Serdica
Bulgariacae mathematicae
publicationes

Сердика
Българско математическо
списание

COMONOTONE APPROXIMATION OF $|x|$ BY RATIONAL FUNCTIONS

G. L. ILIEV, U. OPITZ

1. During the last 15 years a number of papers appeared which investigate the approximation of functions with special properties (monotonicity, convexity, ...) by a smaller class of functions which have the same properties (see [1, 2, 3, 4, 5]). Such approximations were called approximations with constraints. Approximations with constraints were studied in different metrics — uniform, L_p — metric, Hausdorff metric. As systems of approximation mainly polynomial and spline functions were considered. On the other hand in the last 20 years rapid progress was made in the field of rational approximation. The starting point was the famous result of D. Newman [6], which states that the function $|x|$ can be approximated by rational functions in $[-1,1]$ with the order $O(e^{-c_1\sqrt{n}})$. (Here and further on we shall denote by $c_i(*)$, $i=1,\ldots$, some positive constants which will depend on the parameters listed between the brackets while c_i will denote an absolute constant.) In the present paper we link these two directions of research as we prove that $|x|$ can be approximated by a rational function monotone decreasing in $[-1,0]$ and monotone increasing in $[0,1]$ retaining the error of approximation found in [6]. After Newman's work several classes of functions were discovered where using the result of Newman it could be shown that rational approximation is better than polynomial approximation. In all of these problems the questions of comonotone rational approximation remained an open problem.

Let us consider the function

$$x_+ = \begin{cases} 0 & \text{for} \quad x \in [-1,0] \\ x & \text{for} \quad x \in [0,1]. \end{cases}$$

It is natural to ask for the approximation of the function x_+ by rational functions on $[-1,1]$ with positive first derivative. This problem (without the additional condition for the first derivative) is considered in [7]. We show in the present paper that the order of approximation is again $O(e^{-c_1\sqrt{n}})$ as in the theorem of Newman [6]. This result is equivalent to the fact that the order of the Hausdorff monotone rational approximation of the function

$$\sigma(x) = \begin{cases} 0, & x \in [-1,0] \\ 1, & x \in [0,1] \end{cases} \tag{1}$$

is itself $O(e^{-c_1\sqrt{n}})$.

SERDICA Bulgaricae mathematicae publicationes. Vol. 10, 1984, p. 88—105.

In 1975 Vyaceslavov [7] found by a very complicated constructive proof that the constant c_1 in Newman's estimate $O(e^{-c_1\sqrt{n}})$ is equal to π. In the present paper we show that the same constant even holds for comonotone approximation of $|x|$ by rational functions.

The results of this paper were announced in [8].

2. Let R_n be the set of all real rational functions of order n:

$$R_n := \{ r(x) : r(x) = \frac{p_1(x)}{p_2(x)},\ p_i(x) = a_0^i + a_1^i x + \dots + a_n^i x^n,$$

$$i = 1, 2,\ a_k^i \in \mathbf{R},\ k = 0, \dots, n\}.$$

Theorem 1. For every large enough integer $n \geq n_0$ there is $r \in R_n$ such that

(2) $$||x| - r(x)| \leq 3e^{-\sqrt{n}} \quad \text{for } x \in [-1,1]$$

(3) $$r(x) \text{ is an even function}$$

(4) $$r'(x) \geq 0 \quad \text{for } x \in [0,1).$$

We shall prove that (2), (3), (4) are satisfied for the following rational function $r(x)$ introduced by Newman [6]:

$$r(x) = x\frac{p(x) - p(-x)}{p(x) + p(-x)},\quad p(x) = \prod_{k=0}^{n-1} (x + \xi^k),\quad \xi = \exp(-1/\sqrt{n}).$$

The following two lemmas were proved in [6] for every integer $n > 4$:

Lemma A. $\prod_{j=1}^{n} \frac{1-\xi^j}{1+\xi^j} \leq e^{-\sqrt{n}}$.

Lemma B. $\left|\frac{p(-x)}{p(x)}\right| \leq e^{-\sqrt{n}}$ *for* $\xi^{n-1} \leq x \leq 1$.

We also need the next two results.

Lemma 1. $x\left|\frac{p'(x)}{p(x)}\right| \leq n$ *for* $0 \leq x \leq 1$.

Proof.

$$p'(x) = \sum_{k=0}^{n-1} \prod_{\substack{j=0 \\ j \neq k}}^{n-1} (x + \xi^j),\quad x\left|\frac{p'(x)}{p(x)}\right| = \sum_{j=0}^{n-1} \frac{x}{x+\xi^j} = \sum_{j=0}^{n-1} \frac{1}{1+\xi^j/x} \leq n.$$

Lemma 2. *Let* $n > 4$. *Then* $x\left|\frac{p'(-x)}{p(x)}\right| \leq n^2 e^{-\sqrt{n}}$ *for* $\xi^{n-1} \leq x \leq 1$.

Proof.

(5) $$x\left|\frac{p'(-x)}{p(x)}\right| = x\left|\sum_{k=0}^{n-1} \frac{\prod_{\substack{j=0 \\ j \neq k}}^{n-1} (\xi^j - x)}{\prod_{j=0}^{n-1} (\xi^j + x)}\right| \leq: \sum_{k=0}^{n-1} A_k$$

Let $\xi^{i+1} \leq x \leq \xi^i$, $i=0, 1, \ldots, n-1$. Let first $0 \leq k < i$. Then:

$$A_k = x \left| \frac{(\xi^0 - x) \ldots (\xi^{k-1} - x)(\xi^{k+1} - x) \ldots (\xi^i - x)(\xi^{i+1} - x) \ldots (\xi^{n-1} - x)}{\prod_{j=0}^{n-1} (\xi^j + x)} \right| \tag{6}$$

$$= \frac{x}{x+\xi^k} \cdot \frac{(\xi^0 - x) \ldots (\xi^{k-1} - x)(\xi^{k+1} - x) \ldots (\xi^i - x)}{(\xi^0 + x) \ldots (\xi^{k-1} - x)(\xi^{k+1} + x) \ldots (\xi^i + x)} \cdot \frac{(x - \xi^{i+1}) \ldots (x - \xi^{n-1})}{(x + \xi^{i+1}) \ldots (x + \xi^{n-1})}$$

$$\leq 1 \cdot \frac{(\xi^0 - \xi^{i+1}) \ldots (\xi^{k-1} - \xi^{i+1})(\xi^{k+1} - \xi^{i+1}) \ldots (\xi^i - \xi^{i+1})(1 - \xi^{i+1}) \ldots (1 - \xi^{n-1})}{(\xi^0 + \xi^{i+1}) \ldots (\xi^{k-1} + \xi^{i+1})(\xi^{k+1} + \xi^{i+1}) \ldots (\xi^i + \xi^{i+1})(1 + \xi^{i+1}) \ldots (1 + \xi^{n-1})}$$

$$= \frac{(1 - \xi^{i+1}) \ldots (1 - \xi^{i-k+2})(1 - \xi^{i-k}) \ldots (1 - \xi)}{(1 + \xi^{i+1}) \ldots (1 + \xi^{i-k+2})(1 + \xi^{i-k}) \ldots (1 + \xi)} \cdot \frac{(1 - \xi^{i+1}) \ldots (1 - \xi^{n-1})}{(1 + \xi^{i+1}) \ldots (1 + \xi^{n-1})}$$

$$\leq \frac{1 + \xi^{i-k+1}}{1 - \xi^{i-k+1}} \prod_{j=1}^{n} \frac{1 - \xi^j}{1 + \xi^j}.$$

But $i - k > 0$. Then

$$\frac{1 + \xi^{i-k+1}}{1 - \xi^{i-k+1}} \leq \frac{1}{1 - \xi} \leq n \tag{7}$$

and so from Lemma A, (6) and (7) it follows:

$$A_k \leq n e^{-\sqrt{n}}. \tag{8}$$

Let $1 + i < k \leq n - 1$.

