Serdica

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Сердика

Българско математическо списание

ON POINCARE'S TRANSLATION OPERATOR FOR ORDINARY EQUATIONS WITH RETARDS

G. DYLAWERSKI, J. JODEL

The aim of this paper is to generalize Poincare's translation operator along trajectories of ordinary differential equations. We will consider this operator as some multi-valued map. By using the topological degree methods to this operator we are able to prove some results concerning the existence of periodic solutions for ordinary differential equations with retards.

The above results stand some generalization of well known Poincare-Krasnosielski results concerning periodic solutions for ordinary differential equations which right side satisfies the Lipschitz condition with respect to the second variable (cf. [4]). We consider this paper as some extension and continuation of methods presented in [2].

We will use the following notations. Let X be a topological space and $A\subset X$. Then ∂A denotes the boundary of A in X and $\bar{A}$ denotes the closure of A in X. $C([a, b], R^n)$ is the space of all continuous functions from $[a, b]$ into R^n with the norm $\|x\| = \sup_{t\in[a,b]} \|x(t)\|$. By I_X or shortly by I we will denote the identity map over X. By K_r, B_r we will denote, respectively, the closed ball in $C([a, b], R^n)$ and R^n with the center in the point zero and radius r.

Multi-valued maps we will denote by φ, Φ, ψ, H, single-valued by e, S P, V, W, f, x, y, r.

A multi-valued map $\varphi: X\to Y$ is called upper semi-continuous u. s. c. if $\varphi(x)$ is a compact subset of Y for each $x\in X$ and for every open subset $U\subset Y$ the set $\varphi^{-1}(U)=\{x\in X, \varphi(x)\subset U\}$ is an open subset of X.

An u. s. c. multi-valued map $\varphi: X\to Y$ is called acyclic if $\varphi(x)$ is anacyclic set, for each $x\in X$ (in the sense of the Cech cohomology theory with integer coefficients).

Let E be a Banach space, X be a metric space, $K_r\subset E$. A multi-valued map $\Phi: K_r\to E$ is called a pseudoacyclic map if Φ is a composition of an acyclic map $\varphi: K_r\to X$ and a continuous function $r: X\to E$, i. e. $\Phi=r\circ\varphi$.

Let us consider multi-valued map $I-\Phi$: $K_r\to E$, where $\Phi=r\circ\varphi$ is a compact, pseudoacyclic map and $(I-\Phi)(\partial K_r)\subset E\setminus\{0\}$. For such a map is possible to define the topological degree by putting:

$$\mathrm{Deg}(I-\Phi, K_r)=\deg(p, p-r\circ q, K_r),$$

where $p: G_\varphi\to K_r$, $p(x, y)=x$, q: $G_\varphi\to E$, $q(x, y)=y$, $G_\varphi=\{(x, y)\in K_r\times X, y\in\varphi(x)\}$. For the definition of the topological degree see [1; 3; 5]. Below, we will formulate a few properties of the topological degree for pseudoacyclic maps comp. [1; 3; 5] for proofs.

Proposition 1. *If* $\mathrm{Deg}(I-\Phi, K_r)\neq 0$, *then there exists* $x\in K_r$ *such that* $0\in(I-\Phi)(x)$.

Proposition 2. *Let* H: $K_r\times[0, 1]\to X$ *be an acyclic map and* r: $X\to E$ *be a continuous map such that the composition* $r\circ H$ *is a compact* *map. Assume further that* $0 \notin (I - r \circ H)(\partial K_r \times [0, 1])$. *Then* $\mathrm{Deg}(I - r \circ H_0, K_r) = \mathrm{Deg}(I - r \circ H_1, K_r)$, *where* $H_0(x) = H(x, 0)$ $H_1(x) = H(x, 1)$.

Proposition 3. *Let* $\varphi: K_r \to X$ *be an acyclic and compact map and* $R: X \times [0, 1] \to E$ *be a continuous map. Suppose that* $0 \notin (I - r_t \circ \varphi)(\partial K_r)$, *for* every $t \in [0, 1]$ $r_t(x) = R(x, t)$. *Then* $\mathrm{Deg}(I - r_0 \circ \varphi, K_r) = \mathrm{Deg}(I - r_1 \circ \varphi, K_r)$.

