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ON THE LOCAL PROPERTIES OF BELLMAN'S FUNCTION FOR NONLINEAR TIME-OPTIMAL CONTROL PROBLEMS

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The paper studies the minimal time as a function of the initial state of the system for nonlinear time-optimal control problems. It is proved that for systems with controllable linear part this function satisfies the Hoelder condition and estimations of the order are obtained.

1. Introduction. A number of papers have been devoted to the properties of the minimal time as a function of the initial state of the system for time-optimal control problems. Continuity, Lipschitzianity and differentiability of this function are studied for various classes of control systems, cf. e. g. [1—7]. The properties of the Bellman's function provide a basis for a motivation of the dynamic programming for time-optimal control problems. Moreover, the local growth of this function is closely related to the sensitivity of the optimal time with respect to perturbations in the initial state as well as in other systems parameters.

The necessary conditions for Lipschitzianity of the Bellman's function are rather restrictive even in the linear case, since the number of inputs must be no less than the state dimension. The paper [8] shows that for linear systems the optimal time is a Hoelder function of the initial state. In [9; 10] two-dimensional control systems are investigated and estimates of the Hoelder degree are obtained. The paper [11] states a hypothesis that the Bellman's function for analytic local controllable systems with bang-bang controls satisfies the Hoelder condition. This is proved in case of symmetric systems.

In this paper we study the Bellman's function for systems described by the equation

$$\dot{x}=f(x, u), \tag{1}$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^r$ is the control. The initial time $t=0$ is fixed and the target is the origin in $\mathbf{R}^n$. For given $T>0$ the set of feasible controls on the interval $[0, T]$ is

$$\mathcal{U}(T)=\{u(\cdot)\in L_\infty^r(0, T);\quad u(t)\in U\subset \mathbf{R}^r,\ t\in(0, T)\}. \tag{2}$$

By $T(x_0)$ we denote the minimal time for the initial state x_0 (if such exists). The local growth of the function $T(\cdot)$ is studied on the assumption that the system (1) has controllable linear part. It is proved that $T(\cdot)$ is a Hoelder function and estimates of the degree are obtained. As an auxiliary result which strengthens the corresponding result in [8], we give a precise description of the local alteration of the attainable set for linear systems. Examples illustrating the evaluation of the Hoelder degree for nonlinear systems are discussed.

2. On the attainable set for linear systems. Consider a control system described by the linear equation

(3) $$\dot{x}=Ax+Bu, \quad x(0)=0$$

in the set of controls given by (2). Denote by $G(T)$ the attainable set on the interval $[0, T]$ for the system (3), i. e.

$$G(T)=\{\int_0^T \exp(A(T-s))Bu(s)ds\,;\ u(\cdot)\in\mathcal{U}(T)\}.$$

We shall study this set for small but positive T. Let us assume that

A1. The set U is compact and contains the origin in its interior.

A2. The pair (A, B) is controllable, i. e.

(4) $$\operatorname{rank}[B, AB, \ldots, A^{k-1}B]=n,$$

for some $k\leq n$.

Denote by σ the index of controllability of the pair (A, B) which is the smallest integer k such that (4) holds [8]. Let $M_i=\operatorname{Im}A^{i-1}B$ and let $L(M)$ be the linear span of the set $M\subset\mathbb{R}^n$. For a monotone non-decreasing function $\varphi(\cdot):(0, +\infty)\rightarrow(0, 1]$ we denote

$$\mathcal{U}_\varphi(T)=\{u(\cdot)\in\mathcal{U}(T)\,;\ u(t)\in\varphi(T)U,\ t\in[0, T]\}$$

and

$$G_\varphi(T)=\{\int_0^T \exp(A(T-s))Bu(s)ds\,;\ u(\cdot)\in\mathcal{U}_\varphi(T)\}.$$

Obviously $G_\varphi(T)=G(T)$ for $\varphi(\cdot)=1$.

Lemma 1. *Let the conditions A1 and A2 hold. For any unit vector* $l\in M_p\setminus L(\cup_{i=1}^{p-1}M_i)$ *there exist constants* $c_1>0$, c_1' *and* $\varepsilon_1>0$ *such that the relations* $c_1\varepsilon^p\varphi(\varepsilon)\exp(A\varepsilon)l\in G_\varphi(\varepsilon)$, $c_1'\varepsilon^p\varphi(\varepsilon)\exp(A\varepsilon)l\notin G_\varphi(\varepsilon)$ *hold for all* $\varepsilon\in(0, \varepsilon_1)$. *Moreover, the constant* c_1 *does not depend on the vector* l.