$$A_k = \frac{x}{x+\xi^k} \cdot \frac{(\xi^0 - x) \ldots (\xi^i - x)}{(\xi^0 + x) \ldots (\xi^i + x)} \cdot \frac{(x - \xi^{i+1}) \ldots (x - \xi^{k-1})(x - \xi^{k+1}) \ldots (x - \xi^{n-1})}{(x + \xi^{i+1}) \ldots (x + \xi^{k-1})(x + \xi^{k+1}) \ldots (x + \xi^{n-1})} \tag{9}$$

$$\leq \frac{(1 - \xi^{i+1}) \ldots (1 - \xi)}{(1 + \xi^{i+1}) \ldots (1 + \xi)} \cdot \frac{(1 - \xi^{i+1}) \ldots (1 - \xi^{k-1})(1 - \xi^{k+1}) \ldots (1 - \xi^{n-1})}{(1 + \xi^{i+1}) \ldots (1 + \xi^{k-1})(1 + \xi^{k+1}) \ldots (1 + \xi^{n-1})}$$

$$= \frac{1 + \xi^k}{1 - \xi^k} \prod_{j=1}^{n} \frac{1 - \xi^j}{1 + \xi^j} \leq \frac{1}{1 - \xi} \cdot e^{-\sqrt{n}} \leq n e^{-\sqrt{n}}.$$

Let $k = i$

$$A_k = \frac{x}{\xi^i + x} \cdot \frac{(\xi^0 - x) \ldots (\xi^{i-1} - x)}{(\xi^0 + x) \ldots (\xi^{i-1} + x)} \cdot \frac{(x - \xi^{i+1}) \ldots (x - \xi^{n-1})}{(x + \xi^{i+1}) \ldots (x + \xi^{n-1})} \tag{10}$$

$$\leq \frac{(1 - \xi^{k+1}) \ldots (1 - \xi^2)}{(1 + \xi^{k+1}) \ldots (1 + \xi^2)} \cdot \frac{(1 - \xi^{i+1}) \ldots (1 - \xi^{n-1})}{(1 + \xi^{i+1}) \ldots (1 + \xi^{n-1})}$$

$$\leq \frac{1 + \xi}{1 - \xi} \cdot \prod_{j=1}^{n} \frac{1 - \xi^j}{1 + \xi^j} \leq n e^{-\sqrt{n}}.$$

Let $k = i + 1$

$$A_k = \frac{x}{x + \xi^{i+1}} \cdot \frac{(\xi^0 - x) \ldots (\xi^i - x)}{(\xi^0 + x) \ldots (\xi^i + x)} \cdot \frac{(x - \xi^{i+2}) \ldots (x - \xi^{n-1})}{(x + \xi^{i+2}) \ldots (x + \xi^{n-1})} \tag{11}$$

$$\leq \frac{(1 - \xi^{i+1}) \ldots (1 - \xi)}{(1 + \xi^{i+1}) \ldots (1 + \xi)} \cdot \frac{(1 - \xi^{i+2}) \ldots (1 - \xi^{n-1})}{(1 + \xi^{i+2}) \ldots (1 + \xi^{n-1})}$$

$$\leq \frac{1 + \xi^n}{1 - \xi^n} \prod_{j=1}^{n} \frac{1 - \xi^j}{1 + \xi^j} \leq 2 e^{-\sqrt{n}}.$$

From (5), (8), (9), (10) and (11) it follows that $x\left|\frac{p'(-x)}{p(x)}\right|\leq n^2e^{-\sqrt{n}}$, which completes the proof of the lemma.

P r o o f o f T h e o r e m 1. We shall prove that assertions (2), (3) and (4) are true for the rational function constructed by N e w m a n in [6]:

$$r(x)=x\,\frac{p(x)-p(-x)}{p(x)+p(-x)},\ p(x)=\prod_{k=0}^{n-1}(x+\xi^k),\quad \xi=\exp(-1/\sqrt{n}),\ r\in R_n.$$

Using Lemmas A and B it was proved in [6] that $||x|-r(x)|\leq 3e^{-\sqrt{n}}$ for $x\in[-1,1]$. Since it is obvious that $r(x)$ is an even function we only have to prove (4). First let $x\in[0,\xi^{n-1}]$ and let us define:

(12) $$q(x)=\frac{p(x)-p(-x)}{p(x)+p(-x)}\cdot$$

Consider the equation

(13) $$q(x)=1,$$

which is equivalent to $2p(-x)=0$.

We find that the roots of (13) are $x_i=\xi^i$, $i=0,\dots,n-1$. But since $q(x)$ is an odd function, the roots of the equation $q(x)=-1$ are the points $x_{-i}=-\xi^i$, $i=0,\dots,n-1$.

Then from the Theorem of Rolle $q'(x)$ has at least $2n-2$ zeroes for $x\in[-1,-\xi^{n-1}]\cup[\xi^{n-1},1]$.

(14) Let us suppose that $q(x)$ is not a monotone increasing function for $x\in[0,\xi^{n-1}]$. Since $q(0)=0$ $q'(x)$ then must have one more zero in $(0,\xi^{n-1})$ which means at least two more zeroes in $(-\xi^{n-1},\xi^{n-1})$. Then from (14) it follows that $q'(x)$ has $\geq 2n$ zeroes for $x\in[-1,1]$. But

(15) $$q'(x)=\frac{[p(x)-p(-x)]'[p(x)+p(-x)]-[p(x)-p(-x)][p(x)+p(-x)]'}{[p(x)+p(-x)]^2}$$
$$=\frac{Q(x)}{[p(x)+p(-x)]^2}$$

and as $p(x)+p(-x)\neq 0$ for all $x\in\mathbf{R}$, then there must exist $2n$ points $n_i\in[-1,1]$, $i=1,\dots,2n$ for which $Q(n_i)=0$. But from (15) it is clear that Q is an algebraic polynomial of degree less than $2n$. Therefore from (14) it follows that $Q(x)=0$, $q'(x)=0$, which is false.

Therefore

(16) $$\begin{cases} q(x) \text{ is an odd rational function}\\ q(\xi^i)=1 \text{ for } i=0,1,\dots,n-1\\ q(x) \text{ is a monotone increasing function for } x\in[-\xi^{n-1},\xi^{n-1}]\\ 0\leq q(x)\leq 1 \text{ for } x\in[0,\xi^{n-1}].\end{cases}$$

Then for $x\in[0,\xi^{n-1}]$

(17) $$r'(x)=q(x)+q'(x)x\geq 0.$$

On the other hand, for arbitrary x

$$r'(x)=\frac{p(x)-p(-x)}{p(x)+p(-x)}$$

$$+x\frac{[p'(x)+p'(-x)][p(x)+p(-x)]-[p'(x)-p'(-x)][p(x)-p(-x)]}{[p(x)+p(-x)]^2}$$

$$=\frac{p^2(x)-p^2(-x)+2x[p'(-x)p(x)+p'(x)p(-x)]}{[p(x)+p(-x)]^2}$$

$$=\left[\frac{p(x)}{p(x)+p(-x)}\right]^2\left\{1-\left[\frac{p(-x)}{p(x)}\right]^2+2x\,\frac{p'(-x)}{p(x)}+2x\,\frac{p'(x)}{p(x)}\cdot\frac{p(-x)}{p(x)}\right\}$$

$$\geq\left[\frac{p(x)}{p(x)+p(-x)}\right]^2\left\{1-\left[\frac{p(-x)}{p(x)}\right]^2-2x\left|\frac{p'(-x)}{p(x)}\right|-2x\left|\frac{p'(x)}{p(x)}\right|\cdot\left|\frac{p(-x)}{p(x)}\right|\right\}.$$

Now, if $\xi^{n-1}\leq x\leq 1$ then from Lemma B, Lemma 1 and Lemma 2 we have for $n\geq n_0$

$$r'(x)\geq\left[\frac{p(x)}{p(x)+p(-x)}\right]^2\left\{1-e^{-2\sqrt{n}}-2n^2e^{-\sqrt{n}}-2ne^{-\sqrt{n}}\right\}\geq 0. \tag{18}$$

From (17) and (18) it follows that $r'(x)\geq 0$ for $0\leq x\leq 1$, which proves Theorem 1.

