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A NOTE ON A SEMI-INVARIANT SUBMANIFOLD OF A PARA-SASAKIAN MANIFOLD

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In this note we define semi-invariant submanifolds of a para-Sasakian manifold and prove that each semi-invariant submanifold of a para-Sasakian manifold is necessarily an invariant submanifold.

A Sasakian structure on a manifold has been defined in [1]. Later on analogous structure called para-Sasakian structure was introduced and studied in [4].

Recently in (1981) semi-invariant submanifolds of a Sasakian manifold have been defined and studied in [2]. It has been proved that a Sasakian manifold alwalys admits a semi-invariant submanifold. It was then natural to investigate similar properties of semi-invariant submanifolds of a para-Sasakian manifold. In this note we have shown that each semi-invariant submanifold of a para-Sasakian manifold is necessarily an invariant submanifold and consequently a para-Sasakian manifold does not admit any proper semi-invariant submanifold.

1. Preliminaries. An n-dim differentiable manifold $\tilde{M}$ is called an almost paracontact Riemannian manifold [3], if there exists in $\tilde{M}$ a tensor field F of type (1, 1), a positive definite Riemannian metric g, a contravariant vector field ξ and a covariant vector field η satisfying

$$F^2 = I - \eta \otimes \xi, \tag{1.1}$$

$$\eta(\xi) = 1, \tag{1.2}$$

$$g(FX, FY) = g(X, Y) - \eta(X)\eta(Y), \text{ and} \tag{1.3}$$

$$\eta(X) = g(X, \xi). \tag{1.4}$$

The set (F, ξ, η, g) is then called an almost paracontact Riemannian structure on $\tilde{M}$. In such a manifold, the following relations hold:

$$\begin{cases} \text{(i)} & F'(X, Y) = F'(Y, X) \ \text{ where } \ F'(X, Y) = g(FX, Y), \\ \text{(ii)} & F(\xi) = 0, \ \text{(iii)} \ \eta \circ F = 0, \ \text{and} \\ \text{(iv)} & \text{rank } (F) = (n-1). \end{cases} \tag{1.5}$$

An almost paracontact Riemannian structure (F, ξ, η, g) on a manifold is called a para-Sasakian structure [4] if

$$(\tilde{\nabla}_X F)Y = g(X, Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(Y)X, \tag{1.6}$$

where $\tilde{\nabla}$ denotes the Riemannian connection on $\tilde{M}$. On a para-Sasakian manifold we have [4]

$$\tilde{\nabla}_X \xi = -FX. \tag{1.7}$$

2. Semi-invariant submanifold of a para-Sasakian manifold. Let $\tilde{M}$ be an n-dim almost paracontact Riemannian manifold with structure (F, ξ, η, g) and M be an m-dim differentiable manifold isometrically immersed in $\tilde{M}$ such that ξ is tangential to M. Let TM and $TM^{\perp}$ denote the tangent bundle and normal bundle respectively on M. We define a semi-invariant submanifold of $\tilde{M}$ as follows:

A submanifold M of an almost paracontact Riemannian manifold $\tilde{M}$ is called a semi-invariant submanifold of $\tilde{M}$ if the following conditions are satisfied:
(i) $TM = D \oplus D^{\perp} \oplus \{\xi\}$,
(ii) *The distribution D is invariant by F, that is, for each $X \in D$, $FX \in D$, and*
(iii) *the distribution $D^{\perp}$ is anti-invariant by F, that is, for each $X \in D^{\perp}$, $FX \in F(D^{\perp}) \subset TM$,*
where D, $D^{\perp}$ and $\{\xi\}$ are orthogonal distributions on M such that (i) is satisfied.

The distributions D and $D^{\perp}$ are called respectively the invariant distribution and the anti-invariant distribution of M. It is easily seen that invariant (resp. anti-invariant) submanifold is a particular case of semi-invariant submanifolds when $\dim D = 0$ (resp. $\dim D = 0$). A semi-invariant submanifold which is neither invariant nor anti-invariant is called a proper semi-invariant submanifold.

Let M be a semi-invariant submanifold of a para-Sasakian manifold $\tilde{M}$ and g denote the Riemannian metric on $\tilde{M}$ as well as the indiced metric on M. Each $X \in TM$ can be represented by

$$X = PX + QX + \eta(X)\xi, \text{ where } PX \in D \text{ and } QX \in D^{\perp}. \tag{2.1}$$

Thus P and Q are projection morphisms of TM into D and $D^{\perp}$, respectively. Again if $N \in TM^{\perp}$, then FN can be written as

$$FN = BN + CN, \text{ where } BN \in TM \text{ and } CN \in TM^{\perp}. \tag{2.2}$$

It can be easily seen that for $X \in D$, $g(FN, X) = 0$ and also $g(FN, \xi) = 0$, which gives $BN \in D^{\perp}$.

