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A SIMPLE PROOF OF KALANDIYA'S THEOREM IN APPROXIMATION THEORY

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A new proof, simpler than the original one, of Kalandiya's theorem in approximation theory (concerning the approximation of Hölder-continuous functions by polynomials and the Hölder-continuity of the corresponding difference) is proposed.

1. Kalandiya's Theorem. In 1955 A. I. Kalandiya proved (with the help of S. M. Nikol'skii) the following theorem in approximation theory for real Hölder-continuous functions $f(x)$ along $[-1,1]$ [1;2]:

Theorem. *Let a function $f(x)$ of the class H_α be given. Let for every natural n there be an algebraic polynomial $p_n(x)$, for which*

$$|f(x)-p_n(x)|\leq A_1 n^{-\alpha},\quad x\in[-1,1], \tag{1}$$

where A_1 is a constant. Then the following estimate is valid:

$$\max_{x_1,x_2\in[-1,1]}\frac{|r_n(x_2)-r_n(x_1)|}{|x_2-x_1|^\beta}\leq A_2 n^{-\alpha+2\beta}, \tag{2}$$

where

$$r_n(x)=f(x)-p_n(x), \tag{3}$$

β is any positive number such that $2\beta<\alpha$ and A_2 is a constant depending on α and β.

We recall that a function $f(x)$ belongs to the class H_α of Hölder-continuous functions, provided that

$$|f(x_2)-f(x_1)|\leq A_3|x_2-x_1|^\alpha. \tag{4}$$

(Here and in the sequel A_i denote positive constants.) We also recall that (1), that is (because of (3)),

$$|r_n(x)|\leq A_1 n^{-\alpha},\quad x\in[-1,1], \tag{5}$$

for $f\in H_\alpha$ holds always true for some appropriate set of polynomials $p_n(x)$ This is a well-known result of approximation theory [1-3].

The aforementioned Kalandiya's theorem has been widely used not only by Kalandiya [1;2], but also by many other researchers (see, e. g., [4-9]) for the proof of the convergence of quadrature rules for Cauchy type principal value integrals and of the quadrature method for the corresponding singular integral equations. Because of the importance of this theorem and its great applicability, we feel it worth-while to give a second proof of it, in our opinion much simpler that the original one [1;2]. This proof is based on the method of thinking used by Šeško (Sheshko) [10] and Makovoz and Šeško [11] for the estimation of the error in quadrature formulas for Cauchy type principal value integrals.

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2. Proof of the Theorem. By taking into account that

$$|p_n(x_2)-p_n(x_1)|\leq|f(x_2)-p_n(x_2)|+|f(x_1)-p_n(x_1)|$$
$$+|f(x_2)-f(x_1)|,\ x_1,x_2\in[-1,1],$$

for the polynomial $p_n(x)$ approximating $f(x)$ in Kalandiya's theorem, as well as (1), (4) and the result of Stečkin [12] for the derivative of a polynomial $p_n(x)$

$$|p_n'(x)|\leq\frac{n^2}{2}\,\omega\left(2\sin\frac{\pi}{2n},\ p_n\right),\ x\in[-1,1],$$

(where ω denotes a modulus of continuity), we find that

$$|p_n'(x)|\leq A_4n^{2-\alpha},\ x\in[-1,1]. \tag{6}$$

Next, we consider two cases for the difference x_2-x_1 (of course, assuming always that $x_1, x_2\in[-1,1]$):

(i) $|x_2-x_1|\geq\delta_n$, where δ_n is an arbitrary small positive quantity depending on n, which is assumed to be of the form [10;11]

$$\delta_n=n^{-\gamma},\ \gamma>0. \tag{7}$$

(The exponent γ will be determined below.) Then, on the basis of (5), we find that

$$\frac{|r_n(x_2)-r_n(x_1)|}{|x_2-x_1|^\beta}\leq 2A_1n^{\gamma\beta-\alpha},\ x_1,x_2\in[-1,1],\ |x_2-x_1|\geq\delta_n. \tag{8}$$

(ii) $|x_2-x_1|<\delta_n$. In this case, since (see (3))

$$\frac{|r_n(x_2)-r_n(x_1)|}{|x_2-x_1|^\beta}\leq\frac{|f(x_2)-f(x_1)|}{|x_2-x_1|^\beta}+\left|\frac{p_n(x_2)-p_n(x_1)}{x_2-x_1}\right||x_2-x_1|^{1-\beta},$$

taking into account (4), the mean value theorem:

$$\frac{p_n(x_2)-p_n(x_1)}{x_2-x_1}=p_n'(\xi),\ \xi\in(x_1,x_2),$$

as well as (6) and (7), we find that

$$\frac{|r_n(x_2)-r_n(x_1)|}{|x_2-x_1|^\beta}\leq A_3n^{\gamma(\beta-\alpha)}+A_4n^{2-\alpha+\gamma(\beta-1)},\ x_1,x_2\in[1,1],\ |x_2-x_1|<\delta_n. \tag{9}$$

By comparing (8) and (9), we observe directly that the best selection for the exponent γ in (7) is $\gamma=2$. For this selection of γ, Kalandiya's theorem, (2), follows from (8) and (9).□

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