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ON THE APPLICATION OF LYAPUNOV'S SECOND METHOD TO MULTIVALUED DIFFERENCE EQUATIONS

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In the present paper we study the stability of the miltivalued difference equation $x(n+1)\in F(n,x(n))$ via Lyapunov's second method. In particular, we give sufficient conditions for stability, uniform stability, asymptotic stability, uniform-asymptotic stability, as well as the corresponding types of weak stability.

1. Let E be an Euclidean space with norm $|\cdot|$ and $S(r)$ be the open ball in E centered at 0 with radius $r>0$. Let $c(E)$ be the family of all compact nonempty subsets of E. For any $A\in c(E)$ we denote $|A|_s=\sup\{|a|:a\in A\}$.

If n is any integer, we denote by I_n the set $\{n, n+1, \dots\}$.

For some r, $0<r\leq\infty$, let $F:I_0\times S(r)\to c(E)$ be a given multifunction. Fixing $(n_0, x_0)\in I_0\times S(r)$, we consider on I_{n_0} the multivalued difference equation

$$x(n+1)\in F(n, x(n)). \tag{1}$$

By a solution of (1) we understand a function $x:I_{n_0}\to S(r)$ (i.e. a sequence) satisfying (1) an I_{n_0}. We denote by $p(., n, x)$ any solution of (1) passing from $(n, x)\in I_{n_0}\times S(r)$, i. e. taking at $n\in I_{n_0}$ the value $x\in S(r)$. For any $m\in I_{n_0}$, we denote the attainable set from (n, x) at m of (1) by

$$A(m, n, x)=\{y=p(m, n, x):p(\cdot, n, x) \text{ is a solution passing from } (n, x)\in I_{n_0}\times S(r).$$

We remark that the set $A(m, n, x)$ is defined both forwards, i.e. for $m>n$ and backwards, i.e. for $m<n$. Obviously $A(n, n, x)=\{x\}$.

A $\bar{x}\in S(r)$ such that $\{\bar{x}\{=F(n, \bar{x})$, for any $n\in I_{n_0}$, is called an *end-point* o (1). In case $F(n, x)$ is independent of n, i. e. (1) is a discrete multivalued dynamical system, existence of end-points is discussed in [2; 8]. In the present paper we assume that 0 is an end-point of (1) and we determine sufficient conditions for various types of stability (resp. weak stability) of the end point 0 of (1) via Lyapunov's second method. Results of this kind, concerning discrete multivalued dynamical systems, can be found in [2; 4; 6; 8]. They represent the discrete version of current investigations concerning (continuous) multivalued dynamical systems [10; 11] and multivalued differential equations [1; (3; 4; 5].

Finally, let us mention that stability properties of multivalued difference equations find several applications in various fields; for example, in optimization theory [9; 13], in problems of minimization algorithms [12], in game theory [8], in mathematical economics [4], and in numerical analysis. In the last discipline multivalued difference equations are derived from a discretization of multivalued differential equations. But the numerical convergence question of the solutions of the multivalued differential equations is closely linked to numerical stability, that is the stability of the corresponding multivalued difference equations [7].

The following definitions concern the stability (resp. weak stability) of the end-point 0 of (1).

Definition 1. *The end-point* 0 *of* (1) *is stable*), (*resp. weakly stable*), *if for any* $\varepsilon>0$ *and* $n_0 \in I_0$ *there exists a* $\delta(\varepsilon, n_0)>0$ *such that* $|x_0|<\delta$ *implies* $|A(n, n_0, x_0)|_s<\varepsilon$ (*resp. there exists a solution* $p(., n_0, x_0)$ *of* (1) *such that*, $|p(n, n_0, x_0)|<\varepsilon$), *for all* $n \in I_{n_0}$.

Definition 2. *The end-point* 0 *of* (1) *is uniform stable* (*resp. weakly uniformly stable*), *if it is stable and* δ *in Definition* 1 *is independent of* n_0.

Definition 3. *The end-point* 0 *of* (1) *is asymptotically stable* (*resp. weakly asymptotically stable*), *if it is stable* (*resp. weakly stable*) *and for any* $n_0 \in I_0$ *there exists a* $\delta_0(n_0)>0$ *such that* $|x_0|<\delta_0$ *implies* $\lim_{n\to\infty}|A(n, n_0, x_0)|_s=0$ (*resp. there exists a solution* $p(., n_0, x_0)$ *of* (1) *such that* $\lim_{n\to\infty}|p(n, n_0, x_0)|=0$).