Proof. Let $l\in M_p$. Then there exists a vector $w\in\mathbb{R}^r$ such that $l=A^{p-1}Bw$. Define the vectors $v_1, \ldots, v_{p-1}, v_{p+1}, \ldots, v_n$ as the zero vector and $v_p=c\varepsilon^p\varphi(\varepsilon)w/2$. It is proved in [8] that if $c>0$ and $\bar{\varepsilon}>0$ are sufficiently small and $\varepsilon\in(0, \bar{\varepsilon})$, then there exists $u(\cdot)\in\mathcal{U}_{\varphi/2}(\varepsilon)$ such that $\int_0^\varepsilon(-t)^{i-1}u(t)dt=v_i$, $i=1, \ldots, n$. Hence, denoting by $x(\cdot)$ the corresponding solution of (3) we have

(5) $$x(\varepsilon)=\int_0^\varepsilon \exp(A(\varepsilon-t))Bu(t)dt=\exp(A\varepsilon)\sum_{i=1}^{+\infty} A^{i-1}B\int_0^\varepsilon\frac{(-t)^{i-1}}{(i-1)!}u(t)dt$$

$$=\exp(A\varepsilon)(\frac{c}{(p-1)!}\varepsilon^p\varphi(\varepsilon)l/2+\psi_0(\varepsilon)),$$

where $|\psi_0(\varepsilon)|=0(\varepsilon_n\varphi(\varepsilon))$. According to the definition of σ, there exist integers $s_1, \ldots, s_n\in\{1, \ldots, \sigma\}$ and unit vectors $w_1, \ldots, w_n\in\mathbb{R}^r$ such that the vectors $A^{s_1-1}Bw_1, \ldots, A^{s_n-1}Bw_n$ are linearly independent. As above we choose controls $u_i(\cdot)\in\mathcal{U}_\varphi(\varepsilon)$ such that for the corresponding solutions $x_i(\cdot)$ of the equation (3) the following equality holds

(6) $$x_i(\varepsilon)=\exp(A\varepsilon)\Big(\frac{c}{(s_i-1)!}\varepsilon^{s_i}\varphi(\varepsilon)A^{s_i-1}Bw_i+\psi_i(\varepsilon)\Big),$$

where $|\psi_i(\varepsilon)|=0(\varepsilon^n\varphi(\varepsilon))$. The vector $\psi_i(\varepsilon)$ can be represented in the form

$$\psi_i(\varepsilon)=\sum_{j=1}^{n}\beta_j^i(\varepsilon)A^{s_j-1}Bw_j,\qquad i=0,\ldots,n.$$

Then $\beta_j^i(\varepsilon)=0(\varepsilon^n\varphi(\varepsilon))$. Hence, for all sufficiently small $\varepsilon>0$ we can choose the numbers $\alpha_i(\varepsilon)$, $i=1,\ldots,n$ so that $\lim_{\varepsilon\to 0}\alpha_i(\varepsilon)=0$ and

(7) $$\frac{c}{(s_j-1)!}\varepsilon^{s_j}\varphi(\varepsilon)\alpha_j(\varepsilon)+\sum_{i=1}^{n}\beta_j^i(\varepsilon)\alpha_i(\varepsilon)=-\beta_j^0(\varepsilon),\quad j=1,\ldots,n.$$

Define the control

(8) $$\bar{u}(\cdot)=u(\cdot)+\sum_{i=1}^{n}\alpha_i(\varepsilon)u_i(\cdot).$$

Since $u(\cdot)\in\mathcal{U}_{\varphi/2}(\varepsilon)$, we have $\bar{u}(\cdot)\in\mathcal{U}_\varphi(\varepsilon)$ for all sufficiently small $\varepsilon>0$. From (5), (6), (7) and (8) it follows that

$$\bar{x}(\varepsilon)=\exp(A\varepsilon)\frac{c}{2(p-1)!}\varepsilon^p\varphi(\varepsilon)l\in G_\varphi(\varepsilon),$$

where $\bar{x}(\cdot)$ is the solution of (3) resulting from $\bar{u}(\cdot)$.

Now, let $l\in M_p\setminus L(\cup_{i=1}^{p-1}M_i)$. One can find linearly independent vectors $l_1,\ldots,l_n\in\mathbf{R}^n$, such that if $r_1=\operatorname{rank}B$, $r_2=\operatorname{rank}[B,AB],\ldots,r_\sigma=\operatorname{rank}[B,AB,\ldots,A^{\sigma-1}B]=n$, then $l_1,\ldots,l_{r_1}\in M_1$, $l_{r_1+1},\ldots,l_{r_2}\in M_2,\ldots,l_{r_{\sigma-1}+1},\ldots,l_{r_\sigma}\in M_\sigma$ and $l_{r_p}=l$. For an arbitrary control $u(\cdot)\in\mathcal{U}_\varphi(\varepsilon)$ and for the corresponding trajectory $x(\cdot)$ we have

$$\exp(-A\varepsilon)x(\varepsilon)=\sum_{i=1}^{n}A^{i-1}B\int_0^\varepsilon\frac{(-s)^{i-1}}{(i-1)!}u(s)ds+\psi(s)=\sum_{i=1}^{n}\beta_i(\varepsilon)l_i,$$

where $\psi(\varepsilon)=0(\varepsilon^n\varphi(\varepsilon))$ uniformly in $u(\cdot)$. Taking into account that $l_{r_p}=l\notin L(\cup_{i=1}^{p-1}M_i)$ we obtain that $\beta_{r_p}(\varepsilon)\leq c_1'\varepsilon^p\varphi(\varepsilon)$ for an appropriate constant c_1' and for all sufficiently small $\varepsilon>0$. The proof is complete.

