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DIRECT AND CONVERSE THEOREMS FOR APPROXIMATION OF CURVES BY POLYGONS

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1. Let Γ be a curve in the plane. Then we denote by $\Gamma(t)$ the set of all parametric representations of Γ with respect to the interval $[0, 1]$, i. e. $\Gamma(t)$ consists of all couples (φ, ψ) of functions $\varphi(t), \psi(t)$ such that $\Gamma=\{(x, y)\in R^2; x=\varphi(t), y=\psi(t), t\in[0, 1]\}$.

We call a curve Γ bounded if there exists a couple (φ^*, ψ^*) of bounded functions $\varphi^*(t), \psi^*(t)$ in $\Gamma(t)$, and we call Γ continuous if there is a couple (φ^*, ψ^*) of continuous functions in $\Gamma(t)$.

In what follows we consider the approximation of continuous curves by polygonal lines. To be precise we introduce the set S_n^k of splines of degree k with n knots, i. e. $s(t)\in S_n^k$ if $s(t)$ has a $(k-1)$-th derivative in $[0, 1]$ and if there exist points $0=t_0<t_1<\cdots<t_n=1$ such that $s(t)$ restricted to (t_{i-1}, t_i) is an algebraic polynomial of degree $k\,(i=1, \ldots, n)$. The case $k=1$ represents the polygonal lines.

Definition 1. *We call the number*

$$\varepsilon_n(\Gamma):=\inf_{(\varphi,\psi)\in\Gamma(t)}\ \inf_{(f,g)\in S_n^1}\max\{\|\varphi-f\|, \|\psi-g\|\}$$

the best parametric approximation of the curve Γ by polygonal lines of order n, where e. g. $\|\varphi-f\|:=\sup_{0\leq t\leq 1}|\varphi(t)-f(t)|$.

There are other possibilities to define a distance between a curve and a polygonal line. e. g. we could replace $\|\cdot\|$ by some L_p-norm. Still another possibility is the Hausdorff-distance.

Definition 2. *Let Γ and θ be curves. Then for $\alpha>0$*

$$\Gamma^\alpha:=\{(x, y)\in R^2:(x-\xi)^2+(y-\eta)^2\leq\alpha^2, (\xi, \eta)\in\Gamma\}$$

is called an α-neighborhood for Γ. Defining the same for θ we call $r(\Gamma, \theta):=\inf\{\alpha:\Gamma\subset\theta^\alpha, \theta\subset\Gamma^\alpha\}$. the Hausdorff-distance between Γ and θ.

Definition 3. We call the number

$$\varepsilon_n(\Gamma)^r:=\inf_{f,g\in S_n^1} r(\Gamma, (f, g))$$

the best Hausdorff-approximation of Γ by polygonal lines of order n.

There are many papers (see [1—4]) which study the best parametric or Hausdorff approximation of curves by polynomial curves, splines curves, etc. (Definitions 1-3 can be found in [1]). The main results in these papers are upper estimates of the rate of these approximations, called direct theorems. They are established under additional assumptions on the "smoothness" of the curve Γ. Converse theorems, i. e. results giving information on the curve by the rate of the above defined best approximations, are so far not known. The difficulty of their proof lies, on the one hand, in the fact that one has to take into account the various parametrizations of a curve, and, on the other hand, in the nature of the Hausdorff-distance.

The main result in this paper is the following converse theorem:

Theorem 1: *If Γ is a continuous curve satisfying*

$$\sum_{n=1}^{\infty} \varepsilon_n(\Gamma) < \infty \tag{1}$$

then Γ has a finite length.

In order to define the length of a curve it is useful to introduce (cf. [6; 7])

Definition 4. *If f is a bounded function in $[0, 1]$ we define for $n=1, 2, \ldots$*

$$\varkappa(f; n) := \sup_{0 \le x_0 \le \ldots \le x_n \le 1} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \tag{2}$$

and if Γ is a bounded curve with bounded $(\varphi, \psi) \in \Gamma(t)$ we set

$$\varkappa(\Gamma; n) := \sup_{0 \le t_0 \le \ldots \le t_n \le 1} \max \{ \sum_{i=1}^{n} | \varphi(t_i) - \varphi(t_{i-1}) |, \sum_{i=1}^{n} | \psi(t_i) - \psi(t_{i-1}) | \}. \tag{3}$$

We call $\varkappa(f; n)$ and $\varkappa(\Gamma; n)$ the modulus of length of f and Γ, respectively.