3. Theorem 2. *For every integer* $n\geq n_0$ *there is* $s\in R_n$ *such that* $|1-s(x)|\leq e^{-c_2\sqrt{n}}$ *for* $x\in[e^{-c_2\sqrt{n}}, 1]$;

$s(x)$ *is an odd function,* $-1\leq s(x)\leq 1$ *for* $x\in[-1, 1]$;

$s(x)$ *is a monotone increasing function for* $x\in[-1, 1]$.

Proof. Consider

$$\bar{q}(x)=\frac{p(x)-p(-x)}{p(x)+p(-x)},\quad p(x)=\prod_{k=0}^{n-1}(x+\xi^k),\quad \xi=\exp(-1/m),$$

where $m=c_3.\sqrt{n}$.

Similarly as in the paper of D. Newman [6] we see that $|1-\bar{q}(x)|\leq 3e^{-m}$ for $x\in[\exp(-(n-1)/m), 1]$ and $n\geq n_0$.

By the same method we used for the proof of (16) for the function defined by (12), it is easy to verify that $\bar{q}$ is an odd function which must be monotone increasing on $[0, \exp(-(n-1)/m)]$.

We have

$$\bar{q}'(x)=\frac{2[p'(-x)p(x)+p'(x)p(-x)]}{[p(x)+p(-x)]^2}=\frac{2}{x}\cdot\frac{x[p'(-x)p(x)+p'(x)p(-x)]}{[p(x)+p(-x)]^2}.$$

Then, following the same way like in the proof of Theorem 1 we obtain for $x\in[\exp(-(n-1)/m), 1]$ and large n

$$|\bar{q}'(x)|\leq 2(m^2+1)\,n\exp(n/m-m)\leq 3c_3^2n^2\exp\left(\left(\frac{1}{c_3}-c_3\right)\sqrt{n}\right).$$

Now, for $c_3>1$ we see that $1/c_3-c_3<0$. Hence for any $\alpha_1\in(0,1)$, $\beta>1$ we have

$$|\bar{q}'(x)|\leq e^{-\alpha_1(c_3-1/c_3)\sqrt{n}}\quad\text{for}\quad x\in[\exp(-1/(\beta c_3))\sqrt{n}, 1] \tag{19}$$

and n large enough.

Let $s(x)=\bar{q}(x)+e^{-\alpha_1(c_3-1/c_3)\sqrt{n}}\,.\,x$. Then $s(x)$ is an odd rational function of order n that is monotone increasing on $[-1,1]$ by (19) and satisfies

$$|1-s(x)|\leq 3e^{-c_3\sqrt{n}}+e^{-\alpha_1(c_3-1/c_3)\sqrt{n}}\leq e^{-\alpha(c_3-1/c_3)\sqrt{n}}$$

for $x\in[\exp(-1/(\beta c_3)\sqrt{n}),1]$ where $\alpha<\alpha_1$ and $n>n_0$. Finally we choose $\alpha\in(0,1)$ and $\beta>1$ arbitrarily so that $\alpha(c_3-1/c_3)=1/(\beta c_3)=:c_2$ and c_2 is large. The above equality is equivalent to $c_3^2=\sqrt{1+\frac{1}{\alpha\beta}}>1$ and $c_2=\frac{1}{\sqrt{\beta^2+\frac{\beta}{\alpha}}}$. From the restrictions on α and β we see that necessarily $c_2<1/\sqrt{2}$. This construction finishes the proof of Theorem 2.

Theorem 3. *For every pair of integers $n>n_0$, $k>0$, there are a rational function $s_k\in R_n$ and a constant $c_4(k)$ such that*

$$|x_+^k-s_k(x)|\leq\exp[-c_4(k).\sqrt{n}]\quad for\quad x\in[-1,1],$$

$$s_k'(x)\geq 0\quad for\quad x\in[-1,1].$$

Proof. In view of the detailed discussion in the foregoing proofs of Theorems 1 and 2 we shall be rather short here. Let us first consider the case $k\geq 2$ where we shall use the result from Theorem 1. The approximation of x_+ requires an extra construction via the result of Theorem 2.

Let $r(x)\in R_{n+1}$ be the rational approximant to $|x|$ on $[-1,1]$ defined in Theorem 1. Then for $k\geq 2$ set

$$s_k(x):=\frac{x^{k-1}[r(x)+x]}{2}+e^{-c_5(k)\sqrt{n}}\,.\,x,$$

where $c_5(k)$ is a constant to be determined later. Then for $x\geq 0$ we have by virtue of Theorem 1

$$|x^k-s_k(x)|\leq\frac{1}{2}|x^{k-1}|.|x-r(x)|+e^{-c_5(k)\sqrt{n}}\leq 1{,}5e^{-\sqrt{n}}+e^{-c_5(k)\sqrt{n}}$$

and similarly for $x<0$ since $|x+r(x)|=||x|-r(x)|$

$$|s_k(x)|\leq\frac{1}{2}|x^{k-1}|.|x+r(x)|+e^{-c_5(k)\sqrt{n}}\leq 1{,}5\ e^{-\sqrt{n}}+e^{-c_5(k)\sqrt{n}}.$$

Thus $|x_+^k-s_k(x)|\leq 2{,}5e^{-c_5(k)\sqrt{n}}$ for all $x\in[-1,1]$. This proves the first assertion of the theorem for $c_4(k)<c_5(k)$. Let us investigate now the sign of $s_k'(x)$. s_k is a combination of functions which are all non negative and monotone increasing for $x\geq 0$ (remember $r(0)=0$), hence $s_k'(x)\geq 0$ for $x\in[0,1]$. Now consider $x\in[-1,0)$, We have

$$s_k'(x)=\frac{1}{2}\{(k-1)x^{k-2}[r(x)+x]+x^{k-1}[r'(x)+1]\}+e^{-c_5(k)\sqrt{n}}$$

$$\geq e^{-c_5(k)\sqrt{n}}-\frac{1}{2}\{(k-1)||x|-r(x)|+|x|\cdot|1-r'(-x)|\}$$

because r is an even function and $k\geq 2$.

Here $||x|-r(x)|\leqq 3e^{-\sqrt{n}}$, so we have only to estimate $|1-r'(x)|$ for $x\in[0,1]$. If $x\in[\xi^{n-1},1]$ where $\xi=\exp(-1/\sqrt{n})$ then we find by mere repetition of the arguments in the proof o Theorem 1 after formula (17)

$$|1-r'(x)|=\left|1-\frac{p(x)-p(-x)}{p(x)+p(-x)}-2x\,\frac{p'(-x)\,p(x)+p'(x)\,p(-x)}{[p(x)+p(-x)]^2}\right|$$

$$\leqq 2\left\{\left|\frac{p(-x)}{p(x)+p(-x)}\right|+\left|\frac{p(x)}{p(x)+p(-x)}\right|^2\left|x\frac{p'(-x)}{p(x)}+x\,\frac{p'(x)}{p(x)}\,\frac{p(-x)}{p(x)}\right|\right\}$$

$$\leqq 2\left\{\frac{1}{\left|\frac{p(x)}{p(-x)}\right|-1}+\left[\frac{1}{1-\frac{p(-x)}{p(x)}}\right]^2\left\{x\left|\frac{p'(-x)}{p(x)}\right|+x\left|\frac{p'(x)}{p(x)}\right|\left|\frac{p(-x)}{p(x)}\right|\right\}\right\}$$

$$\leqq 2\left\{\frac{1}{e^{\sqrt{n}}-1}+\left[\frac{1}{1-e^{-\sqrt{n}}}\right]^2\left\{n^2\,.\,e^{-\sqrt{n}}+ne^{-\sqrt{n}}\right\}\right\}$$

$$\leqq 3\,[n^2+n+1]\,c^{-\sqrt{n}}\quad \text{if}\quad n\geqq n_0\,.$$

On the other hand for $x\in[0,\xi^{n-1}]$ we have $p(x)\geqq p(-x)\geqq 0$ and so

$$\left|1-\frac{p(x)-p(-x)}{p(x)+p(x)}\right|\leqq 1,\ \left|\frac{p(-x)}{p(x)}\right|\leqq 1,\ x\left|\frac{p'(x)}{p(x)}\right|\leqq n,$$

$$x\,.\left|\frac{p'(-x)}{p(x)}\right|\leqq\sum_{k=0}^{n-1}\frac{1}{1+\frac{\xi^k}{x}}\prod_{\substack{j=0\\ j\neq k}}^{n-1}\frac{|\xi^j-x|}{\xi^j+x}\leqq\sum_{k=0}^{n-1}\frac{1}{1+\frac{\xi_k}{x}}\leqq n\,.$$