3. On the attainable set for nonlinear systems. Consider a control system described by the equation

(9) $$\dot{x}=Ax+Bu+g(x,u),\quad x(0)=0.$$

Denote by $G_\varphi^g(T)$ the attainable set for the system (9) resulting from the set of admissible controls $\mathcal{U}_\varphi(T)$. Assume that

A3. The function g is defined for $|x|<R+1$ $(R>0)$ and $u\in U$, it is continuous and there exists a constant L such that $|g(x',u)-g(x'',u)|\leq L|x'-x''|$ for all $|x'|,|x''|\leq R+1/2$, $u\in U$.

Applying Theorem iv. 3 in [12] and taking into account the compactness of U we obtain immediately that for any closed set $K\subset\{x\in\mathbf{R}^n;\ |x|<R\}$ there exists $\eta>0$ such that if $x_0\in K$, $u(\cdot):[-\tau,\tau]\to U$ is measurable, $\tau\in(0,\eta)$, then the equation (9) has a unique solution $x(\cdot)$, $|x(t)|\leq R$, on the interval $[-\tau, \tau]$, which satisfies the condition $x(0)=x_0$.

We shall use the vectors $l_1, \ldots, l_n$ which have been introduced in the proof of Lemma 1. Let $s_i=r_i-r_{i-1}$, $i=1, \ldots, \sigma$ $(r_0=0)$.

Lemma 2. *Let M be the matrix whose columns are $l_1, \ldots, l_n$. Then the matrices $M^{-1}AM$ and $M^{-1}B$ have the form*

$$M^{-1}AM=\begin{array}{r|ccccc|} s_1+s_2 & \cdots & & & & \\ s_3 & s_1 & \cdots & & & \\ & & s_2 & & & \\ & & 0 & \cdots & & \\ s_{\sigma-1} & & & & \cdots & \\ s_\sigma & & & & s_{\sigma-2} & \cdots \\ & & & & & s_{\sigma-1}+s_\sigma \end{array}, \qquad M^{-1}B=\begin{bmatrix} D \\ 0 \end{bmatrix},$$

where D is a $s_1\times r$ matrix.

The proof uses standard arguments and therefore it is omitted.

We shall introduce another assumption which, as it is further shown, is essential for the obtained Bellman's function estimate. Denote by $j(i)$ the smallest integer j for which $i\leq r_j$.

A4. There exist $\alpha_0>0$, a monotone non-decreasing function $\varphi(\cdot):(0, \alpha_0]\to(0, 1]$ and a function $\mu(\cdot):[0, \alpha_0]\to(0, +\infty)$, $\lim_{\alpha\to 0}\mu(\alpha)=0$ such that the following condition holds: for any $Q>0$ there exists a number $N_g(Q)$ such that if $y\in \mathrm{R}^n$, $|y^i|\leq Q\alpha^{j(i)}\varphi(\alpha)$, $i=1, \ldots, n$, $\alpha\in(0, \alpha_0)$ and $u\in\varphi(\alpha)U$, then

$$|[M^{-1}g(My, u)]^i|\leq C_g(Q)\alpha^{j(i)-1}\varphi(\alpha)\mu(\alpha), \quad i=1, \ldots, n.$$

(We denote by $[z]^i=z^i$ the ith component of the vector z).

Theorem 1. *Let the assumptions A1 — A4 hold. Then there exist $\varepsilon_0>0$ and $c_0>0$ such that for every $\varepsilon\in(0, \varepsilon_0)$ $\{x\in\mathrm{R}^n;\ |x|\leq c_0\varepsilon^\sigma\varphi(\varepsilon)\}\subset G^g_\varphi(\varepsilon)$.*

Proof. According to Lemma 1 there exist constants $c_1>0$ and $\varepsilon_1>0$ such that the inclusion

$$\Omega(\varepsilon)=\{\sum_{i=1}^{n}(\alpha_i-\beta_i)c_1\varepsilon^{j(i)}\varphi(\varepsilon)l_i;\ \ \alpha_i\geq 0,\ \beta_i\geq 0,\ \sum_{i=1}^{n}(\alpha_i+\beta_i)=1\}\subset \mathrm{Int}\, G_\varphi(\varepsilon)$$