Definition 4. The end-point 0 *of* (1) *is uniform-asymptotically stable* (*resp. weakly uniform-asymptotically stable*), *if it is uniformly stable* (*resp. weakly uniformly stable*) *and for any* $\varepsilon>0$ *there exist a* $\delta_0>0$ *and a* $N(\varepsilon)>0$ *such that* $|x_0|<\delta_0$ *implies* $|A(n, n_0, x_0)|_s<\varepsilon$ (*resp. there exists a solution* $p(\cdot, n_0, x_0)$ *of* (1) *such that* $|p(n, n_0, x_0)|<\varepsilon$), *for all* $n \in I_{n_0+N}$.

Clearly, any type of stabilty implies the corresponding type of weak stability. The notion of weak stability, which is pertinent to non-iniqueness of solutions in the multivalued case, was developed by Roxin, [10; 11] in relation to multivalued dynamical systems.

2. Main results. Our stability criteria are based upon the existence of an appropriate scalar function V, $V: I_0\times S(r)\to R$. Moreover, in order to measure the growth or decay of such a function of V along a solution of (1), we define, for any $n \in I_{n_0} x \in S(r)$ and $y \in F(n, x) \cap S(r)$, the following functions

$$\Delta V(n, x, y)=V(n+1, y)-V(n, x)\Delta V(n, x, F)=\sup\{\Delta V(n, x, y): y \in F(n, x)\}$$

We denote by K the class of all monotonically increasing scalar functions on $[0, r)$ vanishing at 0. We consider the following hypotheses:

(H_1) $V(n, 0)=0$, $a(|x|)\leq V(n, x)$, *for some* $a \in K$ *and for all* $(n, x) \in I_0\times S(r)$;

(H_2) $V(n, x)\leq b(|x|)$, *for some* $b \in K$ *and for all* $(n, x) \ni I_0\times S(r)$;

(H_3) $\Delta V(n, x, F)\leq 0$, *for all* $(n, x) \in I_0\times S(r)$;

(H_3^*) *for all* $(n, x) \in I_0\times S(r)$ *there exists* $y_{n,x} \in F(n, x)\cap S(r)$ *such that* $\Delta V(n, x, y_{n,x})\leq 0$;

(H_4) $\Delta V(n, x, F)\leq -c(V(n, x))$, *for some* $c \in K$ *and for all* $(n, x) \in I_0\times S(r)$;

(H_4^*) *for all* $(n, x) \in I_0\times S(r)$ *there exists* $y_{n,x} \in F(n, x)\cap S(r)$ *such that* $\Delta V(n, x, y_{n,x})\leq -c(V(n, x))$, *for some* $c \in K$;

(H_5) $\Delta V(n, x, F)\leq -c(|x|)$, *for some* $c \in K$ *and for all* $(n, x) \in I_0\times S(r)$;

(H_5^*) *for all* $(n, x) \in I_0\times S(r)$, *there exists* $y_{n,x} \in F(n, x)\cap S(r)$ *such that* $\Delta V(n, x, y_{n,x})\leq -c(|x|)$, *for some* $c \in K$.

A scalar function V, which satisfies (H_1) (or (H_1) and (H_2)), is said to be a *Lyapunov function or entropy* (resp. *weak Lyapunov function ar weak entropy*), according to Aubin and Siegel [2] on $I_0\times S(r)$ for the multivalued differnce equation (1), if it also satisfies one of (H_3), (H_4), (H_5) (resp. one of (H_3^*), (H_4^*), (H_5^*)).

Obviously any Lyapunov function is a weak Lyapunov function.

Theorem 1. *The hypotheses* (H_1) (H_3) *imply that the end-point* 0 *of* (1) *is stable.*

Proof. For any $\varepsilon>0$ and $n_0 \in I_0$ we can choose a $\delta(\varepsilon, n_0)>0$ such that $|x_0|<\delta$ implies $V(n_0, x_0)<a(\varepsilon)$. Let $x \in A(n, n_0, x_0)$, $n \in I_{n_0}$, be arbitrary. Then there exists a solution $p(\cdot, n_0, x_0)$ of (1) such that $p(n, n_0, x_0)=x$. Moreover, $V(n+1, p(n+1, n_0, x_0))-V(n, x)\leqq \Delta V(n, x, F)\leqq 0$ implies $V(n+1, p(n+1, n_0, x_0))\leqq V(n, p(n, n_0, x_0))$, for any $n \in I_{n_0}$. By induction, it can be easily proved that $V(n, p(n, n_0, x_0))\leqq V(n_0, x_0)$, for any $n \in I_{n_0}$. Therefore $a(|p(n, n_0, x_0)|) \leqq V(n, p(n, n_0, x_0))\leqq V(n_0, x_0)<a(\varepsilon)$ and, since $a \in K$ we get $|p(n, n_0, x_0)|<\varepsilon$, $n \in I_{n_0}$ i.e. $|A(n, n_0, x_0)|_s<\varepsilon$, since $x \in A(n, n_0, x_0)$ is arbitrary.