Hence $|1-r'(x)|\leqq 1+2[n+n]\leqq 5n$ for $x\in[0,\xi^{n-1}]$. From the above estimates we learn that for all $x\in[0,1]$ and $n\geqq n_0$

$$x\,.\,|1-r'(x)|\leqq 3\,[n^2+n+1]\,.\,e^{-\sqrt{n}}$$

and so finally

$$s_k'(x)\geqq e^{-c_5(k)\sqrt{n}}-\frac{1}{2}\left\{3(k-1)e^{-\sqrt{n}}+3\,[n^2+n+1]e^{-\sqrt{n}}\right\}\geqq 0$$

for $c_5(k)$ sufficiently small and $n\geqq n_0$.

We turn now to the case $k=1$. Let $s(x)$ be the rational approximant to the function sign (x) constructed in Theorem 2. We define

$$s_1(x)=[x+e^{-c_2\sqrt{n}}]\,.\,\frac{s(x)+1}{2}+e^{-c_6\sqrt{n}}\,.\,x,$$

where c_2 denotes the constant introduced in Theorem 2 and $c_6<c_2$ is another constant which will be determined later. In all subsequent computations we shall make thorough use of the proof of Theorem 2. Let us first estimate $|x_+-s_1(x)|$ for $x\in[-1,1]$.

Since $s(0)=0$ and $s'(x)\geqq 0$ we have from Theorem 2 for $x\in[0,1]$

$$|1-s(x)|\leqq\max(|1-s(0)|,\ |1-s(1)|)\leqq\max(1,\ e^{-c_2\sqrt{n}})=1,$$

$$|1+s(x)|\leqq 1+s(1)\leqq 2+e^{-c_2\sqrt{n}}$$

and so by distinction between the cases $x \in [0, e^{-c_2\sqrt{n}}]$ and $x \in [e^{-c_2\sqrt{n}}, 1]$

$$x \cdot |1-s(x)| \leqq e^{-c_2\sqrt{n}} \quad \text{for all} \quad x \in [0,1].$$

From these inequalities we find

$$|x-s_1(x)| \leqq \frac{1}{2}\{x \cdot |1-s(x)| + e^{-c_2\sqrt{n}}|1+s(x)|\} + e^{-c_6\sqrt{n}}$$

$$\leqq \frac{1}{2}\{e^{-c_2\sqrt{n}} + (2+e^{-c_2\sqrt{n}})e^{-c_2\sqrt{n}}\} + e^{-c_6\sqrt{n}} \leqq 3e^{-c_6\sqrt{n}} \quad \text{for} \quad x \in [0,1],$$

$$|s_1(x)| \leqq \frac{1}{2}\{|x| \cdot |1+s(x) + e^{-c_2\sqrt{n}}|1+s(x)|\} + e^{-c_6\sqrt{n}}$$

$$= \frac{1}{2}\{-x \cdot |1-s(-x)| + e^{-c_2\sqrt{n}}|1-s(-x)|\} + e^{-c_6\sqrt{n}}$$

$$\leqq \frac{1}{2}\{e^{-c_2\sqrt{n}} + e^{-c_2\sqrt{n}}\} + e^{-c_6\sqrt{n}} \leqq 2e^{-c_6\sqrt{n}} \quad \text{for} \quad x \in [-1,0].$$

This establishes the first assertion of Theorem 3 for $c_4 = c_4(1) < c_6$.
Now we turn to show that $s_1'(x) \geqq 0$ on $[-1,1]$:

$$2 \cdot s_1'(x) = s(x) + 1 + [x + e^{-c_2\sqrt{n}}] \cdot s'(x) + 2e^{-c_6\sqrt{n}}.$$

As by construction of the function s

$$s(-e^{-(n-1)/m}) = q(-e^{-(n-1)/m}) - e^{-\alpha_1(c_3-\frac{1}{c_3})\sqrt{n}} e^{-(n-1)/m} \geqq -1 - e^{-c_2\sqrt{n}}$$

and $s'(x) \geqq 0$ we have from $e^{-c_2\sqrt{n}} \geqq e^{-(n-1)/m}$ that

$$2s_1'(x) \geqq s(-e^{-(n-1)/m}) + 1 + [-e^{-(n-1)/m} + e^{-c_2\sqrt{n}}]\, s'(x) + 2e^{-c_6\sqrt{n}}$$

$$\geqq -e^{-c_2\sqrt{n}} + 2e^{-c_6\sqrt{n}} \geqq 0 \quad \text{for} \quad x \in [-e^{-(n-1)/m}, 1].$$

On the other hand, for $x \in [-1, -e^{-(n-1)/m}]$ we find

$$2\, s_1'(x) \geqq 2e^{-c_6\sqrt{n}} - |1+s(x)| - |x + e^{-c_2\sqrt{n}}| \cdot |s'(x)|$$

$$= 2e^{-c_6\sqrt{n}} - |1 + [\bar{q}(x) + e^{-\alpha_1(c_3-1/c_3)\sqrt{n}} \cdot x]| - |x + e^{-c_2\sqrt{n}}| \cdot |\bar{q}'(x) + e^{-\alpha_1(c_3-1/c_3)\sqrt{n}}|$$

$$\geqq 2e^{-c_6\sqrt{n}} - |1+\bar{q}(x)| - e^{-\alpha_1(c_3-1/c_3)\sqrt{n}} - |\bar{q}'(x)| - e^{-\alpha_1(c_3-1/c_3)\sqrt{n}}$$

$$\geqq 2[e^{-c_6\sqrt{n}} - e^{-\alpha_1(c_3-1/c_3)\sqrt{n}}] - 3e^{-m} - 3c_3^2 n^2 e^{-(c_3-1/c_3)\sqrt{n}}$$

so that from $e^{-m} = e^{-c_3\sqrt{n}}$, $3e^{-c_3\sqrt{n}} \leqq e^{-c_2\sqrt{n}}$, $3\, c_3^2 n^2 e^{-(c_3-1/c_3)\sqrt{n}} \leqq e^{-c_2\sqrt{n}}$, finally

$$s_1'(x) \geqq e^{-c_6\sqrt{n}} - 2e^{-c_2\sqrt{n}} \geqq 0 \quad \text{for} \quad n \geqq n_0$$

if only c_6 is small enough compared to c_2. This ends the proof of Theorem 3.

By some modifications in the proof of Theorem 2 it should be possible to prove the following assertion:

For every pair of integers $n \geq n_0$, $k>0$ there is $s_k \in R_n$ and a constant $c_7(k)$ such that

$$|x_+^k - s_k(x)| \leq \exp[-c_7(k)\sqrt{n}] \quad \textit{for} \quad x \in [-1,1]$$

$$s_k^{(i)}(x) \geq 0 \quad \textit{for} \quad i=1,\dots,k, \qquad x \in [-1,1].$$

4. Let

$$R_n^* := \{ r(x) : r \in R_n,\ r'(x) \leq 0 \ \text{ for } \ x \in [-1,0],\ r'(x) \geq 0 \ \text{ for } \ x \in [0,1]\},$$

$$R_n(|x|, [-1,1]) = \inf_{r \in R_n} \| |x| - r(x) \| := \inf_{r \in R_n} \max_{x \in [-1,1]} | |x| - r(x)|,$$

$$R_n^*(|x|, [-1,1]) = \inf_{r \in R_n^*} \| |x| - r(x) \|.$$

In [7] Vyaceslavov proves the following result

$$\exp(-\pi\sqrt{n+1} \leq R_n(|x|, [-1,1]) \leq c_8 \exp(-\pi\sqrt{n-1}). \tag{20}$$

The estimate from below was obtained by Bulanov and is also stated in [7]. Vyaceslavov reached the upper estimate by direct construction of a rational function. Our purpose is to prove that this rational function also follows the monotonicity of $|x|$ in $[-1,1]$:

Theorem 4. $\exp(-\pi\sqrt{n+1}) \leq R_n^*(|x|, [-1,1]) \leq c_8 \exp(-\pi\sqrt{n-1})$.