holds for all $\varepsilon\in(0, \varepsilon_1)$. Define in $\Omega(\varepsilon)$ the multivalued function $x\to\mathscr{V}_\varphi(\varepsilon; x)$, where $\mathscr{V}_\varphi(\varepsilon, x)$ is the set of the controls $u(\cdot)\in\mathscr{U}_\varphi(\varepsilon)$ which drive the state in the point x according to the equation (3). Obviously $\mathscr{V}_\varphi(\varepsilon, x)\neq\emptyset$ for $\varepsilon\in(0, \varepsilon_1)$ and $x\in\Omega(\varepsilon)$, it is a convex closed set in $L_2^r(0, \varepsilon)$ (without loss of generality we can assume that the set U is convex and, in view to A1, even that U is a ball with center — the origin in R^r). From Lemma 1 [13] we conclude that the mapping $x\to\mathscr{V}_\varphi(\varepsilon;\ x)$ is lower semicontinuous in $L_2^r(0, \varepsilon)$. Applying the theorem of Michael [14] we obtain that there exists a continuous in L_2^r selection $x\to u(\varepsilon; x; \cdot)\in\mathscr{V}_\varphi(\varepsilon; x)$, $u(\varepsilon, 0; \cdot)=0$. Define the mapping $\Xi_\varepsilon:\Omega(\varepsilon)\to\mathrm{R}^n$ in the following way: if $x(u(\cdot); \cdot)$ is the solution of (9) resulting from $u(\cdot)$ (note that for $\varepsilon<\eta$ it exists), then $\Xi_\varepsilon(x)=x(u(\varepsilon, x; \cdot); \varepsilon)$. We shall prove that the mapping $\Xi_\varepsilon(\cdot)$ is continuous. Let $x_k\in\Omega(\varepsilon)$, $\lim_{k\to+\infty}x_k=\bar{x}$. For $u_k(\cdot)=u(\varepsilon; x_k; \cdot)$ we have $\lim_{k\to+\infty}\|u_k(\cdot)-u(\varepsilon, \bar{x}; \cdot)\|_{L_2^r}=0$ and the control $u(\cdot)=u(\varepsilon, \bar{x}; \cdot)$ drives the system (3) to the point $\bar{x}$. If $\bar{x}_k(\cdot)$ and $\bar{x}(\cdot)$ are the solutions of (9) resulting from $u_k(\cdot)$ and $\bar{u}(\cdot)$, then by definition we have $|\Xi_\varepsilon(x_k)-\Xi_\varepsilon(\bar{x})|=|x_k(\varepsilon)-\bar{x}(\varepsilon)|$. Denote

$$D_k(x)=\int_0^\varepsilon |B(u_k(t)-\bar{u}(t))+g(x, u_k(t))-g(x, \bar{u}(t))|\,dt$$

$$\leq \delta_k+\int_0^\varepsilon |g(x, u_k(t))-g(x, \bar{u}(t))|\,dt,$$

where $\lim_{k\to+\infty}\delta_k=0$. Since $\lim_{k\to+\infty}\|v_k(\cdot)-\bar{u}(\cdot)\|_{L_2^r}=0$, $u_k(\cdot)$ and $\bar{u}(\cdot)$ are bounded uniformly in t and g is continuous, we have $\lim_{k\to+\infty}D_k(x)=0$ uniformly in x, $|x|\leq R+1/2$. From Theorem iv.4 in [12] we obtain that $\lim_{k\to+\infty}\|x_k(\cdot)-\bar{x}(\cdot)\|_c=0$ which proves the continuity of $\Xi_\varepsilon(\cdot)$.

Obviously $G_\varphi^g(\varepsilon)\supset\{\Xi_\varepsilon(x);\ x\in\Omega(\varepsilon)\}$. Let $x\in\partial\Omega(\varepsilon)$ (by ∂ we denote the boundary) and let $x(\cdot)$ and $\bar{x}(\cdot)$ be the solutions of (3) and (9), respectively-resulting from the control $u(\cdot)=u(\varepsilon, x; \cdot)$. Making in (3) and (9) the trans, formation $y=M^{-1}x$, $\bar{y}=M^{-1}\bar{x}$ we obtain

$$\dot{y}=M^{-1}AMy+M^{-1}Bu, \qquad y(0)=0 \tag{10}$$

$$\dot{\bar{y}}=M^{-1}AM\bar{y}+M^{-1}Bu+M^{-1}g(M\bar{y}, u), \ \bar{y}(0)=0. \tag{11}$$

We shall prove that there exist constants d_1 and $\bar{\varepsilon}>0$ such that the inequalities

$$|\bar{y}^i(\varepsilon)|\leq d_1\varepsilon^{j(i)}\varphi(\varepsilon), \qquad i=1,\ldots, n \tag{12}$$

hold for all $\varepsilon\in(0, \bar{\varepsilon})$. Define the operator $\mathscr{F}_\varepsilon : C^n[0, \varepsilon]\to C^n[0, \varepsilon]$ in the following way:

$$[\mathscr{F}_\varepsilon z(\cdot)(t)]^i=[\int_0^t (Az(s)+Bu(s)+M^{-1}g(Mz(s), u(s))ds]^i, \ i=1,\ldots, r_1,$$

$$[\mathscr{F}_\varepsilon z(\cdot)(t)]^i=[\int_0^t Az_1(s)+Bu(s)+M^{-1}g(Mz_1(s), u(s))ds]^i, \ i=r_1+1,\ldots, r_2,$$

where $[z_1(s)]^k=[\mathscr{F}_\varepsilon z(\cdot)(s)]^k$ for $k=1,\ldots, r_1$ and $[z_1(s)]^k=[z(s)]^k$ for $k=r_1+1, \ldots, n$. Let $[\mathscr{F}_\varepsilon z(\cdot)(t)]^i$ be already given for $i=1,\ldots, r_p$ $(p<\sigma)$ jointly with the corresponding $z_1(\cdot),\ldots, z_p(\cdot)$. Then we define

$$[\mathscr{F}_\varepsilon z(\cdot)(t)]^i=[\int_0^t Az_p(s)+Bu(s)+M^{-1}g(Mz_p(s), u(s))ds]^i, \ i=r_p+1,\ldots, r_{p+1}$$

and $[z_{p+1}(s)]^k=[z_p(s)]^k$ for $k=1,\ldots, r_p$, $[z_{p+1}(s)]^k=[\mathscr{F}_\varepsilon z(\cdot)(s)]^k$ for $k=r_p+1, \ldots, r_{p+1}$, $[z_{p+1}(s)]^k=[z(s)]^k$ for $k=r_{p+1}+1,\ldots, n$ (if $p+1<\sigma$).