With the help of Definition 4, it is easy to see that

$$V(f) := \lim_{n\to\infty} \varkappa(f; n) \tag{4}$$

is the variation of f in $[0, 1]$ and that

$$l(\Gamma) := \lim_{n\to\infty} \varkappa(\Gamma; n) \tag{5}$$

is equivalent to the usual Euclidean length of a curve up to factor at most 2.

From these definitions it is also easy to see that for a bounded curve Γ with bounded (φ, ψ) representation we have

$$\varkappa(\Gamma; n) = \max \{\varkappa(\varphi; n), \varkappa(\psi; n)\} \tag{6}$$

and consequently

$$l(\Gamma) = \max \{V(\varphi), V(\psi)\}. \tag{7}$$

The proof of Theorem 1 will be given in the next section. In Section 3 we then comment on the sharpness of this theorem. By a theorem of Korneichuk (see [5]) it is easily establish as a counterpart the following direct theorem [4]:

Theorem 2. *If Γ is a continuous curve with bounded length then*

$$\varepsilon_n(\Gamma) = o(1/n), \quad n \to \infty. \tag{8}$$

However this so-called inverse theorem does not quite match up with Theorem 1. But on the other hand, we will show by an example in Section 3 that

$$\varepsilon_n(\Gamma) = O(\lambda_n), \quad n \to \infty, \tag{9}$$

for a positive sequence $\{\lambda_n\}$ with $\lim_{n\to\infty} n\lambda_n=0$ does not imply bounded length of Γ if λ_n is not sufficiently decreasing in n in the sense that

$$\sum_{n=1}^{\infty} \lambda_n = \infty. \tag{10}$$

Thus Theorem 1 is sharp in the sense that assumption (1) cannot be replaced in the form (9) with λ_n satisfying (10).

Secondly, we show by the same example that Theorem 1 cannot be strengthened in the sense that $\varepsilon_n(\Gamma)$ is replaced by the best Hausdorff-approximation $\varepsilon_n(\Gamma)'$ of Definition 3. In this respect we remark that (see [1]) $\varepsilon_n(\Gamma)'\leq\varepsilon_n(\Gamma)$ and that consequently Theorem 2 holds for the best Hausdorff-approximation of Γ.

2. For the proof of Theorem 1 we need some preliminary results.

Lemma 1. *For $s\in S_n^1$ there holds*

$$V(s)\leq n\,\|s\|. \tag{11}$$

This follows directly from (2) and the definition of S_n^1.

Theorem 3. *For $f\in V\cup C[0,1]$ we have*

$$\varkappa(f;n)\leq 12\{\|f\|+\sum_{k=1}^{n} E_k(f)\}, \tag{12}$$

where

$$E_k(f):=\inf_{s\in S_k^1}\|f-s\|. \tag{13}$$

This result follows from some known inverse theorems but for the sake of completeness we include its proof here.

Proof. We introduce $s_n\in S_n^1$ by $E_n(f)=\|f-s_n\|$ for $n=1,2,\ldots$. Setting $\sigma_l=s_{2^l}-s_{2^{l-1}}$ for $l=0,1,2,\ldots$, where $s_{1/2}:=0$, we obtain

$$\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|\leq\sum_{i=1}^{n}|f(x_i)-s_{2^m}(x_i)|+|f(x_{i-1})-s_{2^m}(x_{i-1})|$$
$$+\sum_{i=1}^{n}\sum_{l=0}^{m}|\sigma_l(x_i)-\sigma_l(x_{i-1})|.$$

But since the function σ_l belongs to $S^1_{2^{l+1}}$ it follows from Lemma 1 that

$$\sum_{i=1}^{n}|\sigma_l(x_i)-\sigma_l(x_{i-1})|\leq V(\sigma_l)\leq 2^{l+1}\|\sigma_l\|\leq 2^{l+1}(\|f-s_{2^l}\|+\|f-s_{2^{l-1}}\|).$$

