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CONVEX DENSITIES AND SELF-DECOMPOSABILITY

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1. A characteristic function $\psi(u)$ is infinitely divisible if and only if it has a unique representation of the form

$$\psi(u)=\exp\{i\alpha u-\frac{\sigma^2}{2}u^2+\int_{R/\{0\}}[e^{ixu}-1-\frac{iux}{1+x^2}]\,dM(x)\}, \tag{1.1}$$

where $M(x)$, σ^2 and α satisfy the following conditions:

(i) $M(x)$ is non-decreasing in the intervals $(-\infty, 0)$ and $(0, +\infty)$.

(ii) The interval $\int_{-\varepsilon}^{\varepsilon}x^2dM(x)$ is finite for every $\varepsilon>0$.

(iii) The constants σ^2, α satisfy the conditions $\sigma^2\geqq 0$, while α is real.

The representation (1.1) is the Levy canonical representation of $\psi(u)$ and the function $M(x)$ is the Levy spectral function. An infinitely divisible characteristc function $\psi(u)$ is self-decomposable if and only if the function $M(x)$ has left and right derivatives everywhere and if the function $xM'(x)$ is non-increasing in the intervals $(-\infty, 0)$ and $(0, +\infty)$. Here $M'(x)$ denotes either the right or left derivative, possibly different ones at different points. Every self-decomposable characteristic function $\psi(u)$ satisfy the functional equation $\psi(u)=\psi(ru)\psi_r(u)$, where $\psi_r(u)$ is an infinitely divisible characteristic function and r any constant with $0<r<1$ [1].

This paper is devoted to the study of a transformation which converts characteristic functions of distributions with convex densities into self-decomposable characteristic functions.

2. **Transformed distributions.** A distribution function $F(x)$ is called unimodal with the mode at $x=0$ if the graph of $F(x)$ is convex on $(-\infty, 0)$ and concave on $(0, +\infty)$. Let $\varphi(u)$ be the characteristic function of $F(x)$. Then

$$\varphi(u)=\frac{1}{u}\int_0^u \alpha(y)\,dy, \tag{2.1}$$

where $\alpha(u)$ is the characteristic function of the distribution function $F(x)-xf(x)$ and $f(x)$ the probability density function of $F(x)$ (see [1, p. 92]).

Theorem 1. *Let $F(x)$ be a distribution function with probability density function $f(x)$. Suppse that $f''(x)$ exists. Then $\gamma(u)=\exp\{-\int_0^u\varphi(y)\,ydy\}$ is a self-decomposable characteristic function if and only if $f(x)$ is convex.*

Proof. Only the sufficiency condition will be proved since the necessity condition can be proved by reversing the argument. Distribution functions having convex densities are unimodal at $x=0$. Hence from (2.1) and Theorem 12.2.8 of [1] it follows that

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$$\gamma(u)=\exp\{-\int_0^u \varphi(y)\,y\,dy\}=\exp\{-\int_0^u\int_0^y \alpha(x)\,dx\,dy\}$$

is an infinitely divisible characteristic function and

$$\gamma(u)=\exp\{\int_{-\infty}^{\infty}[e^{iux}-1-iux]\frac{d[F(x)-xf(x)]}{x^2}\}$$

is the Kolmogorov canonical representation of $\gamma(u)$. From the correspondence between the Levy and the Kolmogorov canonical representations of $\gamma(u)$, the Levy spectral function $M(x)$ of $\gamma(u)$ is given by

$$M(x)=\begin{cases}\int_{-\infty}^{x}\frac{1}{y^2}\,d[F(y)-yf(y)], & x<0,\\ -\int_{x}^{\infty}\frac{1}{y^2}\,d[F(y)-yf(y)], & x>0.\end{cases}$$

The convexity of $f(x)$ implies that $xM'(x)=-f'(x)$ is non-increasing. Hence $\gamma(u)$ is self-decomposable.

Corollary 1. *Let $\varphi(u)$ be characteristic function of a distribution function $F(x)$ with finite first and second moment μ_1,μ_2. Then $\gamma_1(u)=\exp\{\int_0^u\frac{\varphi(u)-1}{y}dy\}$ is a self-decomposable characteristic function.*

Proof. Theorem 1 of [2] implies that $\delta(u)=2[\varphi(u)-i\mu_1u-1]/i^2\mu_2u^2$ is the characteristic function of a distribution with convex density. Hence form Theorem 1 we conclude that

$$\gamma_2(u)=\exp\{-\int_0^u\delta(y)\,y\,dy\},\quad \gamma_1(u)=[\gamma_2(u)]^{\mu_2/2}\exp\{i\mu_1u/2\}$$

are self-decomposable characteristic functions.

The class of distribution functions on $(0,\infty)$ with convex probability density functions includes the class of distribution functions with completely monotone probability density functions. A probability density function $f(x)$ with $(-1)^nf^{(n)}(x)\geq 0$, $n=1,2,\ldots$, is said completely monotone. Here $f^{(n)}(x)$ denotes the nth derivative of $f(x)$.