Clearly, if $y_0(\cdot)$ is a fixed point for $\mathscr{F}_\varepsilon$, then $y_0(\cdot)$ is a solution of (11) and thus $y_0(\cdot)=\bar{y}(\cdot)$. Let us estimate the difference $\|\mathscr{F}_\varepsilon y'(\cdot)-\mathscr{F}_\varepsilon y''(\cdot)\|_C$. For $i=1,\ldots, r_1$ and for appropriate constants d_2 and d_3 we have

$$\|[\mathscr{F}_\varepsilon y'(\cdot)-\mathscr{F}_\varepsilon y''(\cdot)]^i\|_C\leq d_2\varepsilon\|y'(\cdot)-y''(\cdot)\|_C$$
$$+\varepsilon\|M^{-1}\|\,L\,\|M\|\,\|y'(\cdot)-y''(\cdot)\|_C\leq\varepsilon(d_2+d_3)\|y'(\cdot)-y''(\cdot)\|.$$

Applying this estimate in the definition of $\mathscr{F}_\varepsilon$ we obtain successively that for all sufficiently small $\varepsilon>0$, the operator $\mathscr{F}_\varepsilon$ is contractive.

Let $q=(q_1,\dots,q_\sigma)$ be a vector with positive components. Denote

$$\psi_q(\varepsilon)=\{z(\cdot)=(z^1(\cdot),\dots,z^n(\cdot))\in C^n[0,\varepsilon];\ \|z^i(\cdot)\|_C\leq q_{j(i)}\varepsilon^{j(i)}\varphi(\varepsilon)\}.$$

We shall prove that the vector q can be fixed so that

$$\mathscr{F}_\varepsilon\psi_q(\varepsilon)\subset\psi_q(\varepsilon) \tag{13}$$

for all sufficiently small $\varepsilon>0$. Let $1\leq i\leq r_1$ and let $z(\cdot)\in\psi_q(\varepsilon)$. From A4 we get $|[M^{-1}g(Mz(t),\ u(t))]^i|\leq C_g(Q)\varphi(\varepsilon)\mu(\varepsilon)$, where $Q=\max\{q_1,\dots,q_\sigma\}$, $t\in[0,\varepsilon]$. Applying Lemma 2 we estimate

$$|[\mathscr{F}_\varepsilon z(\cdot)(t)]^i|\leq\int_0^t(d_4Q\varepsilon\varphi(\varepsilon)+d_5\varphi(\varepsilon)+C_g(Q)\varphi(\varepsilon)\mu(\varepsilon))ds$$
$$\leq(d_4Q\varepsilon+d_5+C_g(Q)\mu(\varepsilon))\varepsilon\varphi(\varepsilon),$$

where d_4 and d_5 are appropriate constants. Let us fix $q_1=d_5+1$ and let $\bar\varepsilon_1=\bar\varepsilon_1(q_2,\dots,q_\sigma)$ be so small that $C_g(Q)\mu(\varepsilon)+\varepsilon d_4Q\leq1$ for $\varepsilon\in(0,\bar\varepsilon_1)$. Thus the function $z_1(\cdot)$ from the definition of $\mathscr{F}_\varepsilon z(\cdot)$ belongs to $\psi_q(\varepsilon)$. Assume that the numbers $q_1,\dots,q_p$ and $\bar\varepsilon_p=\bar\varepsilon_p(q_{p+1},\dots,q_\sigma)$ are already fixed so that $z_p(\cdot)\in\psi_q(\varepsilon)$ for $\varepsilon\in(0,\bar\varepsilon_p)$ and for arbitrary $q_{p+1},\dots,q_\sigma$. In view of Lemma 2 and A4 we obtain that for $r_p+1\leq i\leq r_{p+1}$

$$|[\mathscr{F}_\varepsilon z(\cdot)(t)]^i|\leq\int_0^t(d_6q_p\varepsilon^p\varphi(\varepsilon)+d_7Q\varepsilon^{p+1}\varphi(\varepsilon)+C_g(Q)\varepsilon^p\varphi(\varepsilon)\mu(\varepsilon))ds$$
$$\leq(d_6q_p+d_7Q\varepsilon+C_g(Q)\mu(\varepsilon))\varepsilon^{p+1}\varphi(\varepsilon).$$

Let us fix $q_{p+1}=d_6q_p+1$ and $\bar\varepsilon_{p+1}=\bar\varepsilon_{p+1}(q_{p+2},\dots,q_r)$ so that $d_7Q\varepsilon+C_g(Q)\mu(\varepsilon)<1$ for $\varepsilon\in(0,\bar\varepsilon_{p+1})$. Thus $z_{p+1}(\cdot)\in\psi_q(\varepsilon)$ for any selection of positive numbers $q_{p+2},\dots,q_\sigma$, if only $\varepsilon\in(0,\bar\varepsilon_{p+1})$ and $z(\cdot)\in\psi_q(\varepsilon)$. Keeping in mind that $z_\sigma(\cdot)=\mathscr{F}_\varepsilon z(\cdot)$ we obtain the inclusion (13).