Furthermore we have trivially

$$\sum_{i=1}^{n}\{|f(x_i)-s_{2^m}(x_i)|+|f(x_{i-1})-s_{2^m}(x_{i-1})|\}\leq 2nE_{2^m}(f)$$

so that with $E_{1/2}(f):=\|f\|$

$$\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|\leq 2nE_{2^m}(f)+\sum_{l=0}^{m}2^{l+1}[E_{2^l}(f)+E_{2^{l-1}}(f)]. \tag{14}$$

Using for $l \geqq 2$

$$2^{l-2}E_{2^{l-1}}(f) \leqq \sum_{k=2^{l-2}+1}^{2^{l-1}} E_k(f) \tag{15}$$

we derive from (14) that

$$\varkappa(f;n) \leqq 2nE_{2^m}(f) + 4\sum_{k=2}^{2^m} E_k(f) + 8\sum_{k=2}^{2^{m-1}} E_k(f) + 6E_1(f) + 2\|f\|. \tag{16}$$

Now we choose m such that $2^m \leqq n < 2^{m+1}$. Then observing $2n \leqq 8 \, . \, 2^{m-1}$ and applying (15) for $l = m+1$ the theorem follows from (16).

A further basic idea for the proof of Theorem 1 is the use of the ν_k-modulus introduced by V. Popov [8].

Definition 5. *For every function f on* $[0, 1]$, $\delta > 0$ *and* $k = 1, 2, \ldots,$ *we define*

$$\nu_k(f;\delta) := \inf_{\mu \in V} \sup_{x, x+kh \in [0,1]} \{ |\Delta_h^k f(x)| : |\mu(x+kh) - \mu(x)| \leqq \delta \}, \tag{17}$$

where $\Delta_h^k f(x)$ denotes the k-th difference of f in $[0, 1]$ *with increment h, and where V is the set of all functions μ with bounded variation in* $[0, 1]$.

The following properties of the modulus ν_k established in [8] will be used later on:

i) the inf in (17) can be taken only over all non-decreasing functions in V, and only over continuous functions in V if f is continuous,

ii) $\nu_k(f;\delta)$ is a non-decreasing function of δ,

iii) $\nu_k(f;\delta) \leqq \omega_k(f;\delta)$,

where $\omega_k(f;\delta)$ is the k-th modulus of continuity defined by

$$\omega_k(f;\delta) := \sup_{x, x+kh \in [0,1]} \{ |\Delta_h^k f(x)| : |h| \leqq \delta \}.$$

The following result proved in [8] will be of primary importance for us (we need only the case $k = 1$):

Theorem 4. *Let $f \in C[0, 1]$ and $k \geqq 1$ be fixed. There exists a constant N_k depending only on k, such that for any integer n and $k \geqq 1$:*

$$2^{-k-1}\nu_{k+1}(f;1/n) \leqq E_n^k(f) \leqq N_k \nu_{k+1}(f;(k+1)/n),$$

$$\textit{where } E_n^k(f) := \inf_{s \in S_n^k} \|f - s\|.$$

We need also the following property of the modulus ν_k:

Lemma 2. *Let $\varphi \in C[0, 1]$ and let p be a monotone increasing function satisfying $p(0) = 0$, $p(1) = 1$. Then, setting $\psi(x) := \varphi(p(x))$,*

$$\nu_k(\psi;\delta) = \nu_k(\varphi;\delta) \tag{18}$$

holds.

Proof. We consider only the case $k = 2$, the general case follows along the same pattern. By definition we have

$$\nu_2(\psi;\delta) = \inf_{\mu \in V} \sup_{0 \leqq x \leqq 1} \sup_{|\mu(x+2h) - \mu(x)| \leqq \delta} |\varphi(p(x+2h)) - 2\varphi(p(x+h)) + \varphi(p(x))|. \tag{19}$$

Now, assuming $h > 0$ (the case $h < 0$ is treated in the same way), we set $t := p(x)$, $p(x+h) := t + h_1$, $p(x+2h) := t + h_1 + h_2$ so that $h_1 > 0$, $h_2 > 0$.

