambda)(e_\lambda*f)=(D-\lambda)T_\lambda f$$

and by (23) if follows (24) for $r=1$.

Note that T_λ^r is the orthogonal generalized inverse of $(D-\lambda)^r$ in the case $X=L_1^2$.

Since $(D-\lambda)^r T_\lambda^r$ is a projector of X onto the closure of $\operatorname{Im}(D-\lambda)^r$, it follows by (23) that

$$\overline{\operatorname{Im}(D-\lambda)^r}=\operatorname{Im}(D-\lambda)^r=\begin{cases} X & \text{if } \lambda\neq 2\pi ik \text{ for all } k\in\mathscr{Z},\\ \{f\in X \mid \hat{f}(k)=0\} & \text{if } \lambda=2\pi ik,\ k\in\mathscr{Z}\end{cases}$$

and that

$$\operatorname{Coker}(D-\lambda)^r=\begin{cases} \{0\} & \text{if } \lambda\neq 2\pi ik \text{ for all } k\in\mathscr{Z},\\ \operatorname{Lim}\{e^{2\pi ikt}\} & \text{if } \lambda=2\pi ik,\ k\in\mathscr{Z}.\end{cases}$$

4. Polynomial operator $p(D)$. With (1) we form the densely defined linear operator $p(D)\colon \operatorname{Dom} D^n\to X$. By assumption this operator has the kernel

$$\operatorname{Ker} p(D)=\begin{cases} \operatorname{Lin}\{e^{2\pi ik_\mu t},\ \mu=1,\dots,m\} & \text{if } K\neq\emptyset,\\ \{0\} & \text{if } K=\emptyset.\end{cases}$$

Then $P\colon X\to \operatorname{Ker} p(D)$ defined by

$$(Pf)(t)=\begin{cases} \sum_{\mu=1}^{m}\hat{f}(k_\mu)\,e^{2\pi ik_\mu t} & \text{if } K\neq\emptyset,\\ 0 & \text{if } K=\emptyset\end{cases} \tag{25}$$

for all $f\in X$, is a projector of X onto $\operatorname{Ker} p(D)$. By (6) this projector has the property

$$P(f*g)=(Pf)*g=f*(Pg) \tag{26}$$

for any $f, g\in X$. Then we obtain the support of $p(D)$

$$\operatorname{Supp} p(D)=\operatorname{Ker} P\cap \operatorname{Dom} D^n=\begin{cases} \{f\in \operatorname{Dom} D^n \mid \hat{f}(k_\mu)=0,\ \mu=1,\dots,m\} & \text{if } K\neq\emptyset,\\ \operatorname{Dom} D^n & \text{if } K=\emptyset\end{cases}$$

The domain $\operatorname{Dom} D^n$ can be represented as direct sum

$$\operatorname{Dom} D^n=\operatorname{Ker} p(D)\dotplus \operatorname{Supp} p(D).$$

The operator $p(D)$ maps $\operatorname{Ker} p(D)$ into $\{0\}$ and $\operatorname{Supp} p(D)$ one-to-one onto the image $\operatorname{Im} p(D)$. Note that P commutes with T_λ by (20) and (26).

Theorem 3. *The operator*

(27) $$R = T_{\lambda_1}^{n_1} \cdots T_{\lambda_s}^{n_s} (I-P) : X \to X$$

is a generalized inverse of $p(D)$ with following properties:

(28) $$\begin{aligned} &p(D)R = I-P \text{ on } X, \\ &R\,p(D) = I-P \text{ on Dom } D^n, \\ &p(D)Rp(D) = p(D) \text{ on Dom } D^n, \\ &R\,p(D)R = R \text{ on } X. \end{aligned}$$

Furthermore

(29) $$R = \sum_{j=1}^{s} \sum_{r=1}^{n_j} \alpha_{jr} T_{\lambda_j}^{r} (I-P)$$

holds, i. e.

(30) $$Rf = \left(\sum_{j=1}^{s} \sum_{r=1}^{n_j} \alpha_{jr} e_{\lambda_j}^{*r} \right) * (I-P) f$$

for all $f \in X$.