From the Banach fixed point theorem and from the uniqueness of the solution of (9) it follows that $\bar y(\cdot)\in\psi_q(\varepsilon)$ for all sufficiently small $\varepsilon>0$, and hence the estimate (12) holds.

We shall estimate the components of the solution $y(\cdot)$ of (10). The vector $x\in\partial\Omega(\varepsilon)$ can be represented in the form

$$x=\sum_{i=1}^n(\alpha_i-\beta_i)c_1\varepsilon^{j(i)}\varphi(\varepsilon)l_i=M\sum_{i=1}^n(\alpha_i-\beta_i)c_1\varepsilon^{j(i)}\varphi(\varepsilon)l_i, \tag{14}$$

where $\alpha_i\geq0$, $\beta_i\geq0$, $\Sigma_{i=1}^n(\alpha_i+\beta_i)=1$, $e_i=M^{-1}l_i=(0,\dots,1,\dots,0)$ with 1 in ith place. Since $x\in\partial\Omega(\varepsilon)$, there exists i_0 such that $|\alpha_{i_0}-\beta_{i_0}|\geq1/n$. From (14) we get

$$|y^{i_0}(\varepsilon)|=|[M^{-1}x]^{i_0}|\geq\frac{c_1}{n}\varepsilon^{j(i_0)}\psi(\varepsilon). \tag{15}$$

Moreover, according to the proof of Lemma 1, there exists a constant d_8 such that

$$|y^i(t)|\leqq d_8 t^{j(i)}\varphi(t), \tag{16}$$

for all sufficiently small $t>0$. Denote $E=M^{-1}AM$, $F=M^{-1}B$. Then

$$\bar{y}(t)=\int_0^t \exp(E(t-s))Fu(s)ds+\int_0^t \exp(E(t-s))M^{-1}g(M\bar{y}(s), u(s))ds$$
$$=y(t)+\int_0^t \exp(E(t-s))M^{-1}g(M\bar{y}(s), u(s))ds.$$

Hence, applying A4 we obtain that for an appropriate constant d_9

$$|\bar{y}(t)-y(t)|\leqq d_9 t\varphi(t)\mu(t).$$

Taking into account Lemma 2, (12) and (16) we get that for $r_1+1\leqq i\leqq r_2$

$$|\dot{\bar{y}}^i(t)-\dot{y}^i(t)|\leqq d_{10}\sum_{j=1}^{r_1}|\bar{y}^j(t)-y^j(t)|+d_{11}t^2\varphi(t)+C_g(d_8)t\varphi(t)\mu(t)$$
$$\leqq(d_9 d_{10} r_1+C_g(d_8))t\varphi(t)\mu(t)+d_{11}t^2\varphi(t).$$

Hence for $\mu_1(t)=\max\{\mu(t), t\}$ we have

$$|\bar{y}^i(t)-y^i(t)|\leqq d_{12}t^2\varphi(t)\mu_1(t), \quad t\in(0, \varepsilon],$$

ε—sufficiently small. Continuing these arguments we obtain that there exist constants $\tilde{d}$ and $\bar{\varepsilon}>0$ and a function $\xi(\cdot)$, $\lim_{t\to 0}\xi(t)=0$, such that

$$|\bar{y}^i(t)-y^i(t)|\leqq\tilde{d}t^{j(i)}\varphi(t)\xi(t) \tag{17}$$

for all $t\in[0, \bar{\varepsilon}]$. Moreover, the constants $\tilde{d}$ and $\tilde{\varepsilon}$ can be chosen independently of x. From (15) and (17) we conclude that $|\bar{y}^{i_0}(\varepsilon)|\geqq c_1\varepsilon^{j(i_0)}\varphi(\varepsilon)/2n$ for all sufficiently small $\varepsilon>0$. Hence,

$$|M^{-1}\bar{x}(\varepsilon)|\geqq|[M^{-1}\bar{x}(\varepsilon)]^{i_0}|=|\bar{y}^{i_0}(\varepsilon)|\geqq c_1\varepsilon^{j(i_0)}\varphi(\varepsilon)/2n.$$

Consequently, there exists $c_0>0$ such that $|\bar{x}(\varepsilon)|\geqq c_0\varepsilon^{j(i_0)}\varphi(\varepsilon)\geqq c_0\varepsilon^{\sigma}\varphi(\varepsilon)$. Moreover, c_0 can be chosen independently of x. Thus for each $x\in\partial\Omega(\varepsilon)$ and $\varepsilon\in(0, \varepsilon_0)$ (ε_0 — sufficiently small) we have

$$|\Xi_\varepsilon(x)|\geqq c_0\varepsilon^\sigma\varphi(\varepsilon). \tag{18}$$

From A4 it follows that $g(0, 0)=0$, thus $\Xi_\varepsilon(0)=0$, which combined with the continuity of $\Xi_\varepsilon(\cdot)$ and (18) gives $\{x\in\mathbb{R}^n;\ |x|\leqq c_0\varepsilon^\sigma\varphi(\varepsilon)\}\subset G^g_\varphi(\varepsilon)$. The proof is complete.