Let $\tilde{\nabla}$ and ∇ be the Riemannian connection on $\tilde{M}$ and M, respectively. Then the equations of Gauss and Weingarten are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.3}$$

and

$$\tilde{\nabla}_X N = -A_N X + \Delta_X^{\perp} N, \tag{2.4}$$

respectively, for all $X, Y \in TM$, $N \in TM^{\perp}$, where h is the second fundamental form of M, $A_N X$ and $\nabla_X^{\perp} N$ are tangential and normal parts of $\tilde{\nabla}_X N$. From (2.3) and (2.4) we get

$$g(h(X, Y), N) = g(A_N X, Y). \tag{2.5}$$

Consequently $g(A_N X, Y)$ is symmetric in X and Y.

L e m m a 2.1. *Let M be a semi-invariant submanifold of a para-Sasakian manifold $\tilde{M}$. Then the following relation holds:*

$$g((\tilde{\nabla}_X F)Y, Z) = \eta(Y)g(X, Z), \text{ for all } Z \in D \oplus D^{\perp}. \tag{2.6}$$

P r o o f. Using (1.6) and the definition of semi-invariant submanifold we obtain $\forall Z \in D \oplus D^{\perp}$

$$g((\tilde{\nabla}_X F)Y, Z) = g(g(X, Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(Y)X, Z) = g(\eta(Y)X, Z) = \eta(Y)g(X, Z),$$

which completes the proof.

In view of Lemma 2.1 we can directly state the following:

C o r o l l a r y (2.1). *Let M be a semi-invariant submanifold of a para-Sasakian manifold $\tilde{M}$. Then the following relations hold:*

$$g((\tilde{\nabla}_X F)Y, Z) = 0, \ \forall\, Y, Z \in D \oplus D^{\perp}, \tag{2.7}$$

and

$$g((\tilde{\nabla}_X F)Y, Z) = 0, \ \forall\, X \perp Z, \text{ where } Z \in D \oplus D^{\perp}. \tag{2.8}$$

We next prove

L e m m a 2.2. *In a semi-invariant submanifold M of a para-Sasakian manifold $\tilde{M}$, we have*

$$A_{FX}Y + A_{FY}X = 0, \quad \forall\, X, Y \in D^{\perp}. \tag{2.9}$$

P r o o f. Since $X, Y \in D^{\perp}$, FX and $FY \in TM^{\perp}$, (2.5) yields $g(A_{FX}Y, Z) = g(h(Y, Z), FX) = g(h(Z, Y), FX)$. Using (3.3) we get

$$g(A_{FX}Y, Z) = g((\tilde{\nabla}_Z Y), FX). \tag{2.10}$$

Now (1.5) (i) we have $g((\tilde{\nabla}_Z Y), FX) = g(F(\tilde{\nabla}_Z Y), X) = g(\tilde{\nabla}_Z(FY)) - (\tilde{\nabla}_Z F)Y, X)$. Using (2.7) and (2.4) the above equation reduces to $g((\tilde{\nabla}_Z Y), FX) = g(\tilde{\nabla}_Z(FY), X) = g(-A_{FY}Z, X) = -g(A_{FY}X, Z)$. The above equation and (2.10) provide the proof.

T h e o r e m 2.1. *A semi-invariant submanifold M of a para-Sasakian manifold $\tilde{M}$ is necessarily an invariant submanifold. Consequently a para-Sasakian manifold $\tilde{M}$ does not admit any proper semi-invariant submanifold.*

P r o o f. Using (2.4), (1.6) and (1.5) (i) we get for all $X, Y \in D^{\perp}$

$$\eta(A_{FY}X) = g(-\tilde{\nabla}_X(FY), \xi) = -g((\tilde{\nabla}_X F)Y + F(\tilde{\nabla}_X Y), \xi)$$

$$= -g(g(X, Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(Y)X, \xi) - g(\tilde{\nabla}_X Y, F\xi) = -g(X, Y).$$

Interchanging X and Y in the above equation and then adding both equations we get $\eta(A_{FY}X + A_{FX}Y) = -2g(X, Y)$. Using (2.9) in the above equation we get $g(X, Y) = 0$ for all $X, Y \in D^{\perp}$. Consequently the dimension of $D^{\perp}$ is zero and the submanifold is invariant submanifold, which completes the proof.

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