Theorem 2. *The hypotheses* (H_1), (H_3^*) *imply that the end-point* 0 *of* (1) *is weakly stable.*

Proof. Let $p(., n_0, x_0); I_{n_0}\to S(r)$ be defined by $p(n_0, n_0, x_0)=x_0$, $p(n+1, n_0, x_0)=y_{n,p(n, n_0,x_0)} \in F(n, p(n, n_0, x_0)) \cap S(r)$, $n \in I_{n_0}$. Clearly $p(., n_0, x_0)$ is a solution of (1) and $p(n, n_0, x_0) \in A(n, n_0, x_0)$. By definition of $p(., n_0, x_0)$ and hypothesis (H_3^*) we get $V(n+1, p(n+1, n_0, x_0))-V(n, p(n, n_0, x_0))=\Delta V(n, p(n, n_0, x_0), y_{n,p(n,n_0,x_0)})\leqq 0$. Then, as in Theorem 1, we get $|p(n, n_0, x_0)|<\varepsilon$, $n \in I_{n_0}$.

Theorem 3. *The hypotheses* (H_1), (H_2), (H_3) *imply that the end-point* 0 *of* (1) *is uniformly stable.*

Proof. For arbitrary $\varepsilon>0$ we choose $\delta=\delta(\varepsilon)>0$ such that $b(\delta)<a(\varepsilon)$. Then, whenever $|x_0|<\delta$, we have $V(n_0, x_0)\leqq b(|x_0|) <b(\delta)<a(\varepsilon)$, $n_0 \in I_0$. Then by the same argument, as in Theorem 1, we have $a(|p(n, n_0, x_0)|)< a(\varepsilon)$, i.e. $|p(n, n_0, x_0)|<\varepsilon$.

Theorem 4. *The hypotheses* (H_1), (H_2), (H_3^*) *imply that the end-point* 0 *of* (1) *is weakly uniformly stable.*

Proof. For arbitrary $\varepsilon>0$ we choose $\delta=\delta(\varepsilon)>0$ such that $b(\delta)<a(\varepsilon)$. If we construct the solution $p(., n_0, x_0)$ as in the proof of Theorem 2, then it is implied $a(|p(n, n_0, x_0)|)\leqq V(n_0, x_0\leqq b(|x_0|)<b(\delta)<a(\varepsilon)$, $n_0 \in I_0$.

Theorem 5. *The hypotheses* (H_1), (H_4) *imply that the end-point* 0 *of* (1) *is asymptotically stable.*

Proof. From Theorem 1 and since (H_4) implies (H_3) we have stability. Also there exists a $\delta_0>0$ such that $n_0 \in I_0$, $|x_0|<\delta_0$ imply $|A(n, n_0, x_0)|_s<r$, $n \in I_{n_0}$. Let $x \in A(n, n_0, x_0)$, $n \in I_{n_0}$, be arbitrary and $p(., n_0, x_0)$ be the corresponding solution of (1) passing from (n_0, x_0). Clearly $V(n, p(n, n_0, x_0))\leqq V(n_0, x_0)$, i.e. there exists $V_0=\lim_{n\to\infty} V(n, p,(n, n_0, x_0))$. If $V_0\neq 0$, we shall show that,

$$V(n, p(n, n_0, x_0))-V(n_0, x_0)\leqq -c(V_0)(n-n_0), \quad n \in I_{n_0}. \tag{2}$$

Actually, (using induction) if we accept that it holds for some n then, $V(n+1, p(n+1, n_0, x_0))-V(n_0, x_0)\leqq V(n, p(n, n_0, x_0))-c(V(n_0, x_0))-V(n_0, x_0)\leqq -c(V_0)(n-n_0)-c(V_0)=-c(V_0)(n+1-n_0)$, since $V(n, p(n, n_0, x_0)$ is non-increasing in n and $c \in K$, i.e. it holds for $n+1$. Thus, taking the limit $n\to\infty$ in (2), we get $V_0=-\infty$, a contradiction. Therefore, $V_0=0$ and $\lim_{n\to\infty} a(|p(n, n_0, x_0)|) \leqq \lim_{n\to\infty}(V(n, p(n, n_0, x_0))=0$, i.e. $\lim_{n\to\infty} a(|p(n, n_0, x_0)|)=0$, i.e. $\lim_{n\to\infty}p(n, n_0, x_0)=0$. Since $x \in A(n, n_0, x_0)$ is arbitrary, it follows $\lim_{n\to\infty}|A(n, n_0, x_0)|_s=0$.