4. The Bellman's function for nonlinear systems. Consider a control system described by the equation

$$\dot{x}=f(x, u), \tag{19}$$

$x\in\mathbb{R}^n$, $u\in\mathbb{R}^r$, in the set $\mathscr{U}(T)$ of feasible controls on $[0, T]$, defined by (2). Further we assume

A5. The function f is continuously differentiable for $|x|<R+1$ and u from a neighbourhood of U; $f(0, 0)=0$; the matrices $A=f_x(0, 0)$ and $B=f_u(0, 0)$ satisfy the condition A2 $f_x(\cdot, \cdot)$ and $f_u(\cdot, \cdot)$ are Hoelder functions in degree $1/m$ in a neighbourhood of $(0, 0)$.

Let $K\subset\{x\in \mathbf{R}^n;\ |x|<R+1\}$ be a closed set. Denote by $D_K(T)$ the set of all initial points x such that there exists a control $u(\cdot)\in \mathscr{U}(T)$ so that the equation (19) has a solution $x(\cdot)$ on $[0, T]$ $x(0)=x$, $x(T)=0$ and $x(t)\in K$ for all $t\in[0, T]$. Let $D_K=\bigcup_{T\geqq 0} D_K(T)$. Clearly the function $T_K(x)=\inf\{T;\ x\in D_K(T)\}$ is defined over D_K.

The equation (19) can be rewritten in the form of (9) with

$$g(x, u)=f(x, u)-f_x(0, 0)x-f_u(0, 0)u. \tag{20}$$

L e m m a 3. *Let the conditions A1 and A5 hold. Then for the function g in* (20) *the condition A4 is fulfilled with*

$$\varphi(\alpha)=\alpha^s, \quad s>m(\sigma-1). \tag{21}$$

P r o o f. For an appropriate $\theta=\theta(x, u)\in[0, 1]$ we have

$$g(x, u)=(f_x(\theta x, \theta u)-f_x(0, 0))x+(f_u(\theta x, \theta u)-f_u(0, 0))u.$$

Let $|y^i|\leqq Q\alpha^{j(i)}\varphi(\alpha)$ and $u\in\varphi(\alpha)U$. Then for $x=My$ we have $|x|\leqq d_1Q\alpha\varphi(\alpha)$ and

$$\begin{aligned}|[M^{-1}g(My, u)]^i| &\leqq d_2|g(x, u)|\leqq d_2H(|\theta x|+|\theta u|)^{1/m}(|x|+|u|)\\ &\leqq d_3(d_1Q\alpha\varphi(\alpha)+\varphi(\alpha))^{1/m}(d_1Q\alpha\varphi(\alpha)+\varphi(\alpha))\\ &\leqq d_4(Q+1)^{1/m}(\varphi(\alpha))^{1/m}(Q+1)\varphi(\alpha)\leqq d_4(Q+1)^{(m+1)/m}\alpha^{j(i)-1}\varphi(\alpha)\mu(\alpha),\end{aligned}$$

i. e. the condition A4 holds for $C_g(Q)=d_4(Q+1)^{(m+1)/m}$. Here H is the Hoelder constant, d_1, d_2, d_3 and d_4 are appropriate constants, $\lim_{\alpha\to 0}\mu(\alpha)=0$. The proof is complete.

Let us assume additionaly that

A6. The function f is defined in $\mathbf{R}^n\times\mathbf{R}^r$. For any compact set $M\subset\mathbf{R}^n$ and a number $T>0$ there exists a compact set $K\subset\mathbf{R}^n$ such that for each $u(\cdot)\in\mathscr{U}(T)$ and $x_0\in M$ the solution of (19) with initial condition x_0 at $t=0$ exists on $[0, T]$ and belongs to K.

The second part of this condition is fulfilled for instance if $|f(x, u)|\leqq c(1+|x|)$, $x\in\mathbf{R}^n$. If A6 holds, then denote $D(T)=D_{\mathbf{R}^n}(T)$ and $T(x)=T_{\mathbf{R}^n}(x)$ (for $x\in D=\bigcup_{T>0}D(T)$).

T h e o r e m 2. *Let the conditions A1 and A5 hold and K' and K'' be closed subsets such that $K'\subset\operatorname{Int}K''\subset\{x;\ |x|<R+1\}$. Then for any $T>0$ there exist constants c and $\delta>0$ such that if $x'\in D_{K'}$, $T_{K'}(x')\leqq T$ and $|x'-x''|\leqq\delta$, then $x''\in D_{K''}$ and*

$$T_{K''}(x'')-T_{K'}(x')\leqq\psi^{-1}(c|x'-x''|), \tag{22}$$

where $\psi(\alpha)=\alpha^\sigma\varphi(\alpha)$. Here $\varphi(\cdot)$ is an arbitrary function such that the condition A4 holds (for instance $\varphi(\alpha)=\alpha^{m\sigma}$). If, additionally, f satisfies the condition A6, then for any compact set $M\subset\mathbf{R}^n$ and $T>0$ there exists a constant c such that

$$|T''(x')-T(x'')|\leqq\psi^{-1}(c|x'-x''|), \tag{23}$$

for every x', $x''\in\bigcup_{s\in[0, T]}D(s)\cap M$.