To prove Theorem 4 we have to take a thorough look at the construction of Vyaceslavov. As in the fundamental paper of Newman [6]:

$$r(x) = x\,\frac{p(x)-p(-x)}{p(x)+p(-x)}, \tag{21}$$

where p is a polynomial of degree n:

$$p(x) = \prod_{j=0}^{n-1} (x+\xi_j). \tag{22}$$

The zeroes $-\xi_j$ are chosen differently from Newman [6]. (In [6] we had $\xi_j = \exp(-j/\sqrt{n})$):

$$\xi_j = \exp[-(A_0 + \cdots + A_j - 2rA_0)], \tag{23}$$

where

$$A_t = \begin{cases} \pi(1-1/e)/(2\sqrt{n}), & t=0,1,\dots,2r-1, \\ \pi(1-1/e^i)/(2\sqrt{n}), & (1+i)r \leq t < (2+i)r,\ i=1,\dots,m-2, \\ \pi/(2\sqrt{n-t}), & t=mr,\dots,n-1, \end{cases} \tag{24}$$

and

$$m = [\pi^{-1}\ln\sqrt{n}+1], \quad r = [2\sqrt{n}+1], \quad n \geq n_0. \tag{25}$$

Here n_0 is a fixed integer chosen large enough. From these settings the following properties are clear:

$$(26)\quad \begin{aligned} &A_t>0;\ A_t \text{ is monotone increasing in } t;\\ &\xi_{j+1}/\xi_j=\exp(-A_{j+1}) \text{ is monotone decreasing in } j;\\ &\xi_0>\xi_1>\cdots>\xi_{2r-1}=1>\xi_{2r}>\cdots>\xi_{n-1}. \end{aligned}$$

Let us define

$$(27)\qquad s(x)=\frac{p(x)-p(-x)}{p(x)+p(-x)}\,.$$

It is easy to see that

$$(28)\quad \begin{aligned} &s(0)=0;\quad s(\pm\xi_j)=\pm 1,\ j=0,\dots,n-1;\\ &s(x)\ \text{ is an odd function in } R_n. \end{aligned}$$

We shall consider only the interval $(0, 1]$. Since $p(x)>0$ for $x\in[0, 1]$ from (21) and (27) we have like in Theorem 1:

$$(29)\qquad r'(x)=s(x)+xs'(x)$$

$$=\frac{1}{[1+\frac{p(-x)}{p(x)}]^2}\ \Big[1-\{\frac{p(-x)}{p(x)}]^2+2x\,[\frac{p(-x)}{p(x)}\cdot\frac{p'(x)}{p(x)}+\frac{p'(-x)}{p(x)}]\Big\}.$$

Moreover by (22):

$$(30)\qquad p'(x)=\sum_{j=0}^{n-1}\prod_{\substack{k=0\\k\neq j}}^{n-1}(x+\xi_k)=\sum_{j=0}^{n-1}\frac{1}{x+\xi_j}\prod_{k=0}^{n-1}(x+\xi_k)=p(x)\sum_{j=0}^{n-1}\frac{1}{x+\xi_j},$$

$$(31)\qquad p'(-x)=\sum_{j=0}^{n-1}\prod_{\substack{k=0\\k\neq j}}^{n-1}(-x+\xi_k)=\begin{cases}\displaystyle\prod_{\substack{k=0\\k\neq l}}^{n-1}(-\xi_l+\xi_k) & \text{when } x=\xi_l,\\[2ex] \displaystyle p(-x)\sum_{j=0}^{n-1}\frac{1}{\xi_j-x} & \text{when } x\neq\xi_l.\end{cases}$$

From (22), (29), (30) and (31) we obtain:

$$(32)\qquad r'(0)=0\quad\text{and}\quad r'(\xi_l)=1+2\xi_l\,\frac{\displaystyle\prod_{\substack{k=0\\k\neq l}}^{n-1}(-\xi_l+\xi_k)}{\displaystyle\prod_{k=0}^{n-1}(\xi_l+\xi_k)}$$

$$=1+\prod_{\substack{k=0\\k\neq l}}^{n-1}\frac{-\xi_l+\xi_k}{\xi_l+\xi_k}>0$$

and for $x\in(0, 1]$, $x\neq\xi_{n-1},\dots,\xi_{2r-1}$,

$$(33)\qquad r'(x)=\frac{1}{[1+\frac{p(-x)}{p(x)}]^2}\Big\{1-[\frac{p(-x)}{p(x)}]^2+2x\frac{p(-x)}{p(x)}\Big[\sum_{j=0}^{n-1}\Big(\frac{1}{x+\xi_j}+\frac{1}{-x+\xi_j}\Big)\Big]\Big\}.$$

We shall now exploit formulas (29) and (33) to prove Theorem 4. We shall distinguish several cases.

Case 1. $x\in(0,\ \xi_{n-1})$. In this case $r'(x)>0$ and this can be proved exactly like in Theorem 1 after assumption (14).

Vyaceslavov notes the following inequality

$$\frac{b-x}{b+x}\ \frac{x-a}{x+a}\leq\left[\frac{1-\sqrt{\frac{a}{b}}}{1+\sqrt{\frac{a}{b}}}\right]^2,\quad 0<a<x<b. \tag{32}$$

A direct consequence of (34) is the estimate ([7], Lemma 3):

$$\frac{|p(-x)|}{p(x)}\leq\left[\frac{1-\sqrt{\frac{\xi_{t+1}}{\xi}}}{1+\sqrt{\frac{\xi_{t+1}}{\xi_t}}}\right]^2<\frac{\pi}{4},\quad x\in(\xi_{t+1},\ \xi_t). \tag{35}$$

From (29) and (33) it is clear that in order to show $r'(x)>0$ we have only to verify

$$\begin{aligned}&1-[\frac{|p(-x)|}{p(x)}]^2+2x[\frac{p(-x)}{p(x)}\ \frac{p'(x)}{p(x)}+\frac{p'(-x)}{p(x)}]=\\ &1-[\frac{|p(-x)|}{p(x)}]^2+2x\frac{p(-x)}{p(x)}[\sum_{j=0}^{n-1}(\frac{1}{x+\xi_j}+\frac{1}{-x+\xi_j})]>0.\end{aligned} \tag{36}$$

Case 2. $x\in(\xi_{t+1},\ \xi_t\}$, $t\in\{2r-1,\ldots,q\}$, $q=[n-\frac{3}{2}-\ln^2 n]$. This case covers almost all intervals except those $\sim\ln^2 n$ which are nearest to the point 0.

We prove the following three inequalities:

$$\frac{|p(-x)|}{p(x)}\leq n^{-(4-\varepsilon)/\pi},\quad x\in[\xi_{q+1},\ 1],\qquad \varepsilon>0\ \text{ arbitrary},\quad n\geq n_0(\varepsilon) \tag{37}$$

$$x\frac{|p'(x)|}{p(x)}\leq n,\quad x\in[0,\ 1] \tag{38}$$

$$x\frac{|p'(-x)|}{p(x)}\leq n^{-(4-\pi-\varepsilon)/\pi},\quad x\in(\xi_{t+1},\ \xi_t)\ \text{ with } t\in\{2r-1,\ldots,q\},\quad \varepsilon>0\ \text{ arbitrary},\quad n\geq n_0(\varepsilon). \tag{39}$$

As a consequence we shall have the desired inequality: For $n\geq n_0$

$$\begin{aligned}&1-[\frac{|p(-x)|}{p(x)}]^2+2x[\frac{p(-x)}{p(x)}\ \frac{p'(x)}{p(x)}+\frac{p'(-x)}{p(x)}]\\ &\geq 1-[\frac{|p(-x)|}{p(x)}]^2-2x\frac{|p(-x)|}{p(x)}\ \frac{|p'(x)|}{p(x)}-2x\frac{|p'(-x)|}{p(x)}\\ &\geq 1-n^{-(8-2\varepsilon)/\pi}-2n^{-(4-\pi-\varepsilon)/\pi}-2n^{-(4-\pi-\varepsilon)/\pi}>0.\end{aligned}$$

This will prove Case 2.