Proposition 4. *Let* $\varphi: K_r \to X$ *be an acyclic and compact map and let* $r: X \to F$ *be a continuous function, where* F *is a finite dimensional subspace of* E. *Assume further that* $0 \notin (I - r \circ \varphi)(\partial K_r)$. *Then* $\mathrm{Deg}(I - r \circ \varphi, K_r) = \mathrm{Deg}(I - r \circ \varphi^n, K_r^n)$, *where* $K_r^n = K_r \cap F$, $\varphi^n = \varphi/_{K_r^n}: K_r^n \to X$.

Proposition 5. *Let* X *be a metric space,* E^n *be a finite dimensional space and* $K_r \subset E^n$. *Let* $\varphi: K_r \to X$ *be a continuous map. Assume further that* $0 \notin (I - r \circ \varphi)(\partial K_r)$ *and* $A: R^n \to E^n$ *is a linear isometry. Then* $\mathrm{Deg}(I - A^{-1} \circ r \circ \varphi \circ A, B_r) = \mathrm{Deg}(I - r \circ \varphi, K_r)$.

Ordinary differential equations with retards. Let T, $h_i \geq 0$, $i = 1, \ldots, m$, $h = \max_i h_i$ and $x \in C([0, T], R^n)$, $y \in C([-h, 0], R^n)$ be two functions such that $x(0) = y(0)$. By yx we will denote a function in $C([-h, T], R^n)$ given by the formula:

$$yx(t) = \begin{cases} y(t) & \text{for} \quad t \in [-h, 0], \\ x(t) & \text{for} \quad t \in [0, T]. \end{cases}$$

Let $f: R \times R^{nm} \to R^n$ and $y \in C([-h, 0], R^n)$ be two given continuous functions. We will consider an equation:

$$(*) \qquad x'(t) = f(t, yx(t - h_1), \ldots, yx(t - h_m)).$$

By $\varphi(y)$ we will denote the set of solutions of this equation on interval $[0, T]$ i. e.

$$\varphi(y) = \{x \in C([0, T], R^n), \ x(0) = y(0) \text{ and } x \text{ satisfies } (*) \text{ for } t \in [0, T]\}.$$

Now we define the Poincare translation operator Φ_T for problem $(*)$. Assume that $\varphi(y)$ is a non-empty set for each $y \in C([-h, 0], R^n)$. Then we have a multi-valued map $\varphi: C([-h, 0], R^n) \to C([0, T], R^n)$ which is given by the following correspondence $y \to \varphi(y)$. Let $S: C([0, T], R^n) \to C([-h, 0], R^n)$ be given by the formula $S(x)(t) = x(T + t)$. We consider the map Φ_T: $C([-h, 0], R^n) \to C([-h, 0], R^n)$ given as follows $\Phi_T(y) = S \circ \varphi(y)$.

Remark. In the case when $h = h_i = 0$, $i = 1, \ldots, m$, then we can identify $C(\{0\}, R^n)$ with R^n and then we will use the following notations $\breve{S}: C([0, T]$ $R^n) \to R^n$, $\breve{\varphi}: R^n \to C([0, T], R^n)$ for S and φ, respectively. So we have, $\breve{S}(x) - x(T)$ and $\breve{\varphi}(y)$ is the set of all solutions on $[0, T]$ of the following: Cauchy problem:

$$x'(t) = f(t, x(t), x(t), \ldots, x(t)),$$

$$x(0) = y.$$

Observe that the Poincare operator $\Phi_T = \breve{S} \circ \breve{\varphi}: R^n \to R^n$ is the same as considered in [2].

The following proposition gives us a connection between the Poincare's operator and periodic solutions of a differential equations with retards:

Proposition 6. *Let* $f: R \times R^{n \cdot m} \to R^n$ *be continuous and* T*-periodic with respect to the first variable and* $0 \in (I - \Phi_T)(y)$ *for some* y. *Then the equation* $x'(t) = f(t, x(t - h_1), \ldots, x(t - h_m))$ *has a* T*-periodic solution on* R.