Proof. Let $f \in X$. Then $g = f - Pf$ is an element of X with $\widehat{g}(k_\mu) = 0$ ($\mu = 1, \ldots, m$). By (23) it follows that

$$(D-\lambda_j)^{n_j} T_{\lambda_j}^{n_j} g = g \quad (j = 1, \ldots, s).$$

Since all operators $T_{\lambda_j}^{n_j}$ commute, this implies

$$p(D)Rf = p(D)\, T_{\lambda_1}^{n_1} \cdots T_{\lambda_s}^{n_s} g = g = (I-P)f$$

and hence $Rp(D)Rf = Rf - RPf = Rf$. Now let $f \in \text{Dom } D^n$. Then $g = f - Pf \in \text{Dom } D^n$ with $\widehat{g}(k_\mu) = 0$ ($\mu = 1, \ldots, m$). By (24) it follows that $T_{\lambda_j}^{n_j} (D-\lambda_j)^{n_j} g = g$ ($j = 1, \ldots, s$). Since $T_{\lambda_j}^{n_j}$ commutes with $(I-P)$, this implies

$$Rp(D)f = R\,p(D)g = (I-P)\, T_{\lambda_1}^{n_1} \cdots T_{\lambda_s}^{n_s} p(D)g = (I-P)g = (I-P)f$$

and hence $p(D)Rp(D)f = p(D)f - p(D)Pf = p(D)f$.

Now we prove formula (29). Let $p_j(\lambda)$ be the polynomials $p_j(\lambda) = (\lambda - \lambda_j)^{-n_j} p(\lambda)$ ($j = 1, \ldots, s$). Then (2) implies that

(31) $$1 = \sum_{j=1}^{s} \sum_{r=1}^{n_j} \alpha_{jr} p_j(\lambda) (\lambda - \lambda_j)^{n_j - r}$$

for all complex λ. Hence

$$I = \sum_{j=1}^{s} \sum_{r=1}^{n_j} \alpha_{jr} p_j(D) (D-\lambda_j)^{n_j - r} \text{ on Dom } D^{n-1}.$$

For arbitrary $f \in X$ is $Rf \in \mathrm{Dom}\ D^n$ such that by (22), (23) and (27) we obtain

$$Rf = \sum_{j=1}^{s} \sum_{r=1}^{n_j} \alpha_{jr} p_j(D)\,(D-\lambda_j)^{n_j-r}\, T_{\lambda_1}^{n_1} \cdots T_{\lambda_s}^{n_s}(I-P)f$$

$$= \sum_{j=1}^{s} \sum_{r=1}^{n_j} \alpha_{jr}\, T_{\lambda_j}^{r}(I-P)f = \Big(\sum_{j=1}^{s} \sum_{r=1}^{n_j} \alpha_{jr}\, e_{\lambda_j}^{*r}\Big) * (I-P)f$$

$$= \sum_{j=1}^{s} \sum_{r=1}^{n_j} \alpha_{jr}(I-P)\, e_{\lambda_j}^{*r} * f.$$

By Theorem 3 it follows that $p(D)$ is normally solvable. The image Im $p(D)$ is closed. Since $p(D)R$ is a projector of X onto Im $p(D)$, it follows by (25) and (28) that

$$\mathrm{Im}\ p(D) = \begin{cases} \{f \in X \mid \hat{f}(k_\mu) = 0, \quad \mu = 1, \ldots, m\} & \text{if } K \neq \emptyset, \\ X & \text{if } K = \emptyset \end{cases}$$

and hence

$$\mathrm{Coker}\ p(D) = \begin{cases} \mathrm{Lin}\ \{e^{2\pi i k_\mu t},\ \mu = 1, \ldots, m\} & \text{if } K \neq \emptyset, \\ \{0\} & \text{if } K = \emptyset. \end{cases}$$

Therefore P is a projector of X onto Coker $p(D)$ parallel to Im $p(D)$ and $X = \mathrm{Im}\ p(D) \dotplus \mathrm{Coker}\ p(D)$ holds.

5. Homogeneous boundary value problems. Now we consider the equation

$$p(D)x = f \tag{32}$$

for given $f \in X$. This equation is solvable if and only if $f \in \mathrm{Im}\ p(D)$, this means $Pf = 0$ or $\hat{f}(k_\mu) = 0$ ($\mu = 1, \ldots, m$) if $K \neq \emptyset$. Then we obtain following theorem from the properties (28) and (30) of R:

Theorem 4. *The equation* (32) *is solvable if and only if* $Pf = 0$. *Under the assumption* $Pf = 0$, *any solution of* (32) *possesses the form*

$$x(t) = \begin{cases} (Rf)(t) + \sum_{\mu=1}^{m} \gamma_\mu e^{2\pi i k_\mu t} & \text{if } K \neq \emptyset, \\ (Rf)(t) & \text{if } K = \emptyset, \end{cases} \tag{33}$$

where γ_μ *are arbitrary complex numbers and*

$$Rf = \sum_{j=1}^{s} \sum_{r=1}^{n_j} \alpha_{jr}\, e_{\lambda_j}^{*r} * (I-P)f.$$