Proof of (37). Note the inequality

$$\frac{1}{\sqrt{x}} \leq \int_{x-1/2}^{x+1/2} \frac{1}{\sqrt{y}} dy = 2\left(\sqrt{x+\frac{1}{2}} - \sqrt{x-\frac{1}{2}}\right), \quad x > \frac{1}{2}. \tag{40}$$

Then by definitions (23), (24) and the mean value theorem we have for an arbitrary pair of indices i, j, $0 \leq i < j \leq q+1$:

$$\xi_j/\xi_i = \exp\left[-(A_{i+1} + \cdots + A_j)\right] \geq \exp\left[-\frac{\pi}{2}\left(\frac{1}{\sqrt{n-(i+1)}} + \cdots + \frac{1}{\sqrt{n-j}}\right)\right]$$

$$\geq \exp\left[-\pi\left(\sqrt{n-\frac{1}{2}-i} - \sqrt{n-\frac{1}{2}-j}\right)\right] \geq \exp\left(-\pi\frac{j-i}{2\sqrt{n-\frac{1}{2}-j}}\right)$$

$$\geq \exp\left(-\frac{\pi}{2}\frac{j-i}{\ln n}\right) = \theta^{j-i},$$

where

$$\theta := \exp\left(-\frac{\pi}{2}\frac{1}{\ln n}\right) \to 1, \quad \text{when } n \to \infty. \tag{41}$$

Therefore for $x \in (\xi_{t+1}, \xi_t)$ (observe $\frac{1-x}{1+x} \leq e^{-2x}$, $x \geq 0$)

$$\frac{|p(-x)|}{p(x)} = \prod_{j=0}^{t} \frac{1-\frac{x}{\xi_j}}{1+\frac{x}{\xi_j}} \prod_{j=t+1}^{n-1} \frac{1-\frac{\xi_j}{x}}{1+\frac{\xi_j}{x}} \leq \prod_{j=0}^{t} \frac{1-\frac{\xi_{t+1}}{\xi_j}}{1+\frac{\xi_{t+1}}{\xi_j}} \prod_{j=t+1}^{q+1} \frac{1-\frac{\xi_j}{\xi_t}}{1+\frac{\xi_j}{\xi_t}}$$

$$\leq \prod_{j=0}^{t} \frac{1-\theta^{t+1-j}}{1+\theta^{t+1-j}} \prod_{j=t+1}^{q+1} \frac{1-\theta^{j-t}}{1+\theta^{j-t}} = \prod_{j=1}^{t+1} \frac{1-\theta^j}{1+\theta^j} \prod_{j=1}^{q+1-t} \frac{1-\theta^j}{1+\theta^j}$$

$$\leq \prod_{j=1}^{t+1} e^{-2\theta^j} \prod_{j=1}^{q+1-t} e^{-2\theta^j} = \exp\left[-2\left(\sum_{j=1}^{t+1} \theta^j + \sum_{j=1}^{q+1-t} \theta^j\right)\right]$$

$$= \exp\left(-2 \cdot \frac{\theta - \theta^{t+2} + \theta - \theta^{q-t+2}}{1-\theta}\right) \leq \exp\left(-2\frac{\theta - \theta^{2r+1}}{1-\theta}\right),$$

because $t \in \{2r-1, \ldots, q\}$.

Now $\theta \to 1$ for $n \to \infty$, but $\theta^{2r+1} \leq \exp\left(-\frac{2\theta\sqrt{n}}{\ln n}\right) \to 0$ for $n \to \infty$, so for any $\varepsilon > 0$ and $n \geq n_0(\varepsilon)$

$$\frac{|p(-x)|}{p(x)} \leq \exp\left[-\left(2-\frac{\varepsilon}{2}\right)\frac{1}{1-\theta}\right] \leq \exp\left[-\left(2-\frac{\varepsilon}{2}\right)\frac{2\ln n}{\pi}\right] = n^{-(4-\varepsilon)/\pi}.$$

Here we have used $\frac{1}{1-e^{-x}} \geq \frac{1}{x}$ for $x > 0$.

Proof of (38). By (30) we have

$$x\frac{|p'(x)|}{p(x)}=x\sum_{j=0}^{n-1}\frac{1}{x+\xi_j}=\sum_{j=0}^{n-1}\frac{1}{1+\xi_j/x}\leq n.$$

Proof of (39). From (31) we have much as in the proof of (37) with $x\in(\xi_{t+1},\xi_t)$:

$$x\frac{|p'(-x)|}{p(x)}=x\left|\sum_{j=0}^{n-1}\frac{\prod\limits_{\substack{k=0\\k\neq j}}^{n-1}(-x+\xi_k)}{\prod\limits_{k=0}^{n-1}(x+\xi_k)}\right|\leq\sum_{j=0}^{n-1}\frac{1}{1+\xi_j/x}\prod_{\substack{k=0\\k\neq j}}^{n-1}\frac{|-x+\xi_k|}{x+\xi_k}$$

(42)

$$\leq\sum_{j=0}^{n-1}\prod_{\substack{k=0\\k\neq j}}^{n-1}\frac{|-x+\xi_k|}{x+\xi_k}\leq n\,.\max_{0\leq j\leq n-1}\prod_{\substack{k=0\\k\neq j}}^{q+1}\frac{|-x+\xi_k|}{x+\xi_k}\,.$$

a) For $j>q+1$ we have as in (37)

$$\prod_{\substack{k=0\\k\neq j}}^{q+1}\frac{|-x+\xi_k|}{x+\xi_k}\leq n^{-(4-\varepsilon)/\pi}\,.$$

b) For $t+1\leq j\leq q+1$ we have

$$\prod_{\substack{k=0\\k\neq j}}^{q+1}\frac{|-x+\xi_k|}{x+\xi_k}=\prod_{k=0}^{t}\frac{\xi_k-x}{\xi_k+x}\prod_{k=t+1}^{j-1}\frac{x-\xi_k}{x+\xi_k}\prod_{k=j+1}^{q+1}\frac{x-\xi_k}{x+\xi_k}$$

$$\leq\prod_{k=0}^{t}\frac{1-\frac{\xi_{t+1}}{\xi_k}}{1+\frac{\xi_{t+1}}{\xi_k}}\prod_{k=t+1}^{j-1}\frac{1-\frac{\xi_k}{\xi_t}}{1+\frac{\xi_k}{\xi_t}}\prod_{k=j+1}^{q+1}\frac{1-\frac{\xi_k}{\xi_t}}{1+\frac{\xi_k}{\xi_t}}$$

$$\leq\prod_{k=0}^{t}\frac{1-\theta^{t+1-k}}{1+\theta^{t+1-k}}\prod_{k=t+1}^{j-1}\frac{1-\theta^{k-t}}{1+\theta^{k-t}}\prod_{k=j+1}^{q+1}\frac{1-\theta^{k-t}}{1+\theta^{k-t}}$$

$$=\prod_{k=1}^{t+1}\frac{1-\theta^k}{1+\theta^k}\prod_{k=1}^{j-1-t}\frac{1-\theta^k}{1+\theta^k}\prod_{k=j+1-t}^{q+1-t}\frac{1-\theta^k}{1+\theta^k}$$

$$\leq\exp\Big[-2\Big(\sum_{k=1}^{t+1}\theta^k+\sum_{k=1}^{j-1-t}\theta^k+\sum_{k=j+1-t}^{q+1-t}\theta^k\Big)\Big]$$

$$=\exp\Big[-2\Big(\frac{\theta-\theta^{t+2}}{1-\theta}+\frac{\theta-\theta^{j-t}}{1-\theta}+\frac{\theta^{j+1-t}-\theta^{q+2-t}}{1-\theta}\Big)\Big]$$

$$\leq\exp\Big(-2\,\frac{\theta-\theta^{t+2}}{1-\theta}\Big)\leq n^{-(4-\varepsilon)/\pi}\,,$$

since $t\geq 2r-1$.