Theorem 2. *$F(x)$ be a distribution function on $(0,\infty)$ with completely monotone probability density function $f(x)$ and characteristic function $\varphi(u)$. Then the self-decomposable characteristic function $\gamma(u)=\exp\{-\int_0^u\varphi(y)y\,dy\}$ satisfies the functional equation $\gamma(u)=\gamma^r(ru)\gamma_r(u)$, where $\gamma_r(u)$ is a self-decomposable characteristic function and $0<r<1$.*

Proof. The function $\gamma_r(u)$ can be written in the form (1.1) with $\alpha=0$, $\sigma^2=0$ and the function $M(x)$ given by

$$M(x)=\begin{cases}0 & x<0,\\ -\int_x^{\infty}\frac{1}{y^2}\,d[F(y)-yf(y)]+r\int_{x/r}^{\infty}\frac{1}{y^2}\,d[F(y)-yf(y)], & x>0.\end{cases}$$

Since $f^{(1)}(x)\leq 0$ and $f^{(2)}(x)\geq 0$ it follows that $f^{(1)}(x)$ is nondecreasing and that

$$M(x)=\frac{rf^{(1)}(x/r)-f^{(1)}(x)}{x}\geq 0,\quad x>0.$$

From here we get that $M(x)$ is non-decreasing. Hence from (1.1) we conclude that $\gamma_r(u)$ is an infinitely divisible characteristic function. Furthermore from the fact that $f^{(3)}(x)\leq 0$ it follows that $f^{(2)}(x)$ is non-increasing and that

$$[xM'(x)]'=f^{(2)}(x/t)-f^{(2)}(x)\leq 0.$$

Form here we get that $xM'(x)$ is non-increasing. Hence from (1.1) we conclude that $\gamma_r(u)$ is a self-decomposable characteristic function.

Below we establish some properties for an infinitely divisible characteristic function related with the self-decomposable characteristic function $\gamma(u)=\exp\{-\int_0^u\varphi(y)\,ydy\}$.

Theorem 3. *$F(x)$ be a distribution function with a twice differentiable convex density $f(x)$. Let $\varphi(u)$ be the characteristic function of $F(x)$. Then $\delta(u)=\exp\{-u^2\varphi(u)\}$ is an infinitely divisible characteristic function and $\gamma(u)=\lim_{n\to\infty}\Pi_{k=1}^n[\delta(\frac{ku}{n})\gamma(\frac{ku}{n})]^{1/n}$, if $\varphi(u)$ is differentiable at $x=0$.*

Proof. From (2.1) we get that $\alpha(u)=\varphi(u)+u\varphi'(u)$ is a characteristic function. Furthermore $\beta(u)=\frac{1}{2}\alpha(u)+\frac{1}{2}\varphi(u)$ is the characteristic function of the distribution function $B(x)=\frac{1}{2}F(x)+\frac{1}{2}[F(x)-xf(x)]$. The convexity of $f(x)$ implies that $B(x)$ is unimodal at $x=0$. From Theorem 1 it follows that

$$\delta(u)=\exp\{-2\int_0^u\beta(y)\,ydy\}=\exp\{-u^2\varphi(u)\}$$

is an infinitely divisible characteristic function. Furthermore $k(u)=\gamma(u)\delta(u)$ is an infinitely divisible characteristic function. Using the partition $\{0, 1/n, 2/n, \ldots, n/n\}$ of the interval $[0, 1]$ and the expression

$$\gamma(u)=\exp\{\int_0^1 k(uy)\,dy\}=\exp\{\frac{1}{u}\int_0^u k(y)dy\}$$

we get that

$$\gamma(u)=\lim_{n\to\infty}\prod_{k=1}^{n}[\delta(\frac{ku}{n})\gamma(\frac{ku}{n})]^{1/n}.$$

In Theorem 4 we establish two properties for the infinitely divisible characteristic function $\gamma(u)=\exp\{-\int_0^u\varphi(y)\,ydy\}$.

Theorem 4. *Let $F(x)$ be a distribution function which is unimodal at $x=0$ and $\varphi(u)$ its characteristic function. Then:*

(i) *$\varphi(u)\gamma(u)$ is the characteristic function of a distribution function which is unimodal at $x=0$.*

(ii) *$\varphi(u)=\exp\{-\int_0^u\varphi(y)ydy\}$ with $F(x)$ infinitely divisible having a finite second moment if and only if $\varphi(u)=2/2+u^2$.*

Proof. (i) From Theorem 1 and Theorem 12.2.8 of [1] we get that $\gamma(u)=\exp\{-\int_0^u\varphi(y)\,ydy\}$ belongs to an infinitely divisible distribution function having a finite second moment. Hence $\gamma''(u)/\gamma''(0)$ is also a characteristic function. Since

$$\varphi(u)\,\gamma(u)=\frac{1}{u}\int_0^u\frac{\gamma''(y)}{\gamma''(0)}\,dy$$

it follows that $\varphi(u)\gamma(u)$ belongs to a distribution function which is unimodal at $x=0$.

(ii) By differentiating $\varphi(u)=\exp\{-\int_0^u \varphi(y)ydy\}$ we get the differential equation $\varphi'(u)=-\varphi^2(u)u$ with $\varphi(0)=1$. Therefore $\varphi(u)=2/2+u^2$.

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