Since R is the orthogonal generalized inverse of $p(D)$ in the case $X = L_1^2$ we obtain under the assumption $Pf = 0$ that

$$x^* = Rf \tag{34}$$

is the unique *solution* of (32) *with minimal norm*, i. e., $\| x \|_2 > \| x^* \|_2$ for any solution $x \neq x^*$ of (32). More general, an element $\tilde{x} \in \text{Dom } D^n$ is called a *least squares solution* of (32), if

$$\| p(D) \tilde{x} - f \|_2 = \inf \{ \| p(D)x - f \|_2 \mid x \in \text{Dom } D^n \}.$$

Evidently, every solution is a least squares solution too. From the properties (28) of the orthogonal generalized inverse R of $p(D)$ follows:

Theorem 5. *Let $X = L_1^2$ and $f \in X$. Then any least squares solution of (32) can be represented in the form* (33). *The element* (34) *is the unique least squares solution of minimal norm, i. e., $\| x \|_2 > \| x^* \|_2$ for any least squares solution $x \neq x^*$.*

6. Inhomogeneous boundary value problems. Now we consider the inhomogeneous boundary value problem

$$(35)\qquad \begin{aligned} & p\left(\frac{d}{dt}\right) y(t) = g(t) \quad (t \in [0, 1]), \\ & y^{(j)} \Big|_0^1 = \eta_j \quad (j = 0, \ldots, n-1), \end{aligned}$$

with given continuous function g, unknown n-times continuously differentiable function y and given complex numbers η_j, where $y \Big|_0^1 = y(1) - y(0)$. By (1) and (35) follows that

$$y^{(n)} \Big|_0^1 = g \Big|_0^1 - \sum_{j=0}^{n-1} a_j \eta_j = \eta_n .$$

Then by Lemma 1 we obtain that

$$(36)\qquad x = y - b \in \text{Dom } D^n$$

with

$$b(t) = \sum_{k=0}^{n} \eta_k \frac{1}{(k+1)!} B_{k+1}(t).$$

Furthermore

$$(37)\qquad f(t) = g(t) - p\left(\frac{d}{dt}\right) b(t) \in C_1.$$

Hence (35) is equivalent to $p(D)x = f$. Thus any solution y of (35) corresponds with a solution $x = y - b$ of (32), and vice versa. By Theorem 4, the problem (32) can be solved if and only if $Pf = 0$. Under the assumption $Pf = 0$, any solution x of (32) possesses the form (33). Let

$$S = \sum_{j=1}^{s} \sum_{r=1}^{n_j} \alpha_{jr} T'_{\lambda_j}.$$

Note that $R = S(I - P)$ by (27) and hence $Rf = Sf$ by $Pf = 0$. Then by (33), (36) and (37), any solution of (35) is given by

$$(38)\qquad y(t)=\begin{cases}(Sg)(t)+b(t)-Sp\left(\frac{d}{dt}\right)b(t)+\sum_{\mu=1}^{m}\gamma_\mu e^{2\pi i k_\mu t} & \text{if } K\neq\emptyset,\\ (Sg)(t)+b(t)-Sp\left(\frac{d}{dt}\right)b(t) & \text{if } K=\emptyset,\end{cases}$$

where γ_μ are arbitrary complex numbers.

Now we shall make use of the simple

Lemma 6 (Taylor's formula). *If z is a r-times continuously differentiable function, then*

$$(39)\qquad T_\lambda^r\left(\frac{d}{dt}-\lambda\right)^r z-z-\sum_{l=0}^{r-1}\left(\left(\frac{d}{dt}-\lambda\right)^l z\right)\Big|_0^1 e_\lambda^{*(l+1)}=\begin{cases}-\hat{z}(k)e^{2\pi i k t} & \text{for } \lambda=2\pi i k\ (k\in\mathscr{Z})\\ 0 & \text{otherwise.}\end{cases}$$