Proof. If $0 \in (I-\Phi_T)(y)$ then for some $x \in \varphi(y)$ we have $y(t)=x(T+t)$ for $t \in [-h, 0]$. We put $x_0(n.T+t)=x(t)$, where $t \in [0, T]$, $n \in Z$. Then x_0 is obviously a T-periodic map on R. Moreover we have $yx(t)=x_0(t)$ for $t \in [-h,T]$. So $x_0'(t)=x_0'(nT+t_1)=x'(t_1)=f(t_1, yx(t_1-h_1), \ldots, yx(t_1-h_m))=f(nT+t_1, x_0(t_1-h_1), \ldots, x_0(t_1-h_m))=f(t, x_0(t-h_1), \ldots, x_0(t-h_m))$ and the proof is completed.

If we assume that the right side of equation (*) is bounded then (comp. [5]) φ is a compact acyclic map. Because S is obviously continuous so Φ_T is a pseudoacyclic operator. Consequently, from Proposition 1 and 6 we obtain automatically the following theorem:

Theorem 1. *If $f: R\times R^{n.m} \to R^n$ is a continuous, bounded T-periodic with respect to the first variable map and for some ball $K_r \subset C([-h, 0], R^n)$ the Poincare operator Φ_T satisfies the following conditions:*

(i) $0 \notin (I-\Phi_T)(\partial K_r)$,

(ii) $\mathrm{Deg}(I-\Phi_T, K_r) \neq 0$,

then the equation $x'(t)=f(t, x(t-h_1), \ldots, x(t-h_m))$ has a T-periodic solution on R.

Theorem 1 gives us a useful method to check whether an ordinary differential equation have a periodic solution. We will show two applications of Theorem 1.

A. C^1-map $V: R^n \to R$ is called a direct potential for $f: R\times R^n \to R^n$ if the following conditions are satisfied:

(i) there exists $r_0>0$ such that if $\|x\| \geqq r_0$, then $\mathrm{grad}\ V(x) \neq 0$

(ii) $\langle \mathrm{grad}\ V(x), f(t, x)\rangle > 0$ for $t \in R$, $\|x\| \geqq r_0$.

The theorem, which is given below, gives us one characterization of the topological degree of Poincare operator $\breve{\Phi}_T$ for the problem: $x'(t)=f(t, x(t))$, $x(0)=x_0$.

Theorem 2 (comp. [2]). *Let $f: R\times R^n \to R^n$ be a continuous and bounded map. Assume further that f has a direct potential V such that* $\mathrm{Ind}\ V \neq 0$; *where* $\mathrm{Ind}\ V = \deg(\mathrm{grad}\ V, B_r)$, $r>r_0$. *Then there exists $r_1>0$ such that $0 \notin (I-\breve{\Phi}_T)(R^n \setminus \mathrm{Int}\, B_{r_1})$ and $\mathrm{Deg}(I-\breve{\Phi}_T, B_r) \neq 0$ for $r>r_1$.*

It is easy to see, that if we assume that f is T-periodic with respect to the first variable then from Theorem 1 follows that the equation $x'(t)=f(t, x(t))$ has a T-periodic solution on R in B_{r_1}.

A. C^1-map $V: R^n \to R$ is called a regular potential for $f: R\times R^n \times R^{n.k} \to R^n$ if the following conditions are satisfied:

(i) there exist $\rho_0>0$, $\alpha>0$ such that $\langle f(t, x, y_1, \ldots, y_k), \mathrm{grad}\ V(x)\rangle > \alpha \cdot \|f(t, x, y_1, \ldots, y_k)\| \cdot \|\mathrm{grad}\ V(x)\|$ for $\|x\| \geqq \rho_0$, $y_i \in R^n$.

(ii) there exists a C^1-map $W: R^n \to R$ such that

$$\|\mathrm{grad}\ W(x)\| < \|\mathrm{grad}\ V(x)\| \text{ for } \|x\| \geqq \rho_0 \text{ and } \lim_{\|x\|\to\infty} |W(x)| = \infty.$$

Let us consider the following equation: $x'(t)=f(t, x(t), yx(t-h_1), \ldots, yx(t-h_k))$, where f has a regular potential V. Then we have:

Lemma 1 (comp. [2] or [4]). *There exists $\rho_1>0$ such that if $x \in C([0, T], R^n)$ is a solution of (**) for some y and $x(0)=x(t_0)$ for some $t_0 \in (0, T]$ then $\|x(t)\| < \rho_1$ for $t \in [0, t_0]$.*