Setting $\eta(t):=\mu(p^{-1}(t))$, $t\in[0,1]$, where p^{-1} is the inverse function of p, we see that the mapping $\mu\in V\to\eta\in V$ is one to one with $\mu(x)=\eta(p(x))$. Hence, it follows from (19) that

$$(20)\qquad \nu_2(\psi;\delta)=\inf_{\eta\in V}\sup_{0\leq t\leq 1}\sup_{|\eta(t+h_1+h_2)-\eta(t)|\leq\delta}|\varphi(t+h_1+h_2)-2\varphi(t+h_1)+\varphi(t)|.$$

But since for fixed $t\in[0,1]$ and $\eta\in V$

$$\sup_{|\eta(t+h_1+h_2)-\eta(t)|\leq\delta}|\varphi(t+h_1+h_2)-2\varphi(t+h_1)+\varphi(t)|$$
$$\geq\sup_{|\eta(t+h+h)-\eta(t)|\leq\delta}|\varphi(t+h+h)-2\varphi(t+h)+\varphi(t)|$$

it follows from (20) that

$$(21)\qquad \nu_2(\psi;\delta)\geq\nu_2(\varphi;\delta).$$

Now we can reverse the roles of φ and ψ (as well as of x and t) by starting from ψ and setting $\varphi(t):=\psi(p^{-1}(t))$ so that (21) follows with ψ replaced by φ and conversely.

This proves then the lemma for $k=2$.

Proof of Theorem 1. We actually prove a slightly stronger statement, namely

$$(22)\qquad \varkappa(\Gamma;n)\leq c\,[\sum_{i=1}^{n}\varepsilon_i(\Gamma)+\max\{\|f\|,\|g\|\}]$$

with a constant c not depending on n, where (f,g) is a parametric representation of Γ. By (5) the assertion of the theorem would then follow.

By Theorems 3 and 4 we get then the following estimates

$$(23)\qquad \varkappa(f;n)\leq 12\{\|f\|+\sum_{i=1}^{n}E_i(f)\}\leq 12\{\|f\|+N_1\sum_{i=1}^{n}\nu_2(f;2/i)\},$$

$$(24)\qquad \varkappa(g;n)\leq 12\{\|g\|+\sum_{i=1}^{n}E_i(g)\}\leq 12\{\|g\|+N_1\sum_{i=1}^{n}\nu_2(g;2/i)\}.$$

On the other hand, we obtain from Definition 1 of $\varepsilon_i(\Gamma)$ and Theorem 4 that there exist $(f_i,g_i)\in\Gamma(t)$ such that

$$(25)\qquad \varepsilon_{[i/2]}(\Gamma)+\frac{\|f\|}{n}\geq E_{[i/2]}(f_i)\geq\frac{1}{4}\nu_2(f_i;\frac{1}{[i/2]})\geq\frac{1}{4}\nu_2(f_i;\frac{2}{i})$$

$$(26)\qquad \varepsilon_{[i/2]}(\Gamma)+\frac{\|g\|}{n}\geq E_{[i/2]}(g_i)\geq\frac{1}{4}\nu_2(g_i;\frac{1}{[i/2]})\geq\frac{1}{4}\nu_2(g_i;\frac{2}{i}).$$

In order to combine (23) with (25) (or (24) with (26), respectively) we will use Lemma 2. Before doing this we observe that the pairs (f,g) and (f_i,g_i) can be assumed as not having periodes (e. g. $(f,g)\in\Gamma(t)$ has a period when there exists a point $t_0\in(0,1)$ and a number $\tau>0$ such that $[t_0-\tau,t_0\in\tau]\subset[0,1]$ and $f(t_0+h)=f(t_0-h)$ or $g(t_0+h)=g(t_0-h)$ for any $h\in[0,\tau]$. Otherwise, if e. g. (f_i,g_i) would have a period, by an appropriate transformation of the parameter we could easily construct $(f_i^*,g_i^*)\in\Gamma(t)$ for which $E_i(f_i)\geq E_i(f_i^*)$, $E_i(g_i)\geq E_i(g_i^*)$.