c) Similarly, for $0\le j\le t$ we obtain

$$\prod_{\substack{k=0\\k\neq j}}^{q+1}\frac{|-x+\xi_k|}{|x+\xi_k|}\le\prod_{k=0}^{j-1}\frac{1-\frac{\xi_{t+1}}{\xi_k}}{1+\frac{\xi_{t+1}}{\xi_k}}\prod_{k=j+1}^{t}\frac{1-\frac{\xi_{t+1}}{\xi_k}}{1+\frac{\xi_{t+1}}{\xi_k}}\prod_{k=t+1}^{q+1}\frac{1-\frac{\xi_k}{\xi_t}}{1+\frac{\xi_k}{\xi_t}}$$

$$\le\prod_{k=t+2-j}^{t+1}\frac{1-\theta^k}{1+\theta^k}\prod_{k=1}^{t-j}\frac{1-\theta^k}{1+\theta^k}\prod_{k=1}^{q+1-t}\frac{1-\theta^k}{1+\theta^k}$$

$$\le\exp\left[-2\left(\frac{\theta^{t+2-j}-\theta^{t+2}}{1-\theta}+\frac{\theta-\theta^{t+1-j}}{1-\theta}+\frac{\theta-\theta^{q+2-t}}{1-\theta}\right)\right].$$

Here, for fixed $t\in\{2r,\dots,q\}$ the exponent is strictly increasing in $j\in\{0,\dots,t\}$, hence we get with $j=t$:

$$\prod_{\substack{k=0\\k\neq j}}^{q+1}\frac{|-x+\xi_k|}{|x+\xi_k|}\le\exp\left(-2\frac{\theta+\theta^2-\theta^{t+2}-\theta^{q+2-t}}{1-\theta}\right)$$

$$\le\exp\left(-2\,\frac{\theta-\theta^{2r+1}}{1-\theta}\right)\le n^{-(4-\varepsilon)/\pi}.$$

Putting together estimates a), b), c) and (42) we arrive at (39):

$$x\,\frac{|p'(-x)|}{p(x)}\le n^{-(4-\pi-\varepsilon)/\pi},\quad x\in(\xi_{t+1},\ \xi_t)\quad\text{with}$$

$$t\in\{2r-1,\dots,q\},\quad \varepsilon>0,\quad n\ge n_0(\varepsilon).$$

Case 3. $x\in(\xi_{t+1},\ \xi_t)$, $t\in\{q+1,\dots,\ n-2\}$; $q=[n-\frac{3}{2}-\ln^2 n]$, and $p(-x)>0$. Then from (22) we have that $t\le n-3$ and (36) is equivalent to

$$\frac{p(x)}{p(-x)}+2x\sum_{j=0}^{n-1}\left(\frac{1}{x+\xi_j}+\frac{1}{-x+\xi_j}\right)>\frac{p(-x)}{p(x)}.$$

From the well-known inequality

$$\prod_{k=0}^{l-1}(1+y_k)\ge 1+\sum_{k=0}^{l-1}y_k,\quad y_k\ge 0, \tag{43}$$

we conclude that

$$\frac{p(x)}{p(-x)}=\prod_{k=0}^{t}\frac{\xi_k+x}{\xi_k-x}\prod_{k=t+1}^{n-1}\frac{x+\xi_k}{x-\xi_k} \tag{44}$$

$$=\prod_{k=0}^{t}\left(1+\frac{2x}{\xi_k-x}\right)\prod_{k=t+1}^{n-1}\left(1+\frac{2\xi_k}{x-\xi_k}\right)$$

$$\ge\left(1+2\sum_{k=0}^{t}\frac{x}{\xi_k-x}\right)\left(1+2\sum_{k=t+1}^{n-1}\frac{\xi_k}{x-\xi_k}\right)$$

$$\ge 1+2\sum_{k=0}^{t}\frac{x}{\xi_k-x}+2\sum_{k=t+1}^{n-1}\frac{\xi_k}{x-\xi_k}.$$

So, together with (35) it is clear that (36) will hold if we only show

$$1+2\sum_{k=0}^{t}\frac{x}{\xi_k-x}+2\sum_{k=t+1}^{n-1}\frac{\xi_k}{x-\xi_k}+2\sum_{k=0}^{n-1}\left(\frac{x}{x+\xi_k}+\frac{x}{-x+\xi_k}\right)$$

$$=1+2\sum_{k=0}^{n-1}\frac{1}{1+\xi_k/x}+4\sum_{k=0}^{t}\frac{1}{\xi_k/x-1}+2\sum_{k=t+1}^{n-1}\left(\frac{\xi_k}{x-\xi_k}+\frac{x}{-x+\xi_k}\right)$$

$$\geqq 1+2\sum_{k=0}^{n-1}\frac{1}{1+\xi_k/\xi_{t+1}}+4\sum_{k=0}^{t}\frac{1}{\xi_k/\xi_{t+1}-1}-2(n-1-t)>\frac{\pi}{4}\,.$$

Now we learn from (26):

$$\frac{1}{1+\frac{\xi_{t+1-k}}{\xi_{t+1}}}+\frac{1}{1+\frac{\xi_{t+1+k}}{\xi_{t+1}}}\geqq\frac{1}{1+\frac{\xi_{t+1-k}}{\xi_{t+1}}}+\frac{1}{1+\frac{\xi_{t+1}}{\xi_{t+1-k}}}=1$$

and therefore, as $t\geqq(n-3)/2$,

$$\sum_{k=0}^{n-1}\frac{1}{1+\frac{\xi_k}{\xi_{t+1}}}\geqq\frac{1}{2}+\sum_{k=1}^{n-1-(t+1)}\left(\frac{1}{1+\frac{\xi_{t+1-k}}{\xi_{t+1}}}+\frac{1}{1+\frac{\xi_{t+1+k}}{\xi_{t+1}}}\right)\geqq\frac{1}{2}+(n-2-t).$$

From this we have finally

$$1+2\sum_{k=0}^{n-1}\frac{1}{1+\frac{\xi_k}{\xi_{t+1}}}+4\sum_{k=0}^{t}\frac{1}{\frac{\xi_k}{\xi_{t+1}}-1}-2(n-1-t)$$

$$\geqq 1+1+2(n-2-t)+4\sum_{k=0}^{t}\frac{1}{\frac{\xi_k}{\xi_{t+1}}-1}-2(n-1-t)$$

$$=4\sum_{k=0}^{t}\frac{1}{\frac{\xi_k}{\xi_{t+1}}-1}\geqq 4\frac{1}{\frac{\xi_t}{\xi_{t+1}}-1}=\frac{4}{\exp(A_{t+1})-1}$$

$$=\frac{4}{\exp\left[\frac{\pi}{2}\,\frac{1}{\sqrt{n-(t+1)}}\right]-1}\geqq\frac{4}{\exp\left(\frac{\pi}{2}\,\frac{1}{\sqrt{2}}\right)-1}\geqq 1.9>\frac{\pi}{4}$$

which was to be shown. We only used here $t\leqq n-3$.

Case 4. $x\in(\xi_{t+1},\ \xi_t)$, $t\in\{q+1,\dots,n-2\}$, $q=[n-\frac{3}{2}-\ln^2 n]$, and $p(-x)<0$.