Theorem 3. *Let us suppose that:*
(i) *$f: R\times R^n\times R^{n\cdot k}\to R^n$ is a continuous bounded and T-periodic with respect to first variable map.*
(ii) *there exists a regular potential V for f and* Ind $V\neq 0$. *Then there exists a T-periodic solution on R for the equation:*

$$x'(t)=f(t, x(t), x(t-h_1), \dots, x(t-h_k)).$$

Proof. Let us consider $K_r\subset C([-h, 0], R^n)$, where $r>\rho_1$ (comp. Lemma 1). It is easy to see that $0\notin (I-\Phi_T)(\partial K_r)$. Now we will prove that Deg $(I-\Phi_T, K_r)\neq 0$ for some $r>\rho_1$. We will do it in three steps:

1. We claim that: Deg $(I-\Phi_T, K_r)=$ Deg $(I-\widehat{\Phi}_T, K_r)$, where $\widehat{\Phi}_T$ is Poincare operator for the problem $x'(t)=f(t, x(t), \dots, x(t))$ defined analogously like Φ_T, i. e. $\widehat{\Phi}_T=\widehat{S}\cdot\widehat{\varphi}$, where $\widehat{\varphi}(y)=\{x\in C([0, T], R^n):\ x(0)=y(0)$ and $x'(t)=f(t, x(t), \dots, x(t))$ for $t\in[0, T]\}$ $\widehat{\varphi}: C([-h, 0], R^n)\to C([0, T], R^n)$ (comp. Remark 1). For the proof of the above equality it is sufficient to construct a pseudoacyclic homotopy between Φ_T and $\widehat{\Phi}_T$. Let Φ_T^λ denotes Poincare's operator for problem: $x'(t)=(1-\lambda)\cdot f(t, x(t), yx(t-h_1), \dots, yx(t-h_k))+\lambda f(t, x(t) \dots, x(t))$. Of course, $\Phi_T^0=\Phi_T$ and $\Phi_T^1=\widehat{\Phi}_T$. Then the homotopy $H(\lambda, y)=\Phi_T^{\lambda}$ connects Φ_T and $\widehat{\Phi}_T$. It is well known (comp. [5]) that H is a pseudoacyclic map which satisfies assumptions of Proposition 2 $(0\notin(I-H)([0, 1]\times\partial K_r)$ from Lemma 1). So we get Deg $(I-\Phi_T, K_r)=$ Deg $(I-\widehat{\Phi}_T, K_r)$.

2. Let $e: C([-h, 0], R^n)\to C([-h, 0], R^n)$ be given by the formula $e(y)(t)=y(0)$ and $\psi=e\circ\widehat{\Phi}_T$. We will prove that Deg$(I-\widehat{\Phi}_T, K_r)=$ Deg$(I-\psi, K_r)$ From the definition we have $\widehat{\Phi}_T=S\circ\widehat{\varphi}$ and $\psi=e\circ S\circ\widehat{\varphi}$. We construct a homotopy between S and $e\circ S$:

$$R(\lambda, y)=(1-\lambda)S(y)+e\circ S(y).$$

Let $r_\lambda(y)=R(\lambda, y)$. From Lemma 1 it follows that $0\notin(I-r_\lambda\circ\widehat{\varphi})(\partial K_r)$ and from Proposition 3 we get the above equality.

3. Let E^n be a space of constant functions on $[-h, 0]$ into R^n. We have a natural isometry $P: E^n\to R_n$ and $\psi/_{E^n}=P^{-1}\circ\check{\Phi}_T\circ P$. From Propositions 4,5 it follows, that

$$\mathrm{Deg}(I-\psi, K_r)=\mathrm{Deg}\,(I-\check{\Phi}_T, B_r).$$

From Theorem 2 we get that if $r>\max\{\rho_1, \tau_1\}$ then Deg$(I-\check{\Phi}_T, K_r)\neq 0$ Hence, using Theorem 1 we get, that the equation $x'(t)=f(t, x(t), x(t-h_1) \dots, x(t-h_k))$ has a T-periodic solution. The proof of theorem 3 is completed

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Received 17. VI. 1982

Institute of Mathematics, University of Gdansk
Gdansk Poland