If (f,g) and (f_i,g_i) do not have periods it is easy to see that there exist monotone increasing functions φ_i, ψ_i on $[0,1]$ with $\varphi_i(0)=\psi_i(0)=0$, $\varphi_i(1)=\psi_i(1)=1$ for $i=1,2,\dots$ and for which $f_i=f(\varphi_i)$, $g_i=g(\psi_i)$.

But by Lemma 2 it follows then that $\nu_2(f_i; 2/i) = \nu_2(f; 2/i)$, $\nu_2(g_i; 2/i) = \nu_2(g; 2/i)$ so that we can combine (23) with (25) and (24) with (26) yielding

$$\varkappa(f; n) \leq 12\{\|f\| + 4N_1 \sum_{i=1}^{n} \varepsilon_{[i/2]}(\Gamma) + 4N_1 \|f\|\},$$

$$\varkappa(g; n) \leq 12\{\|g\| + 4N_1 \sum_{i=1}^{n} \varepsilon_{[i/2]}(\Gamma) + 4N_1 \|g\|\}.$$

Observing

$$\sum_{i=1}^{n} \varepsilon_{[i/2]}(\Gamma) \leq 2 \sum_{i=1}^{n} \varepsilon_i(\Gamma)$$

these inequalities establish (22) with $c = 12(1+4N_1)$ by (6).

3. In this section we show by the example of a curve that Theorem 1 cannot be sharpened within two respects. Its assumption $\Sigma_{n=1}^{\infty} \varepsilon_n(\Gamma) < \infty$ cannot be replaced by $\varepsilon_n(\Gamma) = O(\lambda_n)$ where $\{\lambda_n\}$ is a non-increasing sequence of numbers satisfying $\lim_{n\to\infty} n\lambda_n = 0$, but not $\Sigma_{n=1}^{\infty} \lambda_n = \infty$, nor $\varepsilon_n(\Gamma)$ can be replaced by the (smaller) numbers $\varepsilon_n(\Gamma)'$ of best Hausdorff-approximation. This exhibits an essential difference between this kind of approximation and parametric approximation of curves.

Our example of a curve Γ_0 is simply the polygonal line connecting the points $P_k = (x_k, y_k)$, $k = 1, 2, \ldots$, where $x_k = 1 - 2^{-k}$, $y_k = (-1)^k \lambda_k$.

Obviously, the length of Γ_0 is larger than $2\Sigma_{k=2}^{\infty} \lambda_k = \infty$. On the other hand, Γ_0 is parametrizable in the form $(t, \varphi(t))$, $0 \leq t \leq 1$, where $\varphi(t)$ is a continuous piecewise linear function. As an approximating polygonal line we take the curve $(t, \varphi_n(t))$, where

$$\varphi_n(t) := \begin{cases} \varphi(t), & 0 \leq t \leq x_{n+1}, \\ \dfrac{y_{n+1}(1-t)}{1-x_{n+1}}, & x_{n+1} \leq t \leq 1. \end{cases}$$

Then certainly $\varphi_n(t)$ is of class S_n^1 since it has n knots $x_2, \ldots, x_{n+1}$ and

$$\|\varphi - \varphi_n\| = \lambda_{n+1} \leq \lambda_n \tag{27}$$

since the peaks of $\varphi - \varphi_n$ have height at most λ_{n+1}. Furthermore, we have

$$\varepsilon_n(\Gamma_0)' \leq 2^{-n} \tag{28}$$

since the Hausdorff-distance between the curves $(t, \varphi(t))$ and $(t, \varphi_n(t))$ is less than 2^{-n}. To see this we observe that starting on any point of $(t, \varphi_n(t))$, $t \geq x_{n+1}$, and passing along a line a parallel to the x-axis we hit a point of $(t, \varphi(t))$ within $t \geq x_n$, thus within a distance $\leq 2^{-n}$. Conversely, when starting from a point of $(t, \varphi(t))$ we reach a point of $(t, \varphi_n(t))$ within a distance $\leq 2^{-n}$. Now (27) and (28) show that Γ_0 has the desired properties.

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