Proof. We proceed by induction. The equation

$$(40)\qquad T_\lambda\left(\frac{d}{dt}-\lambda\right)z-z-z\big|_0^1 e_\lambda=\begin{cases}-\hat{z}(k)e^{2\pi i k t} & \text{for } \lambda=2\pi i k\quad (k\in\mathscr{Z})\\ 0 & \text{otherwise}\end{cases}$$

shows that Taylor's formula (39) holds for $r=1$. Suppose that (39) holds for $r\geq 1$. If z is $(r+1)$-times continuously differentiable, then by (5) and (17) we have $e_{2\pi i k} * e^{2\pi i k t}=0$ and hence by (20), (39) and (40) we obtain for $\lambda=2\pi i k$ $(k\in\mathscr{Z})$

$$T_\lambda^{r+1}\left(\frac{d}{dt}-\lambda\right)^{r+1}z=e_\lambda * \left[T_\lambda^r\left(\frac{d}{dt}-\lambda\right)^r(z'-\lambda z)\right]$$

$$=T_\lambda\left(\frac{d}{dt}-\lambda\right)z+\sum_{l=0}^{r-1}\left(\left(\frac{d}{dt}-\lambda\right)^{l+1}z\right)\Big|_0^1 e_\lambda^{*(l+2)}$$

$$=z+\sum_{l=0}^{r}\left(\left(\frac{d}{dt}-\lambda^l z\right)\Big|_0^1 e_\lambda^{*(l+1)}-\hat{z}(k)e^{2\pi i k t}\right..$$

Similarly, the Lemma 6 can be proved in the case $\lambda\neq 2\pi i k$ for all $k\in\mathscr{Z}$. Note that by using of (11) and (18) the formula (39) is the *polynomial expansion of Bernoulli* in the special case $\lambda=0$:

$$z(t)=\hat{z}(0)+\sum_{l=0}^{r-1}z^{(l)}\Big|_0^1\frac{1}{(l+1)!}B_{l+1}(t)-\frac{1}{r!}(B_r * z^{(r)})(t).$$

Applying (31) and (39), we obtain

$$(41)\qquad Sp\left(\frac{d}{dt}\right)b=\sum_{j=1}^{s}\sum_{r=1}^{n_j}\alpha_{jr}T_\lambda^r\left(\frac{d}{dt}-\lambda_j\right)^r\left[\left(\frac{d}{dt}-\lambda_j\right)^{n_j-r}p_j\left(\frac{d}{dt}\right)b\right]$$

$$=b+\sum_{j=1}^{s}\sum_{r=1}^{n_j}\sum_{l=0}^{r-1}\alpha_{jr}\left(\left(\frac{d}{dt}-\lambda_j\right)^{l+n_j-r}p_j\left(\frac{d}{dt}\right)b\right)\Big|_0^1 e_{\lambda_j}^{*(l+1)}+u$$

with certain function $u\in\operatorname{Ker} p(D)$ By Lemma 1 we have $b^{(j)}\big|_0^1=y^{(j)}\big|_0^1=\eta_j$ $(j=0,\ldots,n)$ and hence

$$\zeta_{jr} = [(\tfrac{d}{dt} - \lambda_j)^r p_j(\tfrac{d}{dt})y]\big|_0^1 = [(\tfrac{d}{dt} - \lambda_j)^r p_j(\tfrac{d}{dt})b]\big|_0^1 \quad (j=1, \ldots, s;$$

$$r = 0, \ldots, n_j - 1).$$

Consequently by (38) and (41), any solution of (35) is given by

$$(42) \qquad y = \begin{cases} Sg - \sum_{j=1}^{s} \sum_{r=1}^{n_j} \sum_{l=0}^{r-1} \alpha_{jr} \zeta_{j,n_j-r+l} \overset{*(l+1)}{e_{\lambda_j}} + \sum_{\mu=1}^{m} \gamma_\mu e^{2\pi i k_\mu t} & \text{if } K \neq \emptyset, \\ Sg - \sum_{j=1}^{s} \sum_{r=1}^{n_j} \sum_{l=0}^{r-1} \alpha_{jr} \zeta_{j,n_j-r+l} \overset{*(l+1)}{e_{\lambda}} & \text{if } K = \emptyset, \end{cases}$$

where γ_μ are arbitrary complex numbers.