Theorem 6. *The hypotheses* (H_1), (H_4^*) *imply that the end-point* 0 *of* (1) *is weakly asymptotically stable.*

Proof. Let $p(.,n_0,x_0)$ be the solution of (1) constructed as in the proof of Theorem 2. Clearly $V(n+1, p(n+1, n_0, x_0)) - V(n, p(n, n_0, x_0)) \leq -c(V(n, p(n, n_0, x_0))$. Induction shows that $V(n, p(n, n_0, x_0)) \leq V(n_0, x_0)$, $n \in I_{n_0}$, and as in the proof of Theorem 5 we get $\lim_{n\to\infty} |p(n, n_0, x_0)| = 0$.

Theorem 7. *The hypotheses* (H_1), (H_2), (H_5) *imply that the end-point* 0 *of* (1) *is uniform-asymptotically stable.*

Proof. Uniform stability is established by Theorem 3, since (H_5) implies (H_3). Also there exists a $\delta_0>0$ such that $n_0 \in I_0$, $|x_0|<\delta_0$ imply $|A(n, n_0, x_0)|_s<r$, $n \in I_{n_0}$. Moreover, for any $\varepsilon>0$ there exists a $\delta(\varepsilon)>0$ such that $n_0 \in I_0$, $|x_0|<\delta$ imply $|A(n, n_0, x_0)|_s<\varepsilon$, $n \in I_{n_0}$. We claim that for all $n_0 \in I_0$, $|x_0|<\delta_0$ there exists a $n_1 \in I_{n_0}$ such that $|A(n_1, n_0, x_0)|_s<\delta(\varepsilon)$. Because otherwise, for an arbitrary $x \in A(n, n_0, x_0)$, $n \in I_{n_0}$, and for the corresponding solution $p(n, n_0, x_0)$ of (1) passing from (n_0, x_0), we would have $|p(n, n_0, x_0)| \geq \delta(\varepsilon)$, $n \in I_{n_0}$, i.e. $-c(|p(n, n_0, x)| \leq -c(\delta(\varepsilon))$. Hypotheses (H_5) implies that, for all $n \in I_{n_0}$, $V(n+1, p(n+1, n_0, x_0)) - V(n, p(n, n_0, x_0)) \leq -c(|p(n, n_0, x_0)|)$. One can easily obtain by induction that, for all $n_0 \in I_0$, $V(n, p(n, n_0, x_0)) \leq V(n_0, x_0) - c(\delta(\varepsilon))(n - n_0)$. Let $(b(\delta_0) - a(\delta(\varepsilon))/c(\delta(\varepsilon))$. Since $V(n_0, x_0) \leq b(|x_0|) < b(\delta_0)$ we get, for all $n>n_0+N(\varepsilon)$, $V(n, p(n, n_0, x_0))<a(\delta(\varepsilon))$, a contradiction. Now taking $N(\varepsilon)$ sufficiently large, there exists a $n_1 \in I_{n_0}$, $n_1 \leq n_0+N(\varepsilon)$, such that $|A(n_1, n_0, x_0)|_s <\delta(\varepsilon)$, which implies that $|A(n_1, n_0, x_0)|_s<\varepsilon$, for all $n \geq n_0+N(\varepsilon)$, and the proof is completed.

Theorem 8. *The hypotheses* (H_1), (H_2), (H_5^*) *imply that the end-point* 0 *of* (1) *is weakly uniform-asymptotically stable.*

Proof. If $p(.,n_0,x_0)$ is the solution of (1) constructed as in the proof of Theorem 2, then it can be shown, as in the proof of Theorem 7, that there exists a $N(\varepsilon)>0$ such that $|p(n, n_0, x_0)|<\delta(\varepsilon)$, $n \geq n_0+N(\varepsilon)$.

Finally we give some examples illustrating the above types of stability (resp. weak stability). Consider the multivalued difference equations

$$x(n+1) \in F(n, x) = \left\{\frac{1}{n+1}x(n), \frac{1}{n+2}x(n)\right\}, \quad n \in I_2; \tag{3}$$

$$x(n+1) \in F(n, x) = \left\{x(n), \frac{1}{n+1}x(n), \frac{1}{n+2}x(n)\right\}, \quad n \in I_2. \tag{4}$$

Then, (3) is uniform-asymptotically stable and therefore weakly uniform-asymptotically stable, since if we consider $V(n, x)=x^2$, $a(r)=r^2/2$, $b(r)=3r^2/2$ and $c(r)=-r^2/2$ all hypotheses of Theorem 7 are satisfied. Also (4) is uniformly stable and therefore weakly uniformly stable, since if we consider $V(n, x)=|x|$, $a(r)=r/2$, $b(r)=3r/2$, all hypotheses of Theorem 3 are satisfied. Obviously, (4) is non uniform-asymptotically stable, although it is weakly uniform-asymptotically stable, since if we consider $V(n, x)=|x|$, $a(r)=r/2$, $b(r)=3r/2$, $c(r)=r/2$ and $y_{n,x}=x/(n+2)$ all hypotheses of Theorem 8 are satisfied.

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