Under the above assumptions (36) is equivalent to

$$\frac{p(x)}{|p(-x)|}>2x\left[\sum_{j=0}^{n-1}\left(\frac{1}{x+\xi_j}+\frac{1}{-x+\xi_j}\right)\right]+\frac{|p(-x)|}{p(x)}$$

$$\Leftrightarrow\frac{\xi_t+x}{\xi_t-x}\,\frac{x+\xi_{t+1}}{x-\xi_{t+1}}\prod_{\substack{j=0\\ j\neq t,\,t+1}}^{n-1}\frac{x+\xi_j}{|x-\xi_j|}>2x\left[\sum_{\substack{j=0\\ j\neq t,\,t+1}}^{n-1}\left(\frac{1}{x+\xi_j}+\frac{1}{-x+\xi_j}\right)\right]$$

$$+2x\left[\frac{1}{x+\xi_t}+\frac{1}{-x+\xi_t}+\frac{1}{x+\xi_{t+1}}+\frac{1}{-x+\xi_{t+1}}\right]+\frac{|p(-x)|}{p(x)}$$

(45) $$\Leftrightarrow\left(1+\frac{x}{\xi_t}\right)\left(1+\frac{\xi_{t+1}}{x}\right)\prod_{\substack{j=0\\ j\neq t,\ t+1}}^{n-1}\frac{x+\xi_j}{|x-\xi_j|}$$

$$>2\left(1-\frac{x}{\xi_t}\right)\left(1-\frac{\xi_{t+1}}{x}\right)\left[\sum_{\substack{j=0\\ j\neq t,\ t+1}}^{n-1}\left(\frac{x}{x+\xi_j}+\frac{x}{-x+\xi_j}\right)\right]$$

$$+2\left(1-\frac{x}{\xi_t}\right)\left(1-\frac{\xi_{t+1}}{x}\right)\left[\frac{1}{1+\frac{\xi_t}{x}}+\frac{1}{-1+\frac{\xi_t}{x}}+\frac{1}{1+\frac{\xi_{t+1}}{x}}+\frac{1}{-1+\frac{\xi_{t+1}}{x}}\right]$$

$$+\left(1-\frac{x}{\xi_t}\right)\left(1-\frac{\xi_{t+1}}{x}\right)\frac{|p(-x)|}{p(x)}.$$

We note the following facts in short:

(46) $$\left(1-\frac{x}{\xi_t}\right)\left(1-\frac{\xi_{t+1}}{x}\right)\left[\frac{1}{1+\frac{\xi_t}{x}}+\frac{1}{-1+\frac{\xi_t}{x}}+\frac{1}{1+\frac{\xi_{t+1}}{x}}+\frac{1}{-1+\frac{\xi_{t+1}}{x}}\right]$$

$$=\left(1-\frac{x}{\xi_t}\right)\left(1-\frac{\xi_{t+1}}{x}\right)\left[2\,\frac{\xi_t}{x}\,\frac{1}{\left(1+\frac{\xi_t}{x}\right)\left(-1+\frac{\xi_t}{x}\right)}+2\,\frac{\xi_{t+1}}{x}\,\frac{1}{\left(1+\frac{\xi_{t+1}}{x}\right)\left(-1+\frac{\xi_{t+1}}{x}\right)}\right]$$

$$=2\left[\frac{1-\frac{\xi_{t+1}}{x}}{1+\frac{\xi_t}{x}}-\frac{1-\frac{x}{\xi_t}}{1+\frac{x}{\xi_{t+1}}}\right].$$

(47) The expression $\dfrac{1-\frac{\xi_{t+1}}{x}}{1+\frac{\xi_t}{x}}$ is strictly increasing in x

while $\dfrac{1-\frac{x}{\xi_t}}{1+\frac{x}{\xi_{t+1}}}$ is strictly decreasing in x.

(48) The function $\left(1+\frac{\xi_{t+1}}{x}\right)\left(1+\frac{x}{\xi_t}\right)$ takes its minimum on (ξ_{t+1}, ξ_t) at the point $x=\sqrt{\xi_{t+1}\,\xi_t}$, the function $\left(1-\frac{\xi_{t+1}}{x}\right)\left(1-\frac{x}{\xi_t}\right)$ becomes maximal there.

So from (46) and (35) it is clear that our assertion (45) will follow from

$$(1+\frac{x}{\xi_t})(1+\frac{\xi_{t+1}}{x}\prod_{\substack{j=0\\ j\neq t,\ t+1}}^{n-1}\frac{x+\xi_j}{|x-\xi_j|}$$

$$>2(1-\frac{x}{\xi_t})(1-\frac{\xi_{t+1}}{x})\,[\sum_{\substack{j=0\\ j\neq t,\ t+1}}^{n=1}(\frac{x}{x+\xi_j}+\frac{x}{-x+\xi_j})]+4\left[\frac{1-\frac{\xi_{t+1}}{x}}{1+\frac{\xi_t}{x}}-\frac{1-\frac{x}{\xi_t}}{1+\frac{x}{\xi_{t+1}}}\right]$$

$$+(1-\frac{x}{\xi_t})(1-\frac{\xi_{t+1}}{x})\frac{\pi}{4},$$

and this will, by (47) and (48), be again a consequence of

$$(49)\qquad (1+\rho)^2\prod_{\substack{j=0\\ j\neq t,\ t+1}}^{n-1}\frac{x+\xi_j}{|x-\xi_j|}>2(1-\rho)^2[\sum_{\substack{j=0\\ j\neq t,\ t+1}}^{n-1}(\frac{h}{x+\xi_j}+\frac{x}{-x+\xi_j})]$$

$$+2(1-\rho^2)+(1-\rho)^2\frac{\pi}{4},$$

where we define

$$(50)\qquad \rho=\rho(t)=\sqrt{\xi_{t+1}/\xi_t}=\exp(-\frac{1}{2}A_{t+1})\searrow \text{ in } t \text{ (see (26))}.$$

By the same arguments as in (43), (44) (49) will follow from

$$(1+\rho)^2(1+2\sum_{j=0}^{t-1}\frac{x}{\xi_j-x}+2\sum_{j=t+2}^{n-1}\frac{\xi_j}{x-\xi_j})$$

$$>2(1-\rho)^2[\sum_{\substack{j=0,\\ j\neq t,\ t+1}}^{n-1}(\frac{x}{x+\xi_j}+\frac{x}{-x+\xi_j})]+2(1-\rho^2)+(1-\rho)^2\frac{\pi}{4}.$$

Now let us regard ρ as an independent variable which can take the discrete values $\rho(q+1),\ldots,\rho(n-2)$. Then clearly by (50) with $\rho(n-2)=\exp(-\frac{1}{2}\frac{\pi}{2})\approx 0.4559381$

$$(51)\qquad (1+\rho)^2\geqq[1+\rho(n-2)]^2\geqq 2.1\geqq 1.9\geqq 2[1-\rho^2(n-2)]$$

$$+[1-\rho(n-2)]^2\frac{\pi}{4}\geqq 2(1-\rho^2)+(1-\rho)^2\frac{\pi}{4}.$$

Hence we have only to establish

$$(1+\rho)^2\,[\sum_{j=0}^{t-1}\frac{x}{\xi_j-x}+\sum_{j=t+2}^{n-1}\frac{\xi_j}{x-\xi_j}]$$

$$>(1-\rho)^2\,[\sum_{\substack{j=0\\ j\neq t,\ t+1}}^{n-1}\frac{x}{x+\xi_j}+\sum_{j=0}^{t-1}\frac{x}{\xi_j-x}-\sum_{j=t+2}^{n-1}\frac{x}{x-\xi_j}]$$

$$\Leftrightarrow 4\rho \sum_{j=0}^{t-1} \frac{x}{\xi_j - x} + (1-\rho)^2 \sum_{j=t+2}^{n-1} \frac{x}{x-\xi_j} + (1+\rho)^2 \sum_{j=t+2}^{n-1} \frac{\xi_j}{x-\xi_j}$$

$$> (1-\rho)^2 \sum_{j=t,\ t+1}^{n-1} \frac{x}{x+\xi_j}.$$

Here all sums are positive. Disregarding the third sum on the left our assertion follows from the remark that $\frac{x}{|x-\xi_j|} \geq \frac{x}{x+\xi_j}$ and that as in (51):

$$4\rho \geq 4\rho(n-2) > [1-\rho(n-2)]^2 \geq (1-\rho)^2.$$

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Centre for Mathematics and Mechanics
1090 Sofia P. O. Box 373

Received 20. 09. 1982

Institut für Angewandte Mathematik
der Universität Bonn
Sonderforschungsbereich 72
Wegelerstr. 6 5300 Bonn 1, BRD