Assume that $K \neq \emptyset$. Then by (25) and (37),

$$(Pf)(t) = \sum_{\mu=1}^{m} (\widehat{g}(k_\mu) - (p(\tfrac{d}{dt})b)^\frown(k_\mu))\, e^{2\pi i k_\mu t}.$$

Since $x \in \operatorname{Dom} D^n$ and $(D^k x)^\frown(0) = x^{(k-1)}\big|_0^1 = 0 \quad (k=1, \ldots, n)$ holds, we obtain by (5), (15) and (36)

$$(p(\tfrac{d}{dt})b)^\frown(k_\mu)\, e^{2\pi i k_\mu t} = (p(\tfrac{d}{dt})y - p(\tfrac{d}{dt})x) * e^{2\pi i k_\mu t}$$

$$= (p(\tfrac{d}{dt})y) * e^{2\pi i k_\mu t} - (p(D)x) * e^{2\pi i k_\mu t} = (p(\tfrac{d}{dt})y) * e^{2\pi i k_\mu t} - p(D)(x * e^{2\pi i k_\mu t})$$

$$= (p(\tfrac{d}{dt})y) * e^{2\pi i k_\mu t}.$$

Obviously, a continuously differentiable function h satisfies

$$((\tfrac{d}{dt} - \lambda_j)h) * e^{\lambda_j t} = h\big|_0^1 e^{\lambda_j t}$$

with $\lambda_j = 2\pi i k_\mu$. Choosing

$$h = (\tfrac{d}{dt} - \lambda_j)^{n_j - 1} p_j(\tfrac{d}{dt})y$$

we obtain

$$(Pf)(t) = \sum_{\mu=1}^{m} (\widehat{g}(k_\mu) - \zeta_{j,n_j-1})\, e^{2\pi i k_\mu t} \quad \text{with } \lambda_j = 2\pi i k_\mu.$$

Consequently, $Pf = 0$ holds if and only if

$$(43) \qquad \widehat{g}(k_\mu) = \zeta_{j,n_j-1} \quad \text{for all } \mu = 1, \ldots, m \text{ and } \lambda_j = 2\pi i k_\mu.$$

We summarize:

Theorem 7. *If $K \neq \emptyset$, then the inhomogeneous boundary value problem* (35) *is solvable if and only if the condition* (43) *is fulfilled. Under the assumption* (43), *any solution of* (35) *possesses the form* (42). *If $K = \emptyset$, then the problem* (35) *is uniquely solvable and the solution is given by* (42).

7. Examples. Let us consider the special boundary value problem

$$(44)\qquad \left(\frac{d}{dt}-\alpha\right)\left(\frac{d}{dt}-\beta\right) y(t)=g(t) \quad (t\in[0,1]),$$

$$y\,|_0^1=\eta_0, \quad y'\,|_0^1=\eta_1$$

and apply Theorem 7.

If $\alpha, \beta \neq 2\pi ik$ for all $k\in\mathscr{Z}$ and $\alpha\neq\beta$, then the unique solution of (44) is given by

$$y=y(t\,;\,\alpha,\beta)=\frac{1}{\alpha-\beta}(e_\alpha * g-e_\beta * g+(\beta\eta_0-\eta_1)\,e_\alpha+(\eta_1-\alpha\eta_0)e_\beta).$$

If $\alpha=2\pi il$ $(l\in\mathscr{Z})$ and $\beta\neq 2\pi ik$ for all $k\in\mathscr{Z}$ and if the condition $\hat{g}(k)=\eta_1-\beta\eta_0$ is fulfilled, then any solution of (44) is given by $y(t)=y(t\,;\,2\pi il,\beta)+\gamma e^{2\pi ilt}$ with arbitrary complex number γ.

If $\alpha=2\pi il$, $\beta=2\pi ik$ $(l,k\in\mathscr{Z}$ with $l\neq k)$ and if the conditions

$$\hat{g}(k)=\eta_1-2\pi il\eta_0,$$

$$\hat{g}(l)=\eta_1-2\pi ik\eta_0$$

are fulfilled, then any solution of (44) can be represented by

$$y(t)=y(t\,;\,2\pi il,\,2\pi ik)+\gamma_1 e^{2\pi ilt}+\gamma_2 e^{2\pi ikt}$$

with arbitrary complex numbers γ_1 and γ_2.

If $\alpha=\beta\neq 2\pi ik$ for all $k\in\mathscr{Z}$, then the unique solution of (44) is given by $y=y(t\,;\,\alpha)=e_\alpha^{*2} * g-(\eta_1-\alpha\eta_0)\,e_\alpha^{*2}-\eta_0 e_\alpha$.

Finally, if $\alpha=\beta=2\pi ik$ $(k\in\mathscr{Z})$ and if the condition $\hat{g}(k)=\eta_1-2\pi ik\eta_0$ is satisfied, then any solution of (44) is given by

$$y(t)=y(t\,;\,2\pi ik)+\gamma e^{2\pi ikt}$$

with arbitrary complex number γ.

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