We omit the proof which uses Theorem 1 and some additional standard arguments. The last result has the following meaning: the Bellman's function (the minimal time as a function of the initial state) satisfies the Hoelder condition in degree no less than $1/\sigma(m+1)$. The examples given below show that choosing a function $\varphi(\cdot)$ satisfying A4 for the concrete case one could obtain a better degree.

5. Examples. Let us consider the system

$$\dot{x}_1 = x_2 + x_4^p + x_1 x_2 + au^2,$$
$$\dot{x}_2 = x_3 + x_1^2,$$
$$\dot{x}_3 = x_4 + x_2^3,$$
$$\dot{x}_4 = u,$$

$u \in [-1, 1]$. Applying Theorem 2 we obtain the estimate $c\,|x' - x''|^{1/8}$ in the right-hand side of (22). Let us check the condition A4. The function $\varphi(\cdot)$ is to be chosen such that if

(24) $$|x_1| \leq Qt^4\varphi(t),\quad |x_2| \leq Qt^3\varphi(t),\quad |x_3| \leq Q^2\varphi(t),\quad |x_4| \leq Qt\varphi(t),$$

hen

$$h_1 = |x_4^p + x_1 x_2 + au^2| \leq C(Q)t^3\varphi(t)\mu(t),$$
$$h_2 = |x_1^2| \leq C(Q)t^2\varphi(t)\mu(t),$$
$$h_3 = |x_2^3| \leq C(Q)t\varphi(t)\mu(t).$$

Let $a = 0$, $p > 3$. Then the above inequalities hold (in view of (24)) for $\varphi(\cdot) = 1$. The corresponding estimate in (22) is $C\,|x' - x''|^{1/4}$. If $p \leq 3$ we can take $\varphi(t) = t^s$, where $s < (3-p)/(p-1)$. For instance, if $p = 2$, then we have the estimate $c\,|x' - x''|^k$ for $k < 1/5$.

Let $a = 1$. In this case the degree's estimate $1/4$ does not hold even in case $p > 3$. The condition A4 is fulfilled for $\varphi(t) = t^s$, $s > 3$ which gives the estimate $c\,|x' - x''|^k$ in (22) for $k < 1/7$.

Now, consider the system

$$\dot{x}_1 = \sin x_2 + u^2,$$
$$\dot{x}_2 = u \cos x_1 - u^2,$$
$$\dot{x}_3 = x_1 + x_2 - u,$$

where $u \in [-1, 1]$. The estimate (23) (the condition A6 holds for this system) with $\varphi(\cdot)$ given by (21) becomes

(25) $$|T(x') - T(x'')| \leq c\,|x' - x''|^K,\quad k < 1/5.$$

Checking directly the condition A4 one can easily get that $\varphi(t) = t^s$ $s > 1$ is an appropriate function. Hence we obtain the degree $k < 1/4$ in (25).

REFERENCES

1. Ф. М. Кириллова. О корректности постановки одной задачи оптимального регулирования. *Изв. высш. учебн. заведений, Математика*, **4**, 1958, 113—126.
2. Н. Н. Петров. О функции Белмана для задачи оптимального быстродействия. *Прикл. матем. и мех.*, **34**, 1970, 820—826.
3. Н. Н. Петров. О корректности задачи на быстродействие. *Дифференц. уравнения*, **13**, 1977, 1974—1980.
4. Ю. А. Белов. Локальная липшицовость оптимального времени.— В: *сб. работ по теории оптим. процессов. Калинингр. у-та*, **1**, 1973, 23—31.
5. Л. А. Кун. Об одном классе нелинейных задач оптимального быстродействия. *Автомат. и телемех.*, **5**, 1974, 31—37.
6. G. Olsder. Time-optimal control of multivariable systems near the origin. *JOTA*, **16**, 1975, 497—517.
7. Ю. Пронозин. К задаче предельного быстродействия. *Автоматика и телемеханика*, **9**, 1977, 13—23.
8. В. И. Коробов. О множествах достижимости и об управляемости линейной системы. *Ж. Вычисл. мат. и матем. физ.*, **10**, 1970, 848—856.
9. Т. Ю. Петренко. О свойствах функции Белмана. *Дифференц. уравнения*, **9**, 1973, 1244—1255.
10. А. А. Ливеровский. О гелдеровости функции Белмана плоских систем управления. *Дифференц. уравнения*, **16**, 1981, 604—613.
11. А. А. Ливеровский. Некоторые свойства функции Белмана для линейных и симметрических полисистем. *Дифференц. уравнения*, **16**, 1980, 414—423.
12. Р. В. Гамкрелидзе. Основы оптимального управления. Тбилиси, 1977.
13. A. Dontchev, V. Veliov. Singular perturbations in control problems with fixed final state. *Control and Cybernetics*, **11**, 1982, 19-28.
14. E. Michael. Continuous selections. *Ann. of Math.*, **63**, 1956, 361—381.

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